Applied Mathematical Sciences, Vol. 3, 2009, no. 30, 1473 - 1478

Asymptotic Stability of Nonlinear Delay-Difference Control System with Time-Varying Delay via Matrix Inequalities

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Abstract. In this paper, we obtain some criteria for determining the asymptotic stability of the zero solution of nonlinear delay-difference control system with time-varying delay in terms of certain matrix inequalities by using a discrete version of the Lyapunov second method.

Keywords: Asymptotic stability; Lyapunov function; Nonlinear delay-difference control system with time-varying delay; Matrix inequalities.

1 Introduction

We consider nonlinear delay-difference control system with time-varying delay of the form

$$x(k+1) = Ax(k) + Bx(k-h(k)) + Cu(k) + f(k, x(k), x(k-h(k))), \qquad (1)$$

where $x \in \Omega \subseteq \mathbf{R}^n$, $h(k)$ is a continuous function describing the time-varying transmission delay in the network system and satisfies $0 \le h(k) \le h$, A, B is $n \times n$ constant matrices, C is $n \times m$ constant matrices, $u \in \mathbf{R}^m$ is the control, $f(k, x(k), x(k-h(k)))$ is a nonlinear perturbation satisfying $f(k, 0, 0) = 0$.

The asymptotic stability of the zero solution of the nonlinear delay-differential system has been developed during the past several years. We refer to monographs by Bay and Phat [2] and the references cited therein. Much less is known regarding the asymptotic stability of the zero solution of the nonlinear delay-difference system. Therefore, the purpose of this paper is to establish sufficient conditions for the asymptotic stability of the zero solution of (1) in terms of certain matrix inequalities.

2 Preliminaries

We assume that the n-vector function nonlinear perturbations are bounded and satisfy the following hypotheses, respectively:

$$0 \le \frac{f_i(r_1) - f_i(r_2)}{r_1 - r_2} \le l_i, \quad \forall r_1, r_2 \in \mathbf{R}, \text{ and } r_1 \ne r_2, \tag{2}$$

where $l_i > 0$ are constants for $i = 1, 2, ..., n$.

By assumption (2) we know that the functions $f_i(\cdot)$ satisfy

$$|f_i(x_i)| \le l_i |x_i|, \; i = 1, 2, ..., n,$$

and

$$f^2{}_i(x_i) \le l_i x_i f_i(x_i), \quad i = 1, 2, ..., n. \tag{3}$$

Fact 1 For any positive scalar ε and vectors x and y, the following inequality holds:

$$x^T y + y^T x \le \varepsilon x^T x + \varepsilon^{-1} y^T y.$$

Lemma 2.1 [4] The zero solution of difference system is asymptotic stability if there exists a positive definite function $V(x): \mathbf{R}^n \to \mathbf{R}^+$ such that

$$\exists \beta > 0 : \Delta V(x(k)) = V(x(k+1)) - V(x(k)) \le -\beta \|x(k)\|^2,$$

along the solution of the system. In the case the above condition holds for all $x(k) \in V_\delta$, we say that the zero solution is locally asymptotically stable.

Lemma 2.2 [5] For any constant symmetric matrix $M \in \mathbf{R}^{n \times n}$, $M = M^T > 0$, scalar
$s \in \mathbf{Z}^+ / \{0\}$, vector function $W : [0, s] \to \mathbf{R}^n$, we have

$$s \sum_{i=0}^{s-1} (w^T(i) M w(i)) \ge \left(\sum_{i=0}^{s-1} w(i) \right)^T M \left(\sum_{i=0}^{s-1} w(i) \right).$$

3 Main results

In this section, we present the main results of this paper, which provides a sufficient condition for the asymptotic stability of the zero solution of (1) in terms of certain matrix inequalities.

This is a basic requirement for controller design. Now, we are interested designing a feedback controller for the system (1) as

$$u(k) = Kx(k),$$

where K is $n \times m$ constant control gain matrices.

