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Generalization for Lattice System with Equal Components

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Abstract

This paper proposes a model that generalizes the linear *k*-within-(*m*,*s*)-of-(*m*,*n*):G lattice system to multi-state case. In this model the system consists of *mn* components arranged in *m* rows and *n* columns. Both the system and its components can have different states: from complete failure up to perfect functioning. The system is in state τ or above if and only if the rectangles of dimension $m \times s$ contain at least k_τ components are in state τ or above. An algorithm is provided for evaluating reliability of a special case of multi-state *k*-within-(*m*,*s*)-of-(*m*,*n*):G lattice system. Also numerical results of the formerly published test examples and new examples are given.

Keywords: Linear *k*-within-(*m*,*s*)-of-(*m*,*n*):G lattice system; Multi-state system; System reliability with equal components

Notations

N	: n-s+1.
M+1	: number of states of the system and its components such that state M: perfect functioning and state 0: complete failure.
W	: $\{0,1,\ldots,M\}$.
k_τ	: minimum number of components in state $\geq \tau$ out of $m \times s$ components, which cause system in state $\geq \tau$, $k_\tau \leq ms$.
x_{ij}	: state of component (i,j), $x_{ij} \in W$, for all $i \in \{1,2,\ldots,m\}$ and $j \in \{1,2,\ldots,n\}$.
x	: $(x_{11}, x_{12},\ldots, x_{mn})$ vector of component states.
$\phi(x)$	: System-state structure function, $\phi(x) \in W$.
p_τ	: $\Pr\{x_{ij} = \tau\}$, $\sum_{b=0}^{M} p_b = 1$.

P_τ : Pr $\{x_{ij} \geq \tau\} = \sum_{b=\tau}^{M} p_b$.

R_τ : Pr$\{\phi(x) \geq \tau\}$, $R_0 = 1$, $R_{M+1} = 0$.

r_τ : Pr$\{\phi(x) = \tau\} = R_\tau - R_{\tau+1}$, for $\tau = 0,1,\ldots,M$.

1. Introduction

The consecutive-*k*-out-of-*r*-from-*n*: F system was introduced by Tong [7]. The consecutive-*k*-out-of-*r*-from-*n*: G and consecutive (*r*-*k*+1)-out-of-*r*-from-*n*: F systems are equivalent [8]. The consecutive-*k*-out-of-*r*-from-*n*: G system works if and only if at least *k* components out of *r* consecutive work. An algorithm for evaluating the exact reliability of a consecutive-*k*-out-of-*r*-from-*n*: F system was suggested by A. Habib and T. Szántai [1]. As consecutive-*k*-out-of-*r*-from-*n*: G system involves consecutive-*k*-out-of-*n*: G system (when $k = r$) and *k*-out-of-*n*: G system (when $r = n$) as special cases. The simple *k*-out-*n*: G system was generalized to the multi-state case [5] where the system and its components may be in one of *M*+1 possible states: 0,1,…and *M*. This system is in state τ or above if and only if at least k_τ components are in state τ or above. Generalizations of the consecutive-*k*-out-of-*n*:F system have been reported in a considerable number of papers [6]. Another generalization is multi-state consecutive *k*-out-of-*n*: G system [4] and extended in Ref.[10]. This system is in state τ or above if and only if at least consecutive k_τ components are in state τ or above. Also the consecutive-*k*-out-of-*r*-from-*n*: G system was generalized to the multi-state case where the reliability was calculated for a special case $(k_1 \leq k_2 \leq \ldots \leq k_M)$ [2]. This system is in state τ or above if and only if at least k_τ components out of *r* consecutive are in state τ or above. Another generalization of consecutive-*k*-out-of-*r*-from-*n*: G system is linear *k*-within-(*m,s*)-of-(*m,n*):G lattice system. The linear *k*-within-(*m,s*)-of-(*m,n*):G and linear (*ms*-*k*+1)-within-(*m,s*)-of-(*m,n*):F lattice systems are equivalent [8]. The system works if and only if the rectangles of dimension $m \times s$ contain at least *k* components work. An algorithm for reliability evaluation system was provided by Daming Lin and Ming J. Zuo [3].

This paper proposes a model that generalizes the linear *k*-within-(*m,s*)-of-(*m,n*):G lattice system to multi-state case. In this model the system consists of *mn* components arranged in *m* rows and *n* columns. Both the system and its components can have different states: from complete failure up to perfect functioning:0,1,2,…,*M*. The system is in state τ or above if and only if the rectangles of dimension $m \times s$ contain at least k_τ components are in state τ or above. An algorithm is provided for evaluating reliability of a special case of multi-state *k*-within-(*m,s*)-of-(*m,n*):G lattice system with equal components $(k_1 \leq k_2 \leq \ldots \leq k_M)$. The algorithm is based on the application of a special case taken from A. Habib et al. paper [2]. Also numerical results of the formerly published test examples and new examples are given.

