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Operational Equations for the Five-Point Rectangle, the Geometric Mean, and Data in Prismatic Array

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Abstract

This paper describes the results of three applications of operational calculus: new representations of five data in a rectangular array, new relationships among data in a prismatic array, and the operational analog of the geometric mean.

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1. Introduction

Recent papers have described two series of estimators of central tendency developed by shifting operator [1,2]. They are the operational analogs of the arithmetic and the geometric means. The new estimators are more complicated than the traditional measures and they are restricted to sorted numbers, but these disadvantages are compensated by their increased accuracies in many known cases. This paper illustrates and clarifies the second series of operational center point estimators. The members of that series are exact on linear numbers as well as on those numbers when used as exponents in expressions like 2^x. In the latter respect, the members of the second series are analogous to the geometric mean. Unlike that traditional estimator, the new formulas tolerate zero as a datum. This paper also illustrates relationships among data in a prismatic array and it describes new representations of five data in a rectangular array.

2. Operational analog of the geometric mean

The most popular measure of central tendency is the arithmetic average or mean. Another center point estimator is the geometric mean. It is sensitive to translation of the data and it does not tolerate zero as a datum. New formulas that are analogous to the geometric mean were introduced several years ago [2]. They are presently reformulated and elaborated in the interest of clarity. The estimators apply to numbers sorted by increasing or decreasing magnitude and they tolerate zero as a datum. The first three members of the series appear as Eqs. (1)-(3).

$$(P4E)^2 = [(z3z4-z1z2)(z2z4-z1z3)] / [(z4-z1)^2 - (z3-z2)^2] \tag{1}$$

$$\begin{aligned}(P6E)^2 =\ &[2(z6-z1)^2(z5-z2)^2(z4+z3)^2 - (z6-z1)^2(z5+z2)^2(z4-z3)^2 - (z6+z1)^2(z5-z2)^2(z4-z3)^2] \\ &/ [8(z6-z1)^2(z5-z2)^2 - 4(z6-z1)^2(z4-z3)^2 - 4(z5-z2)^2(z4-z3)^2]\end{aligned} \tag{2}$$

$$\begin{aligned}(P8E)^2 =\ &[3(z8-z1)^2(z7-z2)^2(z6-z3)^2(z5+z4)^2 - (z8-z1)^2(z7-z2)^2(z6+z3)^2(z5-z4)^2 \\ &- (z8-z1)^2(z7+z2)^2(z6-z3)^2(z5-z4)^2 - (z8+z1)^2(z7-z2)^2(z6-z3)^2(z5-z4)^2] \\ &/ [12(z8-z1)^2(z7-z2)^2(z6-z3)^2 - 4(z8-z1)^2(z7-z2)^2(z5-z4)^2 \\ &- 4(z8-z1)^2(z6-z3)^2(z5-z4)^2 - 4(z7-z2)^2(z6-z3)^2(z5-z4)^2)]\end{aligned} \tag{3}$$

The number 4 in the notation P4E indicates the expression on the right hand side of Eq. (1) applies to four sorted numbers. The letter E indicates that P4E is an exponential-type estimator. The numbers to which the expressions apply are denoted z1, z2, z3, … where z represents a datum and the appended integer represents its position rank in the ordered list [1]. P4E is the first member of a series of analogous formulas. Successive members of the series are generated by rules that apply at least up to sixteen ordered numbers. This property suggests the series can be extended indefinitely. Like the traditional geometric mean, the new formulas are sensitive to translation of the data. Unlike that mean, they tolerate zero as one datum.

The operational formulas also apply to odd numbers of sorted data. First, remove the median of the ordered set. Second, apply the appropriate operational formula to the remaining numbers. The operational estimate of the center of the original data is the square root of the product of the removed median and the center of the remaining data as estimated by one of the suffix-E formulas as typified by Eqs. (1)-(3) [1,2].