The new form of (1) is now given by

$$x(k+1) = Ax(k) + Bx(k - h(k)) + CKx(k) + f(k, x(k), x(k - h(k))). \tag{4}$$

Theorem 3.1 The zero solution of the nonlinear delay-difference control system with time-varying delay (4) is asymptotic stable if there exist symmetric positive definite matrices P, G, W and $L_1 = diag[l_{11},\dots,l_{1n}] > 0,\ L_2 = diag[l_{21},\dots,l_{2n}] > 0$ satisfying the following matrix inequalities:

$$\psi = \begin{pmatrix} (1,1) & 0 & 0 \\ 0 & (2,2) & 0 \\ 0 & 0 & (3,3) \end{pmatrix} < 0, \tag{5}$$

where

$$\begin{aligned}(1,1) = {} & A^T PA + A^T PCK + K^T C^T PCK + K^T C^T PA - P \\ & + h(k)G + W + \varepsilon A^T PPA + \varepsilon_1 A^T A + (\varepsilon_1^{-1} + \varepsilon_2^{-1} + \varepsilon_4^{-1} + 1)L_1 PL_1 \\ & + \varepsilon_4 K^T C^T CK + \varepsilon_3^{-1} K^T C^T PPCK,\end{aligned}$$

$$(2,2) = B^T PB + (\varepsilon^{-1} + \varepsilon_2 + \varepsilon_3)B^T B + (\varepsilon_1^{-1} + \varepsilon_2^{-1} + \varepsilon_4^{-1} + 1)L_2 PL_2 - W,$$

$(3,3) = -h(k)G$, and

$$f = f(k, x(k), x(k-h(k))).$$

Proof Consider the Lyapunov function $V = V_1 + V_2 + V_3$, where

$$V_1 = x^T(k)Px(k),$$

$$V_2 = \sum_{i=k-h(k)}^{k-1} (h(k) - k + i)x^T(i)Gx(i),$$

$$V_3 = \sum_{i=k-h(k)}^{k-1} x^T(i)Wx(i),$$

P, G and W being symmetric positive definite solutions of (5). Then difference of V along trajectory of solution of (4) is given by $\Delta V = \Delta V_1 + \Delta V_2 + \Delta V_3$.

Where

$$\begin{aligned}\Delta V_1 &= V_1(x(k+1)) - V_1(x(k)) \\ &= [Ax(k) + Bx(k-h(k)) + CKx(k) + f]^T P \\ &\quad \times [Ax(k) + Bx(k-h(k)) + CKx(k) + f] \\ &\quad - x^T(k)Px(k) \\ &= x^T(k)[A^T PA + A^T PCK + K^T C^T PCK + K^T C^T PA - P]x(k) \\ &\quad + x^T(k)A^T PBx(k-h(k)) + x^T(k-h(k))B^T PAx(k) \\ &\quad + x^T(k)A^T Pf + f^T PAx(k) \\ &\quad + x^T(k-h(k))B^T PCKx(k) + x^T(k)K^T C^T PBx(k-h(k)) \\ &\quad + x^T(k)K^T C^T Pf + f^T PCKx(k) \\ &\quad + x^T(k-h(k))B^T Pf + f^T PBx(k-h(k)) \\ &\quad + x^T(k-h(k))B^T PBx(k-h(k)) + f^T Pf,\end{aligned}$$

$$\Delta V_2 = \Delta\left(\sum_{i=k-h(k)}^{k-1} (h(k) - k + i)x^T(i)Gx(i)\right) = h(k)x^T(k)Gx(k) - \sum_{i=k-h(k)}^{k-1} x^T(i)Gx(i),$$

and

$$\Delta V_3 = \Delta\left(\sum_{i=k-h(k)}^{k-1} x^T(i)Wx(i)\right) = x^T(k)Wx(k) - x^T(k-h(k))Wx(k-h(k)). \qquad (6)$$

Where **(3**) and **Fact 1** is utilized in (6), respectively.