2. Assumptions

- The system is multi-state monotone [9]:
  - $\phi(x)$ is nondecreasing in each argument.
  - $\phi(\tau) = \phi(\tau, \tau, ..., \tau) = \tau$ for $\tau \in W$.
- The x_{ij} are mutually S-independent.
- The possible states of each component and of the system are ordered: State 0 $\le$State 1$\le$ $\dots$$\le$State M.
- $k_1 \le k_2 \le \dots \le k_M$.

3. System Reliability Evaluation

This section calculates the reliability of multi-state linear k-within-(m,s)-of-(m,n):G lattice system with equal component probability by the following two cases:-

Case 1:- When $s < N$ then

$$R_\tau = \prod_{i=1}^{n} \sum_{y_i=t_i}^{m} \binom{m}{y_i} P_\tau^{y_i} (1 - P_\tau)^{m-y_i} \ , \tag{1}$$

$$P_\tau = \sum_{b=\tau}^{M} p_b \ , \tag{2}$$

$$t_i = \max\left(0,\ k_\tau - m(s-i) - \sum_{b=1}^{i-1} y_b\right), \qquad i = 1,2,\dots,s, \tag{3}$$

$$t_i = \max\left(0,\ k_\tau - \sum_{b=i-s+1}^{i-1} y_b\right), \qquad i = s+1, s+2,\dots,n. \tag{4}$$

Further we can find the probabilities that the system is exactly in state τ for $\tau = 0, 1, \dots, M$ by using the following equations:

$$r_\tau = \Pr(\phi = \tau) = R_\tau - R_{\tau+1}\,, \qquad \text{for } \tau = 0,1,\dots,M\ , \tag{5}$$

$$r_0 = \Pr(\phi = 0) = R_0 - R_1 = 1 - R_1\,, \tag{6}$$

$$r_M = \Pr(\phi = M) = R_M\,, \tag{7}$$

Note:- The equation (1) contains n summations. The lower bounds of these summations depend on others, so we can not calculate these summations as separated. As we will illustrate by the following example.

Illustrated example:

Consider a multi-state linear k-within-(m,s)-of-(m,n):G lattice system with the following data:

m=3, n=4, s=2, M=3, N=3, k_1=3, k_2=4, k_3=5,

p_0=0.1, p_1=0.3, p_2=0.4, p_3=0.2, P_1=0.9, P_2=0.6, P_3=0.2

This system contains three rectangles of dimension 3×2. The system is in state τ or above if and only if every rectangle contains at least k_τ components are in state τ or above.

For state 3:

$$R_3 = \prod_{i=1}^{4}\sum_{y_i=t_i}^{3}\binom{3}{y_i}(0.2)^{y_i}(0.8)^{3-y_i} \qquad \text{Using (1)}$$

$$= \sum_{y_1=t_1}^{3}\sum_{y_2=t_2}^{3}\sum_{y_3=t_3}^{3}\sum_{y_4=t_4}^{3}\binom{3}{y_1}\binom{3}{y_2}\binom{3}{y_3}\binom{3}{y_4}(0.2)^{y_1+y_2+y_3+y_4}(0.8)^{12-(y_1+y_2+y_3+y_4)}$$

$$= \binom{3}{2}\binom{3}{3}\binom{3}{2}\binom{3}{3}(0.2)^{2+3+2+3}(0.8)^{12-10} + \binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{2}(0.2)^{2+3+3+2}(0.8)^{12-10}$$

$$+ \binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{3}(0.2)^{2+3+3+3}(0.8)^{12-11} + \binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{2}(0.2)^{3+2+3+2}(0.8)^{12-10}$$

$$+ \binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{3}(0.2)^{2+3+3+3}(0.8)^{12-11} + \binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{3}(0.2)^{3+3+2+3}(0.8)^{12-11}$$

$$+ \binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{2}(0.2)^{3+3+3+2}(0.8)^{12-11} + \binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}(0.2)^{3+3+3+3}(0.8)^{12-12} = 0.000002,$$

t_1 = max (0,5-3)=2, t_2 = max (0,5-y_1), Using (3)

t_3 = max (0,5-y_2), t_4 = max (0,5-y_3) Using (4)