Suffix-E formulas, illustrated by Eqs. (1)-(3), can be prepared from the analogous suffix-P formulas [1,2]. Both series of center point estimators display patterns in their numerators. For example, the numerator of P6E contains three members. The first member has a positive sign and leading coefficient 2. The remaining members have negative signs. Each member consists of three terms and each term is squared. The parenthesized numbers are equidistant from the center of the list. Their position numbers have the same sum. One member of the parenthesized numbers has a positive sign whereas the others contain minus signs. The plus sign migrates in a predictable manner. Analogous patterns are observed in the suffix-P series of estimators [1,2]. The estimators on the right-hand sides of Eqs. (1)-(3) are the squares of P4E, P6E, and P8E, respectively.

The denominators of the new formulas also display patterns. The denominator of P6E contains three members. The first one has a positive sign and coefficient 8. The remaining members have negative signs and coefficients 4. Each member has two terms and each term is squared. The denominator of P8P has four members. The first one has a positive sign and the coefficient is 12. The remaining members have negative signs and coefficients 4. Each member has three terms and each term is squared. Such rules can be used to generate successive estimators up to 16 numbers at least. This observation suggests the possibility that the series can be represented in terms of Sigma and integral notations. The proposed formulations are not known but their discovery promises to simplify the generation of successive members and perhaps reveal information about their properties. These putative representations are opportunities for research.

For the sake of illustration, let eight basis numbers be chosen as the first eight ordered integers. Their "true" center is taken as 4.5. Operations M can be applied to 4.5 as well to all the basis numbers. Thus, M^2 represents the squaring operation so the true center of the trial data is M(4.5) or 20.25. Table 1 illustrates that both the median and the operational formula P8E render estimates of the center of the data that are closer to the true values than the traditional geometric mean. The entries in Table 1 suggest the operational estimates are often more accurate than the geometric mean in the context of the illustration. The operational estimators of central tendency do not reduce to zero when zero is encountered as one datum. The traditional geometric mean reduces to zero in that case. Other properties of P4E and P6E have been described in Ref. [2]. The third and fourth column headings in Table 5 of Ref. [2] should read Eq. (6) and Eq. (7), respectively. The title of Table 6 in Ref. [2] should read Eqs. (8) and (10).

3. Numerical relationships for data in prismatic array

Recent papers have illustrated equations that relate trilinear numbers positioned at various points within a prismatic array as illustrated in Fig. 1 [1,3,4]. In these equations, a single letter such as A represents a number at a vertex of the rectangular prism, a double letter combination such as AB represents a number at the midpoint of edge AB of the prism, and a four letter combination such as ABDC represents a number at the center point of the bottom face of the prism. The illustrated equations are potentially useful for estimating data that are missing from the prismatic array in Fig. 1. The equations are derived by combining typical textbook identities with the shifting operator [5]. Three examples are listed as Eqs. (5), (7), and (9). Their precursor identities appear as Eqs. (4), (6), and (8), respectively. The identities are converted to their Euler forms before multiplying them by the appropriate power of the unknown function at center point E in Fig. 1 [5]. This operation generates the numerical relationships.

$$\sin(y-z)\sin(x) + \sin(z-x)\sin(y) - \sin(-x+y)\sin(z) = 0 \tag{4}$$

$$(CD-FG)(BDIG-ACHF) + (FH-BD)(CDIH-ABGF) - (CH-BG)(FGIH-ABDC) = 0 \tag{5}$$

$$\sin(y-z)\sin(y)\sin(z) + \sin(z-x)\sin(z)\sin(x) - \sin(-x+y)\sin(x)\sin(y) - \sin(y-z)\sin(z-x)\sin(-x+y) = 0 \tag{6}$$

$$(CD-FG)(CDIH-ABGF)(FGIH-ABDC) + (FH-BD)(FGIH-ABDC)(BDIG-ACHF) - (CH-BG)(BDIG-ACHF)(CDIH-ABGF) - (CD-FG)(FH-BD)(CH-BG) = 0 \tag{7}$$

$$\sin(y-z)\sin(y)^2\sin(z)^2\sin(y+z) + \sin(z-x)\sin(z)^2\sin(x)^2\sin(z+x) - \sin(-x+y)\sin(x)^2\sin(y)^2\sin(x+y) - \sin(y-z)\sin(z-x)\sin(-x+y)\sin(y+z)\sin(z+x)\sin(x+y) = 0 \tag{8}$$

$$(CD-FG)(CDIH-ABGF)^2(FGIH-ABDC)^2(HI-AB) - (BD-FH)(FGIH-ABDC)^2(BDIG-ACHF)^2(GI-AC) + (BG-CH)(BDIG-ACHF)^2(CDIH-ABGF)^2(DI-AF) - (CD-FG)(BD-FH)(BG-CH)(HI-AB)(GI-AC)(DI-AF) = 0 \tag{9}$$