Note that

$$x^T(k)A^TPBx(k-h(k)) + x^T(k-h(k))B^TPAx(k) \le \varepsilon x^T(k)A^TPPAx(k)$$
$$+\varepsilon^{-1}x^T(k-h(k))B^TBx(k-h(k)),$$
$$x^T(k)A^TPf + f^TPAx(k) \le \varepsilon_1 x^T(k)A^TAx(k) + \varepsilon_1^{-1} f^TPf,$$
$$x^T(k-h(k))B^TPf + f^TPBx(k-h(k)) \le \varepsilon_2 x^T(k-h(k))B^TBx(k-h(k)) + \varepsilon_2^{-1} f^TPf,$$
$$x^T(k-h(k))B^TPCKx(k) + x^T(k)K^TC^TPBx(k-h(k)) \le$$
$$\varepsilon_3 x^T(k-h(k))B^TBx(k-h(k)) + \varepsilon_3^{-1}x^T(k)K^TC^TPPCKx(k),$$
$$x^T(k)K^TC^TPf + f^TPCKx(k) \le \varepsilon_4 x^T(k)K^TC^TCKx(k) + \varepsilon_4^{-1} f^TPf,$$
$$f^TPf \le x^T(k)L_1PL_1x(k) + x^T(k-h(k))L_2PL_2x(k-h(k)),$$
$$\varepsilon_1^{-1} f^TPf \le \varepsilon_1^{-1}x^T(k)L_1PL_1x(k) + \varepsilon_1^{-1}x^T(k-h(k))L_2PL_2x(k-h(k)),$$
$$\varepsilon_2^{-1} f^TPf \le \varepsilon_2^{-1}x^T(k)L_1PL_1x(k) + \varepsilon_2^{-1}x^T(k-h(k))L_2PL_2x(k-h(k)),$$
$$\varepsilon_4^{-1} f^TPf \le \varepsilon_4^{-1}x^T(k)L_1PL_1x(k) + \varepsilon_4^{-1}x^T(k-h(k))L_2PL_2x(k-h(k)),$$

hence

$$\Delta V_1 \le x^T(k)[A^TPA + A^TPCK + K^TC^TPCK + K^TC^TPA - P + \varepsilon A^TPPA + \varepsilon_1 A^TA$$
$$+(\varepsilon_1^{-1} + \varepsilon_2^{-1} + \varepsilon_4^{-1} + 1)L_1PL_1 + \varepsilon_4 K^TC^TCK + \varepsilon_3^{-1}K^TC^TPPCK]x(k)$$
$$+x^T(k-h(k))[B^TPB + (\varepsilon^{-1} + \varepsilon_2 + \varepsilon_3)B^TB + (\varepsilon_1^{-1} + \varepsilon_2^{-1} + \varepsilon_4^{-1} + 1)L_2PL_2]$$
$$\times x(k-h(k))$$

Then we have

$$\Delta V \le x^T(k)[A^TPA + A^TPCK + K^TC^TPCK + K^TC^TPA - P + h(k)G + W + \varepsilon A^TPPA + \varepsilon_1 A^TA$$
$$+(\varepsilon_1^{-1} + \varepsilon_2^{-1} + \varepsilon_4^{-1} + 1)L_1PL_1 + \varepsilon_4 K^TC^TCK + \varepsilon_3^{-1}K^TC^TPPCK]x(k)$$
$$+x^T(k-h(k))[B^TPB + (\varepsilon^{-1} + \varepsilon_2 + \varepsilon_3)B^TB + (\varepsilon_1^{-1} + \varepsilon_2^{-1} + \varepsilon_4^{-1} + 1)L_2PL_2 - W]$$
$$\times x(k-h(k))$$
$$-\sum_{i=k-h(k)}^{k-1} x^T(i)Gx(i).$$

