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# An Algorithm for Solving Initial Value Problems Using Laplace Adomian Decomposition Method

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**Abstract**

In this paper, we present a symbolic algorithm for solving a class of n$^{th}$ order linear or nonlinear ordinary differential equations with initial conditions. This algorithm is based on Laplace Adomian Decomposition Method (LADM). Numerical results will be presented.

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## 1 Introduction

The Adomian Decomposition Method (ADM) has been applied to a wide class of problems in physics, biology and chemical reactions. The method provides the solution in a rapid convergent series with computable terms. This method was successfully applied to nonlinear differential delay equations [1], a nonlinear dynamic systems [2], the heat equation [3,4], the wave equation [5], coupled nonlinear partial differential equations [6,7], linear and nonlinear integro-differential equations [8] and Airy's equation [9]. Different modifications of this method and their applications are given in [10-13].

The Laplace Adomian Decomposition Method (LADM) is a combination of ADM and placeLaplace transforms. This method was successfully used for solving Bratu and Duffing equations in [14,15].

In this paper, we developed a symbolic algorithm to find the solution of

$$\left\{ \begin{array}{l} a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y + F(y) = G(x) \\ y(x_0) = \alpha_0, y'(x_0) = \alpha_1, \ldots, y^{(n)}(x_0) = \alpha_n \end{array} \right. \quad (1)$$

initial value problem by LADM where $a_i$ and $\alpha_i$ $i = 0, 1, 2, \ldots, n$ are real constants.

# 2 Laplace Adomian Decomposition Method

In this section we want to describe how to use LADM for problem (1). We will study the second order equations for simplicity.

Now, consider the initial value problem

$$\begin{cases} a_2x^2\frac{d^2y}{dx^2} + a_1x\frac{dy}{dx} + a_0y + F(y) = G(x) \\ y(x_0) = \alpha_0, y'(x_0) = \alpha_1 \end{cases} \tag{2}$$

First, we use a transformation $x = e^t$ to reduce the equation in (2) to a differential equation with constant coefficients. So problem (2) turn into

$$\begin{cases} \frac{d^2y}{dt^2} + b_1\frac{dy}{dt} + b_0y + F(y) = G(t) \\ y(t_0) = \alpha_0, y'(t_0) = \alpha_1 \end{cases} \tag{3}$$

where $t_0 = \ln x_0$. Applying the placeLaplace transform to both sides of equation (2) we get

$$L\{y''\} + L\{b_1y'\} + L\{b_0y\} + L\{F(y)\} = L\{G(t)\}.$$

By using linearity of Laplece transforms the result is,

$$L\{y''\} + b_1L\{y'\} + b_0L\{y\} + L\{F(y)\} = L\{G(t)\}.$$

Applying the formulas of placeLaplace transform, we get

$$\left(s^2L\{y\} - sy(t_0) - y'(t_0)\right) + b_1(sL\{y\} - y(t_0)) + b_0L\{y\} + L\{F(y)\} = L\{G(t)\}.$$

Using the initial conditions in (3), we obtain

$$\left(s^2 + b_1s\right)L\{y\} = \alpha_0(s + b_1) + \alpha_1 - b_0L\{y\} - L\{F(y)\} + L\{G(t)\}$$

$$L\{y\} = \frac{\alpha_0}{s} + \frac{\alpha_1}{s^2 + b_1s} - \frac{b_0}{s^2 + b_1s}L\{y\} - \frac{1}{s^2 + b_1s}L\{F(y)\} + \frac{1}{s^2 + b_1s}L\{G(t)\}. \tag{4}$$

The LADM represents the solution as an infinite series

$$y = \sum_{i=0}^{\infty} y_i \tag{5}$$

where the terms $y_i$ calculated recursively.

Similarly, the nonlinear term $L\{F(y)\}$ decomposes as

$$F(y) = \sum_{i=0}^{\infty} A_i \tag{6}$$

where the $A_i$s are Adomian polynomials of $y_i$s. One can calculate the polynomials by the following formula.

$$A_i = \frac{1}{i!}\frac{d^i}{du^i} F\left(\sum_{j=0}^{\infty} u^j y_j\right)_{u=0}, i = 0, 1, 2, \ldots$$

Substituting (5) and (6) into (4), we have

$$\begin{aligned} L\left\{\sum_{i=0}^{\infty} y_i\right\} &= \frac{\alpha_0}{s} + \frac{\alpha_1}{s^2 + b_1 s} - \frac{b_0}{s^2 + b_1 s} L\left\{\sum_{i=0}^{\infty} y_i\right\} - \frac{1}{s^2 + b_1 s} L\left\{\sum_{i=0}^{\infty} A_i\right\} \\ &+ \frac{1}{s^2 + b_1 s} L\{G(t)\}. \end{aligned}$$

Now, we can obtain the following algorithm

$$\begin{aligned} L\{y_0\} &= \frac{\alpha_0}{s} + \frac{\alpha_1}{s^2 + b_1 s} + \frac{1}{s^2 + b_1 s} L\{G(t)\} \\ L\{y_1\} &= -\frac{b_0}{s^2 + b_1 s} L\{y_0\} - \frac{1}{s^2 + b_1 s} L\{A_0\} \\ L\{y_2\} &= -\frac{b_0}{s^2 + b_1 s} L\{y_1\} - \frac{1}{s^2 + b_1 s} L\{A_1\} \\ &\vdots \\ L\{y_{i+1}\} &= -\frac{b_0}{s^2 + b_1 s} L\{y_i\} - \frac{1}{s^2 + b_1 s} L\{A_i\} \end{aligned} \tag{7}$$

Then, applying the inverse placeLaplace transform to (7) we obtain the values $y_i$ recursively. The last step of the algorithm is using the transformation $t = \ln x$ to get the solution of (2).

