Applied Mathematical Sciences, Vol. 2, 2008, no. 44, 2165 - 2171

On the Eigenvalues of Integral Operators[1]

Yüksel Soykan, Melih Göcen and Arzu Aylin Bostancı

Zonguldak Karaelmas University
Art and Science Faculty
Department of Mathematics
67100, Zonguldak, Turkey
yuksel_soykan@hotmail.com
gocenm@hotmail.com
arzuaylin78@hotmail.com

Abstract

In this paper, we find the numbers of positive and negative eigenvalues of integral operators with certain rational kernels.

Mathematics Subject Classification: 45C05, 45H05

Keywords: Integral operators, eigenvalues

1 Introduction and Preliminaries

In this work we give some examples regarding the numbers of eigenvalues of integral operators with some certain rational kernels k. In this section we begin by setting up some notation given by G.Little in [2] with some modifications and we introduce some definitions and theorems.

For any compact symmetric operator T on a Hilbert space H let us denote:

(a) $\lambda_n^+(T)$ the positive eigenvalues of T in decreasing order

$$\lambda_1^+(T) \geq \lambda_2^+(T) \geq \lambda_3^+(T) \geq ...$$

with repetitions to account for multiplicities and $\lambda_n^-(T)$ the negative eigenvalues in increasing order.

(b) $N^+(T)$, $N^-(T)$ the number of positive and the number of negative eigenvalues of T, respectively, $0 \leq N^+(T), N^-(T) \leq +\infty$.

[1]This work was supported by the Zonguldak Karaelmas University research project 2007/2-13-04-15.

Let $k : I \times I \to \mathbb{C},\ I = [a, b]\,, -\infty < a < b < +\infty$ be a continuous function. Then the integral operator $K : L^2(I) \to L^2(I)$ with kernel k is defined by

$$Kf(s) = \int_I k(s,t)f(t)dt.$$

It is well known that K is compact and the adjoint K^* of K is an integral operator with kernel k^*, where

$$k^*(t,s) = \overline{k(s,t)}.$$

So if $k(t,s) = \overline{k(s,t)}$, then K is a self-adjoint (symmetric) operator.

Definition 1 ([2]) *An open set $E \subseteq \mathbb{C}^2$ is called symmetric if $(z,w) \in E$ implies $(w,z) \in E$. A complex-valued function k on E is called a symmetric analytic kernel on E if*

(a) k is continuous on E,

(b) $k\,(z,w) = \overline{k\,(w,z)}$, for all $(z,w) \in E$,

(c) k is analytic in its first variable at all points of E.

If I is a (bounded) closed real interval such that $I \times I \subseteq E$ and k is a symmetric analytic kernel on E then I is called admissable for k. In this case the integral operator K on $L^2(I)$ with kernel k is compact and symmetric. We use $Ad(k)$ to denote the set of all admissable intervals for k, $N^+(k,I)$ for the sum of the multiplicities of the (strictly) positive eigenvalues of K, and $N^-(k,I)$ for the sum of the multiplicities of the negative eigenvalues of K. In the previous context $N^+(k,I) = N^+(K)$ ve $N^-(k,I) = N^-(K)$. Also we will use $\lambda^+(k,I)$ to denote $\lambda^+(K)$ and $\lambda^-(k,I)$ to denote $\lambda^-(K)$.

Remark 2 ([1]) (a) *A symmetric operator T on a Hilbert space H is called strictly positive if it is positive (i.e. $(Tf,f) \geq 0$ for all $f \in L^2(J)$, written $T \geq 0$) and injective.*

(b) *If T is a strictly positive compact linear operator on a Hilbert space H, then $\overline{ranT} = H$, where $ranT$ is the range of operator T, so that $N^+(T) = \dim H$, in particular if $H = L^2(I)$ for some interval then $N^+(T) = \infty$.*

(c) *If T, S are compact positive operators on a Hilbert Space H and $T - S \geq 0$ (written $T \geq S$) then S strictly positive implies T strictly positive.*