Using **Lemma 2.2**, we obtain

$$\sum_{i=k-h(k)}^{k-1} x^T(i)Gx(i) \ge \left(\frac{1}{h(k)}\sum_{i=k-h(k)}^{k-1} x(i)\right)^T (h(k)G)\left(\frac{1}{h(k)}\sum_{i=k-h(k)}^{k-1} x(i)\right).$$

From the above inequality it follows that :

$$\Delta V \le x^T(k)[A^TPA + A^TPCK + K^TC^TPCK + K^TC^TPA - P + h(k)G + W + \varepsilon A^TPPA + \varepsilon_1 A^TA$$
$$+(\varepsilon_1^{-1} + \varepsilon_2^{-1} + \varepsilon_4^{-1} + 1)L_1PL_1 + \varepsilon_4 K^TC^TCK + \varepsilon_3^{-1}K^TC^TPPCK]x(k)$$
$$+x^T(k-h(k))[B^TPB + (\varepsilon^{-1} + \varepsilon_2 + \varepsilon_3)B^TB + (\varepsilon_1^{-1} + \varepsilon_2^{-1} + \varepsilon_4^{-1} + 1)L_2PL_2 - W]$$
$$\times x(k-h(k))$$
$$-\left(\frac{1}{h(k)}\sum_{i=k-h(k)}^{k-1} x(i)\right)^T (h(k)G)\left(\frac{1}{h(k)}\sum_{i=k-h(k)}^{k-1} x(i)\right)$$

$$= \left(x^T(k), x^T(k-h(k)), (\frac{1}{h(k)}\sum_{i=k-h(k)}^{k-1} x(i))^T \right)\begin{pmatrix} (1,1) & 0 & 0 \\ 0 & (2,2) & 0 \\ 0 & 0 & (3,3) \end{pmatrix}$$
$$\times \begin{pmatrix} x(k) \\ x(k-h(k)) \\ (\frac{1}{h(k)}\sum_{i=k-h(k)}^{k-1} x(i)) \end{pmatrix}$$

$$= y^T(k)\psi y(k),$$

where

$$(1,1) = A^TPA + A^TPCK + K^TC^TPCK + K^TC^TPA - P + h(k)G$$
$$+W + \varepsilon A^TPPA + \varepsilon_1 A^TA + (\varepsilon_1^{-1} + \varepsilon_2^{-1} + \varepsilon_4^{-1} + 1)L_1PL_1$$
$$+\varepsilon_4 K^TC^TCK + \varepsilon_3^{-1}K^TC^TPPCK,$$
$$(2,2) = B^TPB + (\varepsilon^{-1} + \varepsilon_2 + \varepsilon_3)B^TB + (\varepsilon_1^{-1} + \varepsilon_2^{-1} + \varepsilon_4^{-1} + 1)L_2PL_2 - W,$$
$$(3,3) = -h(k)G,$$

and $y(k) = \begin{pmatrix} x(k) \\ x(k-h(k)) \\ \left(\frac{1}{h(k)}\sum_{i=k-h(k)}^{k-1} x(i)\right) \end{pmatrix}$.

By the condition (5), ΔV is negative definite, namely there is a number $\beta > 0$ such that $\Delta V(y(k)) \le -\beta \|y(k)\|^2$, and hence, the asymptotic stability of the system immediately follows from **Lemma 2.1**. This completes the proof. □

Remark 3.1 Theorem 3.1 gives a sufficient condition for the asymptotic stability of delay-difference control system (4) via matrix inequalities. These conditions are described in terms of certain diagonal matrix inequalities, which can be realized by using the linear matrix inequality algorithm proposed in [5]. But Bay and Phat [2] these conditions are described in terms of certain symmetric matrix inequalities, which can be realized by using the Schur complement lemma and linear matrix inequality algorithm proposed in [5].

4 Conclusions

In this paper, based on a discrete analog of the Lyapunov second method, we have established a sufficient condition for the asymptotic stability of nonlinear delay-difference control system with time-varying delay in terms of certain matrix inequalities.

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Received: November, 2008