Equations (5), (7), and (9) are exact on trilinear numbers, denoted x, on the squares of those numbers, denoted x^2, and those numbers used as exponents in expressions like 2^x. They are also exact on sin(x), cos(x), sinh(x), and cosh(x). The numerical relationships maintain their properties under translation of the data. A potential application of the equations is the estimation of a missing or corrupted datum from the remaining trustworthy data [1,3,4]. The accuracies of the equations are difficult to predict in laboratory situations but their versatility suggests they are potentially useful to the experimentalists.

4. Operational representations of the five-point rectangle

Three operational, polynomial equations for the five-point rectangle have been described by means of their quadratic-term coefficients [6]. The equations are exact on bilinear numbers and their squares. They differ primarily by the forms of the cited coefficients. Eq. (10) is a symbolic form of a polynomial-type interpolating equation for the five-point rectangle in Fig. 2. The letter E currently represents the center point of that design, not the center point of the prism in Fig. 1. V is an independent parameter that can be varied at pleasure. V can sometimes be changed to provide a better fit to laboratory results while still maintaining the traditional form of the interpolating equation.

$$R = E + (xc)(x) + (yc)(y) + (xyc)(xy) + (Qx4)x^2 + (Qy4)y^2 \quad (10)$$
$$xc = (1/4)(C + I – A – G) \quad (11)$$
$$yc = (1/4)(G + I – A – C) \quad (12)$$
$$xyc = (1/4)(A + I – C – G) \quad (13)$$
$$Qx4 = (1/4)(A + C + G + I – 4E) – V \quad (14)$$
$$Qy4 = V \quad (15)$$

Another approach involves estimating the quadratic-term coefficients by means the pair of equations in Eqs. (16) and (17). In these equations, P is (+/–)1. The proper value of P can usually be determined by comparison to the numerical values of the analogous coefficients in Ref. [6]. Alternatively, Eqs. (18) and (19) can be used for that purpose.

$$Qx5 = (1/8)(A – C – G + I)^2 / [G + I + A + C – 4E + 2P(C – 2E + G)^{(1/2)}(A – 2E + I)^{(1/2)}] \quad (16)$$
$$Qy5 = (1/8)(A – C – G + I)^2 / [G + I + A + C – 4E – 2P(C – 2E + G)^{(1/2)}(A – 2E + I)^{(1/2)}] \quad (17)$$
$$Qx6 = (1/8)(A + 3G + I – C – 4E) + (1/4)(IC – CG – IG + G^2) / (I – A + C – G) + (1/4)(C^2 – IC – CG + IG) / (A – I – G + C) \quad (18)$$

$$Qy6 = (1/8)(A + 3C + I – G – 4E) – (1/4)(IC – CG – IG + G^2) / (I – A + C – G) – (1/4)(C^2 – IC – CG + IG) / (A – I – G + C) \quad (19)$$

These quadratic-term coefficients, and those illustrated elsewhere [6], raise a question: How many such coefficients can be prepared? The question does not seem to have been addressed. The preceding “Q” notation adheres to the notation in Ref. [6].

The five-point diamond array can be represented in terms of exponential-type equations [7]. The five-point rectangle can be represented in a similar manner. Choose the form given as Eq. (11) in Ref. [7]. It is rewritten as Eq. (20) where P now represents an interpolated number as in rectangle ACEIG shown in Fig. 2. The coordinate system is

-1 .. 1 in both the x- and y-directions. The terms in Eq. (20) are defined by Eqs. (21)-(25). For example, if A=12, C=18, E=42, G=138, and I=522, then $P=(32)(2^x)(8^y)+10$.