(d) *Let T and S be two integral operators on $L^2(I)$ with continuous kernels k and l on $I \times I$ respectively. In the sequel we shall say that k is positive definite when $T \geq 0$. We shall use the notation $k\,(s,t) \overset{\circ}{\geq} 0$ and we shall always understand that $s, t \in I$. In general we will write $k\,(s,t) \overset{\circ}{\geq} l\,(s,t)$ when $T \geq S$.*

Definition 3 (1, Definition 1.4.2) *Let T be an integral operators on $L^2(J)$ with kernel k and T' be an integral operator on $L^2(I)$ with kernel k'.*

(a) *We say that T is symmetrically equivalent to T' if*

$$T' = MTM^* \tag{1.1}$$

for some continuous and invertible linear operator $M : L^2(J) \to L^2(I)$.

(b) *If M in (1.1) is a unitary operator then we say that T is unitarily equivalent to T'.*

Note that unitary equivalence implies symmetric equivalence.

Lemma 4 (1, Lemma 1.4.3) **(a)** *T unitarily equivalent to T' implies $\lambda_n^{\pm}(T) = \lambda_n^{\pm}(T')$.*

(b) *T symmetrically equivalent to T' implies $N^{\pm}(T) = N^{\pm}(T')$.*

Let $I = [-\rho, \rho]$, $(\rho > 0)$ and let $L_e^2(I)$, $L_o^2(I)$ be the spaces of even, odd functions in $L^2(I)$ respectively. Given $f \in L^2(I)$, write $f_e = projection\ of\ f$ on $L_e^2(I)$ and $f_o = projection\ of\ f$ on $L_o^2(I)$. That is, we can write $f = f_e + f_o$.

The following theorem will play a central role while calculating the numbers of eigenvalues.

Theorem 5 (1, Theorem 1.2.3) *Let $I = [-\rho, \rho]$, where $\rho > 0$, and suppose that k is a continuous symmetric kernel on $I \times I$ satisfying*

$$k(s,t) = k(-s,-t) \qquad (s,t \in I).$$

If T is the integral operator on $L^2(I)$ corresponding to k, then

(a) *$L_e^2(I)$ and $L_o^2(I)$ are invariant subspaces of T (i.e $T(L_e^2(I)) \subseteq L_e^2(I)$ and $T(L_o^2(I)) \subseteq L_o^2(I)$).*

(b) *if $T_e : L_e^2(I) \to L_e^2(I)$ is the restriction of T to $L_e^2(I)$ then T_e is unitarily equivalent to the integral operator T_+ on $L^2(0,\rho)$ given by*

$$T_+ g(s) = \int_0^{\rho} (k(s,t) + k(s,-t)) g(t) dt \qquad \left(g \in L^2(0,\rho), 0 \le s \le \rho\right).$$

(c) *if $T_o : L_o^2(I) \to L_o^2(I)$ is the restriction of T to $L_o^2(I)$ then T_o is unitarily equivalent to the integral operator T_- on $L^2(0,\rho)$ given by*

$$T_- g(s) = \int_0^{\rho} (k(s,t) - k(s,-t)) g(t) dt \qquad \left(g \in L^2(0,\rho), 0 \le s \le \rho\right).$$

The next remark is very useful to calculate the positive and negative numbers of eigenvalues.

Remark 6 (1, Remark 1.2.4) *If $T_e \geq 0$ and $T_o \leq 0$ then*

$$N^{+}(T) = N^{+}(T_e) \text{ and } N^{-}(T) = N^{-}(T_o).$$

Lemma 7 (3, Lemma 2) *Let S, T be symmetric integral operators on $L^2(I)$ with kernels*

$$\frac{1}{q(s,t)}, \ \frac{1}{q(s,t) - r(s,t)}$$

respectively, and suppose that

(a) *q and r are continuous on $I \times I$;*

(b) *$q(s,t) = \overline{q(t,s)}$, $r(s,t) = \overline{r(t,s)}$ on $I \times I$;*

(c) *$|r(s,t)| < q(s,t)$ on $I \times I$.*

If S is positive and r is a positive definite kernel on I, then T is positive and $S \leq T$; in particular rank $T \geq$ rank S and $0 \leq \lambda_n(S) \leq \lambda_n(T)$ for all $n \geq 0$.