And so, **For state 2:**

$$R_2 = \prod_{i=1}^{4}\sum_{y_i=t_i}^{3}\binom{3}{y_i}(0.6)^{y_i}(0.4)^{3-y_i} \qquad \text{Using (1)}$$

$$= \sum_{y_1=t_1}^{3}\sum_{y_2=t_2}^{3}\sum_{y_3=t_3}^{3}\sum_{y_4=t_4}^{3}\binom{3}{y_1}\binom{3}{y_2}\binom{3}{y_3}\binom{3}{y_4}(0.6)^{y_1+y_2+y_3+y_4}(0.4)^{12-(y_1+y_2+y_3+y_4)} = 0.257586,$$

t_1 = max (0,4-3)=1, t_2 = max (0,4-y_1), Using (3)

t_3 = max (0,4-y_2), t_4 = max (0,4-y_3) Using (4)

Similarly, **For state 1:**

$$R_1 = \prod_{i=1}^{4}\sum_{y_i=t_i}^{3}\binom{3}{y_i}(0.9)^{y_i}(0.1)^{3-y_i} \qquad \text{Using (1)}$$

$$= \sum_{y_1=t_1}^{3}\sum_{y_2=t_2}^{3}\sum_{y_3=t_3}^{3}\sum_{y_4=t_4}^{3}\binom{3}{y_1}\binom{3}{y_2}\binom{3}{y_3}\binom{3}{y_4}(0.9)^{y_1+y_2+y_3+y_4}(0.1)^{12-(y_1+y_2+y_3+y_4)} = 0.996380,$$

t_1 = max (0,3-3)=0, t_2 = max (0,3-y_1), Using (3)

t_3 = max (0,3-y_2), t_4 = max (0,3-y_3) Using (4)

So we can find the probabilities that the system is exactly in state τ for τ = 0,1,2,3 by using the equations (5-7) as the following:

$r_0 = 1 - R_1 = 1 - 0.996380 = 0.003620,$

$r_1 = R_1 - R_2 = 0.996380 - 0.257586 = 0.738794,$

$r_2 = R_2 - R_3 = 0.257586 - 0.000002 = 0.257584,$

$r_3 = R_3 = 0.000002$

Case 2:- When $s \geq N$ then

$$R_\tau = \prod_{i=1}^{2N-1} \sum_{y_i=t_i}^{m_i} \binom{m_i}{y_i} P_\tau^{y_i} (1 - P_\tau)^{m_i - y_i} , \qquad (8)$$

$m_i = m$, i = 1,2,…,N-1,N+1,…,2N-1,

$m_N = m\,(2s - n)$,

$$t_i = \max\left(0,\ k_\tau - m(s-i) - \sum_{b=1}^{i-1} y_b\right), \quad i = 1,2,\ldots,N-1, \qquad (9)$$

$$t_i = \max\left(0,\ k_\tau - \sum_{b=i-N+1}^{i-1} y_b\right), \quad i = N,\ldots,2N-1. \qquad (10)$$

Illustrated example:

Consider a multi-state linear k-within-(m,s)-of-(m,n):G lattice system with the following data:

m=3, n=5, s=4, M=3, N=2, k_1=8, k_2=10, k_3=12,

p_0=0.1, p_1=0.2, p_2=0.3, p_3=0.5, P_1=0.9, P_2=0.8, P_3=0.5

This system contains two rectangles of dimension 3×4. The system is in state τ or above if and only if every rectangle contains at least k_τ components are in state τ or above.

For state 3:

$$R_3 = \prod_{i=1}^{3} \sum_{y_i=t_i}^{m_i} \binom{m_i}{y_i} (0.5)^{y_i} (0.5)^{m_i - y_i} \qquad \text{Using (8)}$$

$$= \sum_{y_1=t_1}^{3} \sum_{y_2=t_2}^{9} \sum_{y_3=t_3}^{3} \binom{3}{y_1}\binom{9}{y_2}\binom{3}{y_3} (0.5)^{y_1+y_2+y_3} (0.5)^{15-(y_1+y_2+y_3)}$$

$$= \binom{3}{3}\binom{9}{9}\binom{3}{3} (0.5)^{3+9+3} (0.5)^{12-12} = 0.000031,$$

t_1 = max (0,12-9)=3, Using (9)

t_2 = max (0,12-y_1), t_3 = max (0,12-y_2) Using (10)

And so, **For state 2:**

$$R_2 = \prod_{i=1}^{3} \sum_{y_i=t_i}^{m_i} \binom{m_i}{y_i} (0.8)^{y_i} (0.2)^{m_i - y_i}$$

