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Parallel AGEI Method for Fourth Order Parabolic Equations

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Abstract

Based on a high order absolutely stable implicit scheme, we construct the alternating group explicit iterative method (AGEI) for four order parabolic equations. The method is verified to be convergent, and has the intrinsic parallelism. Results of the convergence analysis shows that the method is of high accuracy.

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1 Introduction

Many finite difference methods have been presented for 1D and 2D parabolic equations so far, which are sorted by explicit and implicit methods in general. Most of explicit methods are short in stability and accuracy, while implicit methods are unadaptable for parallel computing, and need to solve large equation set. Thus it is necessary to construct parallel methods with absolute stability. With the development of parallel computer, research on parallel numerical methods is getting more and more popular. A class of AGE method is derived in [1-2] for 1D parabolic equations. The AGE method is of intrinsic parallelism and unconditional stability. Soon the AGE method is applied to other equations by many authors such as in [3-4]. R. Tavakoli derived a class of domain-split method based on the AGE method for 1D and 2D diffusion equation in [5-6]. We notice that almost all the methods have less than four order accuracy in spatial step, and research on alternating group explicit iterative method for fourth order parabolic equations has been scarcely presented.

In this paper we consider the following periodic boundary value problem of fourth order parabolic equations:

$$\begin{cases} \dfrac{\partial u}{\partial t}+\dfrac{\partial^4 u}{\partial x^4}=0, \quad -\infty<x<\infty, \quad 0\le t\le T \\ u(x,0)=f(x), \\ u(x+l,t)=u(x,t). \end{cases} \tag{1.1}$$

We organize this paper as follows: In section 2, we construct the AGEI method based on an $O(\tau^2+h^6)$ order unconditionally stable symmetry implicit scheme in [7]. In section 3, convergence analysis and stability analysis are given. Results of several numerical examples are presented in section 4.

2 The Alternating Group Iterative Method(AGEI)

The domain $\Omega:(0,1)\times(0,T)$ will be divided into $(m\times\xi)$ meshes with spatial step size h=$\dfrac{l}{m}$ in x direction and the time step size τ=$\dfrac{T}{\xi}$. Grid points are denoted by (x_i,t_n), $x_i=ih(i=0,\ 1,\cdots,m)$, $t_n=n\tau(n=0,\ 1,\cdots,\frac{T}{\tau})$. The numerical solution of (1.1) is denoted by u_i^n, while the exact solution $u(x_i,\ t_n)$. Let r=$\frac{360\tau}{h^4}$.

We present the implicit finite difference scheme [7] for solving (1.1) as below:

$$-(u_{j-2}^{n+1}+u_{j+2}^{n+1})+124(u_{j-1}^{n+1}+u_{j+1}^{n+1})+474u_j^{n+1}+r(u_{j-2}^{n+1}-4u_{j-1}^{n+1}+6u_j^{n+1}-4u_{j+1}^{n+1}+u_{j+2}^{n+1})=$$

$$-(u_{j-2}^{n}+u_{j+2}^{n})+124(u_{j-1}^{n}+u_{j+1}^{n})+474u_j^{n}-r(u_{j-2}^{n}-4u_{j-1}^{n}+6u_j^{n}-4u_{j+1}^{n}+u_{j+2}^{n}) \tag{2.1}$$

Applying Taylor's formula to the scheme at (x_i,t_n). Considering $\frac{\partial^k u}{\partial t^k}=(-1)^k\frac{\partial^{4k}u}{\partial x^{4k}}$, we have the truncation error $O(\tau^2+h^6)$.

Let $U^n=(u_1^n,\ u_2^n,\ \cdots,\ u_m^n)^T$, then we can denote (2.1) as $AU^{n+1}=\widetilde{F}^n$. here $\widetilde{F}^n=FU^n$.

$$A=\begin{pmatrix} 474+6r & 124-4r & -1+r & & & & & -1+r & 124-4r \\ 124-4r & 474+6r & 124-4r & -1+r & & & & & -1+r \\ -1+r & 124-4r & 474+6r & 124-4r & -1+r & & & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & -1+r & 124-4r & 474+6r & & 124-4r & -1+r & \\ -1+r & & & -1+r & 124-4r & & 474+6r & 124-4r & \\ 124-4r & -1+r & & & -1+r & & 124-4r & 474+6r & \end{pmatrix}_{m\times m}$$

$$F=\begin{pmatrix}
474-6r & 124+4r & -1-r & & & & -1-r & 124+4r \\
124+4r & 474-6r & 124+4r & -1-r & & & & -1-r \\
-1-r & 124+4r & 474-6r & 124+4r & -1-r & & & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & -1-r & 124+4r & 474-6r & & 124+4r & -1-r \\
-1-r & & & -1-r & 124+4r & & 474-6r & 124+4r \\
124+4r & -1-r & & & -1-r & & 124+4r & 474-6r
\end{pmatrix}_{m\times m}$$

Let $m=8k,\ k$ is an integer. Then the alternating group iterative method will be constructed as follows:

Let $A=\dfrac{1}{2}(G_1+G_2)$, here

$$G_1=\begin{pmatrix}
G_{11} & & & \\
 & \cdots & & \\
 & & \cdots & \\
 & & & \cdots & \\
 & & & & G_{11}
\end{pmatrix}_{m\times m},\quad G_2=\begin{pmatrix}
G_{21} & & & & \overline{G} \\
 & G_{11} & & & \\
 & & \cdots & & \\
 & & & G_{11} & \\
\overline{G}^T & & & & G_{21}
\end{pmatrix}_{m\times m}$$

$$\overline{G}=\begin{pmatrix}
0 & 0 & -2+2r & 248-8r \\
0 & 0 & 0 & -2+2r \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},\quad G_{21}=\begin{pmatrix}
474+6r & 124-4r & -1+r & 0 \\
124-4r & 474+6r & 124-4r & -1+r \\
-1+r & 124-4r & 474+6r & 124-4r \\
0 & -1+r & 124-4r & 474+6r
\end{pmatrix}$$

$$G_{11}=\begin{pmatrix}
474+6r & 124-4r & -1+r & & & & & \\
124-4r & 474+6r & 124-4r & -1+r & & & & \\
-1+r & 124-4r & 474+6r & 124-4r & -2+2r & & & \\
 & -1+r & 124-4r & 474+6r & 248-8r & -2+2r & & \\
 & & -2+2r & 248-8r & 474+6r & 124-4r & -1+r & \\
 & & & -2+2r & 124-4r & 474+6r & 124-4r & -1+r \\
 & & & & -1+r & 124-4r & 474+6r & 124-4r \\
 & & & & & -1+r & 124-4r & 474+6r
\end{pmatrix}$$

Then the alternating group iterative method can be derived as below:

$$\begin{cases}
(\rho I+G_1)U^{n+\frac{1}{2}}=(\rho I-G_2)U^n+2\widetilde{F}^n \\
(\rho I+G_2)U^{n+1}=(\rho I-G_1)U^{n+\frac{1}{2}}+2\widetilde{F}^n
\end{cases}\tag{2.2}$$

From the construction of (2.2), we can see the AGEI method is of obvious parallelism.

3 Convergence Analysis

Lemma 1 Let $\theta>0$, and $G+G^T$ is nonnegative, then $(\theta I+G)^{-1}$exists, and

$$\begin{cases}
\|(\theta I+G)^{-1}\|_2\le\theta^{-1} \\
\|(\theta I-G)(\theta I+G)^{-1}\|_2\le 1
\end{cases}\tag{3.1}$$

From the construction of the matrices we can see that $G_1,\ G_2,\ (G_1 + G_1^T),\ (G_2 + G_2^T)$ are all nonnegative matrices. Then we have

$$\|(\rho I - G_1)(\rho I + G_1)^{-1}\|_2 \leq 1,\ \|(\rho I - G_2)(\rho I + G_2)^{-1}\|_2 \leq 1$$

From (2.2), we can obtain $U^{n+1} = GU^n + 2(\rho I + G_2)^{-1}[(\rho I - G_1)(\rho I + G_1)^{-1}\widetilde{F}^n + \widetilde{F}^n]$. here $G = (\rho I + G_2)^{-1}(\rho I - G_1)(\rho I + G_1)^{-1}(\rho I - G_2)$ is growth matrix.

Let $\widetilde{G} = (\rho I + G_2)G(\rho I + G_2)^{-1} = (\rho I - G_1)(\rho I + G_1)^{-1}(\rho I - G_2)(\rho I + G_2)^{-1}$, then $\rho(G) = \rho(\widetilde{G}) \leq \|\widetilde{G}\|_2 \leq 1.$, which shows the AGEI method given by (2.2) is convergent. Then we have:

Theorem 1 The alternating group iterative method is convergent.

4 Numerical Experiments

We consider the following example:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^4 u}{\partial x^4},\ 0 \leq x \leq 2\pi,\ 0 \leq t \leq T \\ u(x,0) = sin(x), \\ u(0,t) = u(2\pi,t) = 0. \end{cases} \tag{4.1}$$

The exact solution for the problem is $u(x,t) = e^{-t}sinx$. Let A.E. denote maximum absolute error, while P.E. denote maximum relevant error. A.E=$|u_i^n - u(x_i,t_n)|$, P.E=$100 \times |u_i^n - u(x_i,t_n)/u(x_i,t_n)|$. Let $\rho = 1$. In order to verify the AGEI method, we present the following examples in variant conditions. We will use the iterative error 1×10^{-10} to control the A.E. in the process of iterativeness, and the results of numerical experiments is listed in the following tables:

Table 1: The numerical results for the iterative method I

	$m = 16, \tau = 10^{-3}, t = 100\tau$	$m = 16, \tau = 10^{-4}, t = 100\tau$
A.E.	1.734×10^{-6}	1.701×10^{-6}
P.E.	1.917×10^{-4}	1.718×10^{-4}

Table 2: The numerical results for the iterative method I

	$m = 24, \tau = 10^{-3}, t = 100\tau$	$m = 24, \tau = 10^{-4}, t = 100\tau$
A.E.	5.676×10^{-8}	1.772×10^{-6}
P.E.	2.021×10^{-5}	1.817×10^{-4}

Table 3: The numerical results for the iterative method II

	$m = 20, \tau = 10^{-3}, t = 100\tau$	$m = 20, \tau = 10^{-4}, t = 100\tau$
A.E.	1.706×10^{-6}	1.686×10^{-6}
P.E.	1.890×10^{-4}	1.702×10^{-4}

Table 4: The numerical results for the iterative method II

	$m = 28, \tau = 10^{-3}, t = 100\tau$	$m = 28, \tau = 10^{-4}, t = 100\tau$
A.E.	7.811×10^{-7}	1.722×10^{-6}
P.E.	2.069×10^{-4}	1.761×10^{-4}

5 Conclusions

In this paper, we present AGEI method for four order parabolic equations. The method is based on an $O(\tau^2 + h^6)$ order implicit scheme, which is of absolute stability. Results of table 1-4 show that the numerical solution by the AGEI method can converge to the exact solution with high accuracy, which accords to the conclusion of convergence and error analysis.

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