2 Main Result

We are ready to give our main result.

Theorem 8 *Let*

$$k(s,t) = \frac{1}{(A+st)^{2n}}, \qquad n \in \mathbb{N}$$

where $A > 0$ and suppose $I = [-\rho, \rho]$ is a symmetric interval where $\rho > 0$ is small enough to ensure that $I \in Ad(k)$. Let T be the integral operator on $L^2(I)$ corresponding to k. Then $N^{+}(T) = N^{-}(T) = \infty$.

Proof. Since $I \in Ad(k)$,

$$q(s,t) = (A+st)^{2n} \neq 0$$

for all $s, t \in I$ so that the integral operator T on $L^2(I)$ with kernel k is compact and symmetric. Clearly $k(s,t) = k(-s,-t)$. By Theorem 5, $L_e^2(I)$ and $L_o^2(I)$ are invariant subspaces of T where $T_e : L_e^2(I) \to L_e^2(I)$ is a operator with kernel

$$k_e(s,t) = \frac{1}{2}[k(s,t) + k(s,-t)]$$

and $T_o : L_o^2(I) \to L_o^2(I)$ is a operator with kernel

$$k_o(s,t) = \frac{1}{2}[k(s,t) - k(s,-t)]$$

and $T = T_e + T_o$. Firstly, we will show that $T_e \geq 0$ with $N^+(T_e) = \infty$ by Remark 6 and then we will investigate the number of negative eigenvalues of this operator. Now note that

$$(s+t)^{2n} = C(2n,0)\, s^{2n} t^0 + C(2n,1)\, s^{2n-1} t^1 + C(2n,2) s^{2n-2} t^2 + ... + C(2n,2n)\, s^0 t^{2n}.$$

Here $C(n,r) = \dfrac{n!}{(n-r)!r!}$ where n and r are integers. We have

$$\begin{aligned} k_e(s,t) &= \frac{1}{2}[k(s,t) + k(s,-t)] \\ &= \frac{1}{2}\left[\frac{1}{(A+st)^{2n}} + \frac{1}{(A-st)^{2n}}\right] \\ &\geq \frac{1}{C(2n,0)\, A^{2n} + C(2n,2) A^{2n-2} s^2 t^2 + ... + C(2n,2n)\, A^0 s^{2n} t^{2n}} \\ &= l(s,t) \overset{\circ}{\geq} 0 \end{aligned}$$

so that $T_e \geq 0$ and $N^+(T_e) = \infty$ because of Theorem 7, Remark 2(c) and the operator L corresponding to kernel $l(s,t)$ is positive. To complete the proof let us investigate the case of operator T_o. Now we want to show that $T_o \leq 0$ with $N^-(T_o) = \infty$. To do this let $T' = -T_o$ and $k' = -k_o$. It is sufficient to show that $T' \geq 0$. Since

$$k_o(s,t) = \frac{1}{2}[k(s,t) - k(s,-t)]$$

we have

$$\begin{aligned} k'(s,t) &= \frac{1}{2}[k(s,-t) - k(s,t)] \\ &= \frac{1}{2}\left[\frac{1}{(A-st)^{2n}} - \frac{1}{(A+st)^{2n}}\right] \\ &= \frac{C(2n,1)\, A^{2n-1} st + C(2n,3) A^{2n-3} s^3 t^3 + ... + C(2n,2n-1)\, A s^{2n-1} t^{2n-1}}{(A+st)^{2n}(A-st)^{2n}} \\ &= \frac{C(2n,1)\, A^{2n-1} st}{(A^2 - s^2 t^2)^{2n}} + \frac{C(2n,3) A^{2n-3} s^3 t^3}{(A^2 - s^2 t^2)^{2n}} + ... + \frac{C(2n,2n-1)\, A s^{2n-1} t^{2n-1}}{(A^2 - s^2 t^2)^{2n}} \\ &= l_1 + l_3 + ... + l_k. \end{aligned}$$

where

$$l_k = \frac{C(2n,k)\,A^{2n-k}s^k t^k}{\left(A^2 - s^2t^2\right)^{2n}}, \qquad k = 1, 3, 5, ..., 2n-1.$$