# 3 The Algorithm of the Method

Here we present an algorithm that compute the solution of problem (1).

**Algorithm 1** *Assume that problem (1) is given.*

1. *Apply the transformation* $x = e^t$ *to problem (1).*
2. *Split the equation given in (1) into two pieces. The first part is*

$$-\left(F(y) + b_0 y\right)$$

*and the second part is*

$$\frac{d^n y}{dx^n} + b_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + b_1\frac{dy}{dx} = G(t)$$

3. *Apply the Laplace transform to second part, determine the coefficient of* $L\{y\}$ *and solve this equation for* $L\{y\}$. *So, you can get* $L\{y_0\}$.
4. *Calculate the Adomian polynomials for the function* $F(y)$. *Apply the Laplace transform to these polynomials.*
5. *Divide the first part to the coefficient of* $L\{y\}$. *In a loop, calculate* $L\{y_{i+1}\}$*with substituting the* $L\{A_i\}$*and* $L\{y_i\}$*in the first part.*
6. *Construct the solution using inverse Laplace transform to* $L\{y_i\}$ *.Apply the transformation* $t = \ln x$ *to the solution.*

# 4 Numerical Examples

In this section, we will briefly describe the implementation of the Algorithm 3.1 by following examples.

**Example 2** *Consider the following linear problem.*

$$\begin{cases} x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2 \\ y(1) = \frac{4}{3} \\ y'(1) = \frac{5}{3} \end{cases}$$

*The analytic solution of this IVP is* $y(x) = x + \frac{x^2}{3}$. *Lets apply the Algorithm 1*

1. *After the transformation* $x = e^t$ *we get the equation* $\frac{d^2y}{dt^2} - y = e^{2t}$*and initial conditions* $y(0) = \frac{4}{3},\ y'(0) = \frac{5}{3}$.

2. *The first part of the equation is* $y$ *and the second part of the equation is* $\frac{d^2y}{dt^2} = e^{2t}$.

3. *From the second part we get* $L\{y_0\} = \frac{1}{s^2(s-2)} + \frac{4s+5}{3s^2}$.

4. *We do not need to calculate Adomian polynomials for this example because there is no non-linear term in this equation.*

5. *From the first part we get* $L\{y_{i+1}\} = \frac{1}{s^2} L\{y_i\}$. *So we can calculate* $L\{y_{i+1}\}$*s in a loop.*

6. *With inverse Laplace transform we get* $y_i$*s and with a little calculation we can find the series expansion below*

$$\frac{1}{3}\left(4 + 5t + \frac{7}{2}t^2 + \frac{11}{6}t^3 + \frac{19}{24}t^4 + \frac{7}{24}t^5 + \dots\right) \cong e^t + \frac{e^{2t}}{3}$$

*Finally with applying the transformation* $t = \ln x$ *we get the solution as* $y(x) = x + \frac{x^2}{3}$.

This result can be easily obtained from the computer algebra system Maple with five calculations.

**Example 3** *Consider the following non-linear problem.*

$$\begin{cases} x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y + y^2 = (x \ln x)^2 \\ y(1) = 0 \\ y'(1) = 1 \end{cases}$$

*The analytic solution of this IVP is* $y(x) = x \ln x$.

1. *After the transformation* $x = e^t$ *we get the equation* $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} + y + e^y = e^t t$ *and initial conditions* $y(0) = 0,\ y'(0) = 1$.

2. *The first part of the equation is* $-y - y^2$ *and the second part of the equation is* $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} = te^t$.

3. *From the second part we get* $L\{y_0\} = \dfrac{1}{s(s-2)} + \dfrac{2}{s(s-2)^4}$.

4. *The non-linear term of this equation is* $y^2$. *The first four Adomian polynomials for this term are*

$$A_0 = y_0^2$$
$$A_1 = 2y_0 y_1$$
$$A_2 = y_1^2 + 2y_0 y_2$$
$$A_3 = 2y_1 y_2 + 2y_0 y_3$$

5. *From the first part we get* $L\{y_{i+1}\} = -\dfrac{L\{y_i\} + L\{A_i\}}{s(s-2)}$. *So we can calculate* $L\{y_{i+1}\}$*s in a loop.*

6. *With inverse Laplace transform we get* $y_i$*s and we can get the series expansion below.*

$$t + t^2 + \frac{t^3}{2} + \frac{t^4}{6} + \frac{t^5}{24} + \frac{t^6}{120} + \ldots \cong te^t$$

*Finally with applying the transformation* $t = \ln x$ *we get the solution of the problem with regarding the noise terms as* $y(x) = x \ln x$*.*

In this paper, we used the LADM for solving equations given as (2). This method is to be considered an effective method for solving many problems. From these examples we can see that this method works efficiently.

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