Using (8)

$$= \sum_{y_1=t_1}^{3} \sum_{y_2=t_2}^{9} \sum_{y_3=t_3}^{3} \binom{3}{y_1}\binom{9}{y_2}\binom{3}{y_3} (0.8)^{y_1+y_2+y_3} (0.2)^{15-(y_1+y_2+y_3)} = 0.453680,$$

t_1 = max (0,10-9)=1, Using (9)

t_2 = max (0,10-y_1), t_3 = max (0,10-y_2) Using (10)

Similarly, **For state 1:**

$$R_1 = \prod_{i=1}^{3} \sum_{y_i=t_i}^{m_i} \binom{m_i}{y_i} (0.9)^{y_i} (0.1)^{m_i - y_i} \qquad \text{Using (8)}$$

$$= \sum_{y_1=t_1}^{3} \sum_{y_2=t_2}^{9} \sum_{y_3=t_3}^{3} \binom{3}{y_1}\binom{9}{y_2}\binom{3}{y_3} (0.9)^{y_1+y_2+y_3} (0.1)^{15-(y_1+y_2+y_3)} = 0.992814,$$

t_1 = max (0,8-9)=0, Using (9)

t_2 = max (0,8-y_1), t_3 = max (0,8-y_2) Using (10)

So we can find the probabilities that the system is exactly in state τ for $\tau = 0,1,2,3$ by using the equations (5-7) as the following:

$r_0 = 1 - R_1 = 1 - 0.992814 = 0.007186,$

$r_1 = R_1 - R_2 = 0.992814 - 0.453680 = 0.539134,$

$r_2 = R_2 - R_3 = 0.257586 - 0.000031 = 0.257555,$

$r_3 = R_3 = 0.000031$

5. Numerical Results

The same results of some test examples given by A. Habib et al. [2] for calculation the reliability of multi-state consecutive k-out-of-m-from-n:G system be in Table 1 when the components have the same probabilities (such that p_0=0.1, p_1=0.3, p_2=0.4 and p_3=0.2) by using equations (1) and (8) to show that the multi-state consecutive k-out-of-m-from-n:G system is a special case of the multi-state linear k-within-(m,s)-of-(m,n):G lattice system when m=1. A new examples for calculation the reliability of the multi-state linear k-within-(m,s)-of-(m,n):G lattice system with W={0,1,2,3}given in Table 2 where the components have the same probabilities (p_0=0.1, p_1=0.2, p_2=0.3 and p_3=0.4) by using equations (1) and (8).

6. Conclusions

This paper proposes a model that generalizes linear k-within-(m,s)-of-(m,n):G lattice system to multi-state case. An algorithm for evaluating reliability of a special case of multi-state linear k-within-(m,s)-of-(m,n):G lattice system with equal components $(k_1 \le k_2 \le ... \le k_M)$. The numerical results showed that:

1. The multi-state consecutive k-out-of-r-from-n: G system and linear k-within-(m,s)-of-(m,n):G lattice system are special cases of the multi-state linear k-within-(m,s)-of-(m,n):G lattice system (when m=1 and M=1 respectively).
2. For all τ we have $\sum_{b=0}^{M} r_b = 1$
3. r_τ or R_τ satisfies the probability distribution of the system in various states.

Table 1:-

Multi-state Linear k-within-(1,s)-of-(1,n):G Lattice System (p_0=0.1, p_1=0.3, p_2=0.4, p_3=0.2).

n	s	k_1	k_2	k_3	R_1	R_2	R_3
20	8	3	5	7	0.9997903	0.1614302	0.0000000
40	15	12	13	14	0.7513515	0.0000194	0.0000000
50	20	17	18	19	0.5520580	0.0000002	0.0000000
15	10	4	6	7	0.9999643	0.4171952	0.0000273
30	20	16	17	18	0.8988619	0.0014037	0.0000000
50	30	27	28	29	0.3888692	0.0000000	0.0000000

Table 2:-

Multi-state Linear k-within-(3,s)-of-(3,n):G Lattice System (p_0=0.1, p_1=0.2, p_2=0.3 and p_3=0.4).

s=2, k_1=4, k_2=5, k_3=6				s=5, k_1=12, k_2=13, k_3=14			
n	R_1	R_2	R_3	n	R_1	R_2	R_3
5	0.9464318	0.0924460	0.0000011	5	0.9444444	0.1268277	0.0000252
10	0.8866368	0.0069671	0.0000000	10	0.8402642	0.0104280	0.0000000
15	0.8306196	0.0005264	0.0000000	15	0.7487573	0.0007965	0.0000000
20	0.7781416	0.0000398	0.0000000	20	0.6672703	0.0000607	0.0000000

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