$$P = R(J^x)(K^y)(M^Z) + Q \tag{20}$$
$$R = [(G - I)^{(1/2)}(C - I)^{(1/2)}(A - C)^{(1/2)}(A - G)^{(1/2)}] / (A + I - C - G) \tag{21}$$
$$J = (C - I)^{(1/2)} / (A - G)^{(1/2)} \tag{22}$$
$$K = (G - I)^{(1/2)} / (A - C)^{(1/2)} \tag{23}$$
$$M = [CG - IA - E(C + G - A - I)] / [(A - C)^{(1/2)}(C - I)^{(1/2)}(A - G)^{(1/2)}(G - I)^{(1/2)}] \tag{24}$$
$$Q = (AI - CG) / (A - C + I - G) \tag{25}$$
$$Z = 1 - x^2/2 - y^2/2 \tag{26}$$

Interesting situations appear when one or more of R, J, K, M contain imaginary numbers. In such cases, the temptation to reject Eq. (20) arises. If exponential-type representations remain desirable, more than one approach to the problem is available. An easy alternative is to interpolate with the real parts of the predictions rendered by Eq. (20). If those parts reproduce the original data, and if the real parts are likewise acceptable estimations of laboratory results at other points within the rectangle, then the Eq. (20) may be a suitable representation of the experimental space. The letter P in Eq. (20) is unrelated to the P in Eqs. (16) and (17).

If the preceding approach is not satisfactory, the following method may be useful. Change every imaginary number in Eq. (20) into a real number, and then change every negative number into a positive number. Modified Eq. (20) now contains only positive numbers and the exponents x and y. In this form, however, the new equation does not reproduce the original data. To overcome this difficulty, prefix the first term in Eq. (20) by a multiplier selected from Table 2. The table contains entries for 20 commonly arising cases but it is not represented as complete [8].

For example, suppose the laboratory data are A=128, C=512, E=2, G=32, I=8 in Fig. 2. By the present method, the interpolating equation for the rectangle is Eq. (27). (The coefficients have been rounded.) The italic letter *I* represents the imaginary unit. R, K, and M are negative and imaginary but J is real and positive. The entry on the last line of Table 2 applies in these circumstances so the multiplier is $(-1-y+y^2)$. Eq. 27 is thereby transformed into Eq. (28). The new equation should reproduce all of the original data. It is advisable to verify that when applying the method. The suitability of Eq. (28) for the experimental space is determined in the laboratory. A commentary on this approach to the five-point rectangle, including different formulations of J, K, M, R, and a table for comparison to the present Table 2, is found in Ref. [8]. Note that the last line of Table 2 herein corrects a misprint in the corresponding Table 3 in Ref. [8].

$$P = (-51.76I)(2.29)^x(-0.25I)^y(-0.69I)^Z + 37.65 \quad (27)$$
$$P = (51.76)(-1-y+y^2)(2.29)^x(0.25)^y(0.69)^Z + 37.65 \quad (28)$$

Suppose the data at vertices A,C,E,G,I in Fig. 2 are sinh(X/4), where X=1,3,5, 7,9, respectively. The quadratic-term, $[x^2,y^2]$ coefficients as estimated by Eqs. [16, 17] (with P=+1), Eqs. [18, 19], and Eq. (20) are about [0.0577, 0.479], [0.0849, 0.452], and [0.0612, 0.449], respectively. The latter numbers are obtained by Taylor expansion of substituted Eq. (20). The true values of the coefficients are about [0.0501, 0.451].

5. Discussion

Two new series of estimators of central tendency have been illustrated in recent times. The first series is the operational analog of the arithmetic mean. They are suffix-P formulas [1,2]. The second series is a list of operational analogs of the geometric mean. They are suffix-E formulas as in Eqs. (1)-(3) above [1,2]. The members of both series require the data to be sorted before they can be applied. Successive members of the suffix-P and suffix-E series are generated from preceding members by observed rules. Both series appear to be indefinitely extensible.