Firstly we investigate the kernel $l_1(s,t)$. Let

$$M_1 f(s) = \sqrt{C(2n,1)\,A^{2n-1}}s$$

be the multiplication operator then we have

$$\begin{aligned} l_1(s,t) &= \frac{C(2n,1)\,A^{2n-1}st}{\left(A^2 - s^2t^2\right)^{2n}} \\ &= \sqrt{C(2n,1)\,A^{2n-1}}s \frac{1}{\left(A^2 - s^2t^2\right)^{2n}} \sqrt{C(2n,1)\,A^{2n-1}}t. \end{aligned}$$

If we take

$$m_1(s,t) = \frac{1}{\left(A^2 - s^2t^2\right)^{2n}}$$

then we obtain

$$l_1 = M_1 m_1 M_1^*.$$

By $m_1(s,t) \overset{\circ}{\geq} 0$ and the unitarily equivalence in Definition 3, we can see that $l_1 = M_1 m M_1^* \geq 0$. Now we do the same operations to the kernel $l_3(s,t)$. Let

$$M_3 f(s) = \sqrt{C(2n,3)A^{2n-3}}s^3$$

be the multiplication operator then we have

$$\begin{aligned} l_3(s,t) &= \frac{C(2n,3)A^{2n-3}s^3t^3}{\left(A^2 - s^2t^2\right)^{2n}} \\ &= \sqrt{C(2n,3)A^{2n-3}}s^3 \frac{1}{\left(A^2 - s^2t^2\right)^{2n}} \sqrt{C(2n,3)A^{2n-3}}t^3 \end{aligned}$$

If we set

$$m_3(s,t) = \frac{1}{\left(A^2 - s^2t^2\right)^{2n}}$$

we have $l_3 = M_3 m_3 M_3^*$. Again by $m_3(s,t) \overset{\circ}{\geq} 0$ and the unitarily equivalence in Definition 3, we can see that

$$l_3 = M_3 m_3 M_3^* \geq 0.$$

Finally we take the kernel $l_k(s,t)$ for any $k = 2n-1$, $n \in \mathbb{N}$. Let

$$M_k f(s) = \sqrt{C(2n, 2n-1)A} s^{2n-1}$$

be the multiplication operator then we have

$$\begin{aligned} l_k(s,t) &= \frac{C(2n, 2n-1)As^{2n-1}t^{2n-1}}{(A^2 - s^2t^2)^{2n}} \\ &= \sqrt{C(2n, 2n-1)A}s^{2n-1} \frac{1}{(A^2 - s^2t^2)^{2n}} \sqrt{C(2n, 2n-1)A}t^{2n-1}. \end{aligned}$$

If we let

$$m_k(s,t) = \frac{1}{(A^2 - s^2t^2)^{2n}}$$

then it can be seen that $l_k = M_k m_k M_k^*$. Since $m_k(s,t) \overset{\circ}{\geq} 0$ and the unitarily equivalance in Definition 3, we have

$$l_k = M_k m_k M_k^* \geq 0.$$

Thus we obtain $l_1, l_3, ..., l_k \geq 0$. Hence

$$k'(s,t) = l_1(s,t) + l_3(s,t) + ... + l_k(s,t) \geq 0$$

so $T' \geq 0$. Because of the $T' = -T_o$, we have $T_o \leq 0$ and $N^-(T_o) = \infty$. Now we obtained the desired conclusion, i.e $N^+(T) = N^-(T) = \infty$.

References

[1] Al Abbas, M.A., Integral Operators with Rational Kernels, University of Manchester, PhD Thesis (1997)

[2] Little, G., The spectral type of symmetric analytic kernel, *Bull. London Math. Soc.*, **24** (1992), 169-175

[3] Little, G., Eigenvalues of positive power series kernels, *Bull. London Math. Soc.*, **28** (1996), 43-50

Received: February 28, 2008