The collection of four data in rectangular array, in diamond array, or eight data in prismatic array, presents the analyst with a problem: What is the best way to represent them? The traditional answers for the four-point rectangle and the eight-point prism in Fig. 1 are the bilinear and trilinear equations, respectively. There is little to question. The diamond array does not enter the discussion. No tests seem be known that assess the comparative advantages of polynomial forms, exponential forms, or trigonometric forms for these designs [9]. The definitive but expensive test for any representation is adherence to laboratory results. The adverse economic impact of more experiments suggests the need for better insights but these considerations have not generated productive interest.

The variety of applications suggested by the shifting operator is manifested in many ways. Eq. (29) is an alternative rendering of the traditional expression for estimating the first derivative on three equidistant, curvilinear data denoted B, C, D that are separated by distance h. The operator generalizes Eq. (29) as illustrated by Eq. (30). For example, let the trial function be 3^x so that $B=3^2$, $C=3^3$, $D=3^4$. In the test case, h=1 and Eq. (29) yields C'=36.00. The true value of C' is about 29.66. When P=10, Eq. (30) yields C'=29.66 (nearly), a recognizable estimate of the true value. In the present illustration, the function is a simple one and the data are uncorrupted by errors. When presented with three equidistant, curvilinear laboratory measurements, but no other information, an easy way to assign the value of P in Eq. (30) may not be apparent. The letter P appearing in Eqs. (16), (17), (20), (27) and (28) is unrelated to the P in Eq. (30).

$$C' = [F(C + h) - F(C - h)] / (2h) \qquad (29)$$
$$C' = 2^{(P-2)}[(F(C + h/2^{(P-1)}) - F(C - h/2^{(P-1)})] / h \qquad (30)$$

A recent paper described a method for constructing equations for six-point grids by means of equations for two four-point sub-grids [10]. The six-point grid was defined by numbers at vertices A=1, B=2 C=3, D=7, E=8, F=9 in Fig. 1. The parameter (y) in the equations for both sub-grids was changed to (2y+1) [10]. To transform the y-coordinate system from –1 .. 0 into –1 .. 1 do not make the change. That is, take (y) as (y). Now A=1, B=2 C=3, G=7, H=8, I=9 and bilinear data yield R=5+x+3y instead of R=8+x+6y.

Table 2 presents twenty multipliers that can be used to represent five-point rectangles by means of the exponential-type equation in Eq. (20). Suppose the data at vertices A,C,E,G,I are 1,3,9,4,5, respectively, as in Fig. 2. An equation representing them is Eq. (31). The exponent Z is $1-x^2/2-y^2/2$. Change all negative numbers to positive numbers and prefix the first term by an unknown multiplier (W) as in Eq. (32).

$$P = -3.464(0.8165)^x(0.7071)^y(-0.5774)^Z + 7 \qquad (31)$$
$$P = 3.464(W)(0.8165)^x(0.7071)^y(0.5774)^Z + 7 \qquad (32)$$

Form five equations by substituting the coordinates of a design point into Eq. (32) and then subtract the datum at the same point from substituted Eq. (32). Solve each equation for (W). The solutions at rectangle vertices [A,C,E,G,I] are [–1,–1,1,–1, –1], respectively. Substitute these numbers for the letters [A,C,E,G,I], respectively, into a polynomial equation for the five-point rectangle. For this purpose, use coefficients Qx1 and Qy1 in Ref. [6]. (Coefficients Qx6 and Qy6 above can also be used.) The result is the multiplier for substituted Eq. (20). In the present case, the new interpolating equation for the rectangle is Eq. (33). It may be useful if it reproduces the original data and estimates data at arbitrary points within the rectangle. The described method checks the entries in Table 2 for potential errors.

$$P = 3.464(1-x^2-y^2)(0.8165)^x(0.7071)^y(0.5774)^Z + 7 \qquad (33)$$

Table 1. Comparisons of the centers of eight ordered data generated by applying selected functions M to the integers 1-8. The centers of the data are estimated by the geometric mean, the median, and formula P8E in the text. The true value is taken as M(4.5).

Function	Geometric Mean	Median	Formula P8E	True value
M	3.76	4.50	4.50	4.50
M^2	14.2	20.5	20.2	20.3
M^3	53.3	94.5	90.4	91.1
2^M	22.6	24.0	22.6	22.6
Ln(M)	0	1.50	1.50	1.50
(M)Ln(M!)	0	18.3	17.7	17.8
100/M	26.6	22.5	22.3	22.2
$100/M^2$	7.06	5.13	4.97	4.94
$M^M/100$	16.5	16.9	8.94	8.70
$Ln(M^M)$	0	6.80	6.76	6.77

Table 2. A list of the properties of J, K, M, and R that are used to assign the multiplier to R in Eq. (20). The abbreviations +Re and –Re represent positive and negative real numbers, respectively. The abbreviations +Im and –Im represent positive and negative imaginary numbers, respectively [8]. The table illustrates twenty commonly arising cases.

R	J	K	M	Multiplier
+Re	–Im	–Im	+Re	$1–xy–(x^2+y^2)/2$
+Re	–Im	–Im	–Re	$–1–xy+(x^2+y^2)/2$
–Re	+Im	+Im	+Re	$–1+xy+(x^2+y^2)/2$
–Re	+Im	+Im	–Re	$1+xy–(x^2+y^2)/2$
+Im	+Re	+Im	+Im	$–1–y+y^2$
+Im	+Re	+Im	–Im	$1–y–y^2$
+Im	+Re	–Im	+Im	$–1+y+y^2$
+Im	+Re	–Im	–Im	$1+y–y^2$
+Im	+Im	+Re	+Im	$–1–x+x^2$
+Im	+Im	+Re	–Im	$1–x–x^2$
+Im	–Im	+Re	+Im	$–1+x+x^2$
+Im	–Im	+Re	–Im	$1+x–x^2$
–Im	+Im	+Re	+Im	$1+x–x^2$
–Im	+Im	+Re	–Im	$–1+x+x^2$
–Im	–Im	+Re	+Im	$1–x–x^2$
–Im	–Im	+Re	–Im	$–1–x+x^2$
–Im	+Re	+Im	+Im	$1+y–y^2$
–Im	+Re	+Im	–Im	$–1+y+y^2$
–Im	+Re	–Im	+Im	$1–y–y^2$
–Im	+Re	–Im	–Im	$–1–y+y^2$

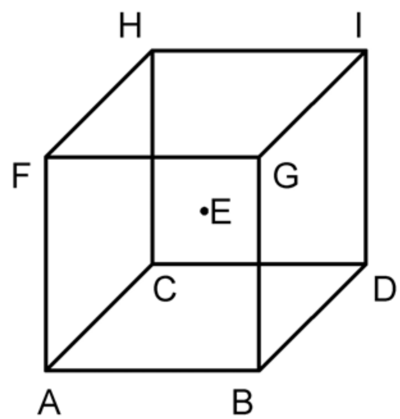

Fig. 1. The nine-point cube.

G	H	I
D	E	F
A	B	C

Fig. 2. The nine-point rectangle.

References

1. G. L. Silver, Numerical relations within cubes and the estimation of central tendency. Appl. Math. Sci. 3 (2009) 119-128.

2. G. L. Silver, Operational measures of central tendency, Appl. Math. Comput. 186 (2007) 1379-1384.

3. G. L. Silver, Analysis of three-dimensional grids: Interpolation of data in prismatic array and the estimation of a missing datum. Appl. Math. Comput. 174 (2006) 1398-1407.

4. G. L. Silver, Analysis of three-dimensional grids: The estimation of two missing data. Appl. Math. Comput. 184 (2007) 743-747.

5. G. L. Silver, Operational method of data treatment. J. Comput. Chem. 6 (1985) 229-236.

6. G. L. Silver, Analysis of three-dimensional grids: Alternative methods for five- and nine-point cubes. Appl. Math. Comput. 176 (2006) 37-43.

7. G. L. Silver, Analysis of five-point grids: the diamond configuration. Appl. Math. Comput. 144 (2003) 389-395.

8. G. L. Silver, Applications of operational equations. Los Alamos National Laboratory report LA-UR-01-6949. See pp. 16-19.

9. G. L. Silver, Analysis of three-dimensional grids: the eight-point cube. Appl. Math. Comput. 153 (2004) 467-473.

10. G. L. Silver, Center points and grid equations from sub-grid equations. Appl. Math. Comput. 189 (2007) 1467-1475.

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