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Sensitivity Analysis of Righthand-Side Parameter in Transportation Problem

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Abstract. The balanced relation between supply and demand in transportation problem makes it difficult to use traditional sensitivity analysis methods. Therefore, in the process of changing supply or demand resources, at least one more resource needs to be changed to make the balanced relation possible. In this paper, utilizing the concept of complete differential of changes for sensitivity analysis of righthand-side parameter in transportation problem, a method is set forth. This method examines simultaneous and related changes of supply and demand without making any change in the basis. The mentioned method utilizes Arasham and Kahn's simplex algorithm to obtain basic inverse matrix. The validity of mentioned method's results are compared to and inspected by the well-known transportation problems in the literature review.

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1. Introduction

The transportation problem (TP) is one of the fundamental problems of network flow optimization. A surprisingly enough large number of real-life applications can be formulated as a TP [2,6,11,13,14]. Transportation models have become quite popular during the past years. TP is a network structured linear programming problem that has deservedly received an important attention in the literature. The most famous algorithm for solving this problem is stepping stone (SS) [5]. Transportation problems have been studied by various

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authors [1,7-9,] and introduce new approach for TP, but none of them didn't consider sensitivity analysis of TP. As soon as a transportation problem gets its optimal solution, its sensitivity analysis is proposed and this is due to the managers who are facing with changes in demand or warehouse inventory. It is possible to measure the ratio of optimal solution's changes (optimal transportation) to the parameters' changes. For instance, study of increase or decrease in supply and demand may result in discussions with department stores and storekeepers to improve the optimal solution. Formalizing and solving a transportation problem is as boring and difficult as a linear programming one. Because, to make a simplex table, artificial variables are needed that make the calculations increased. Therefore, the SS algorithm based on specific structure of transportation problem model is the best one to solve the problem. Sensitivity analysis of transportation problem, concerning the obtained solution from SS is rarely done. Concerning the obtained solution from SS, Srinivasan and Thompson set forth an approach to analyze the sensitivity of transportation problem which was boring and needed many calculations[12]. Since, the final table's information, represented by SS algorithm is not enough for the sensitivity analysis (there is not any basic inverse matrix in the final table), thus, this algorithm can not be utilized to analyze sensitivity of the transportation problem. Utilizing symmetric tolerance analysis and successful networked algorithm, Ravi and Wendell represented an approach to analyze the sensitivity of the transportation problem. But their sensitivity analysis is highly limited and causes the unreal bounds [10]. Unfortunately, in the mentioned approach, the question about the extra value of supply and demand which results in an optimal solution is responded by try and error. But Arasham and Kahn's algorithm which is designed to solve the transportation problem by simplex solves this problem as well. They represented a simple simplex algorithm to solve the transportation problem utilizing coefficient matrix characteristics [3]. A few years later, in 1992, using Arsham and Kahn algorithm, Arsham analyzed the sensitivity of righthand-side values of the transportation problem [4]. In the sensitivity analysis which introduced by Winston later, the equal increase of only one demand and supply is considered [13]. This sensitivity analysis will not be effective if the value of multiple demand and supply change simultaneously.

Because of using the complete differential calculus, the represented method in this paper is a new approach in sensitivity analysis. The mentioned sensitivity analysis in this paper utilizes the basic inverse matrix from Arsham - Kahn algorithm. This paper is continued with four more sections. In Section 2, the transportation problem and its characteristics and also Arsham - Kahn algorithm are introduced. In Section 3, the sensitivity analysis of transportation problem using complete differential calculus method is defined. Section 4 includes few numerical examples to indicate the efficiency of the represented approach and also their comparison with Arsham's results. Finally, Section 5 is the conclusion.

2. Transportation Problem

Transportation problem (TP) is a specific model in the general image of linear programming problems. The components of this problem are as follow:

- A set of m elements;$O_1, O_2, ..., O_m$, for example depository or farm;
- A set of n elements;$D_1, D_2, ..., D_n$, for example shops or chain stores;
- A set of supplies with certain units; s_1 units from the resource O_1, s_2 units from the resource O_2, , s_m units from the resource O_m;
- A set of needs(demand); d_1 units required for D_1, d_2 units required for D_2, d_n units required for D_n
- A set of transportation costs, c_{ij} is the cost of transporting a product unit from O_i to D_j and there are mnnumbers of these costs.

The aim is to find a transportation design which is applicable to all constraints and has the minimum transportation cost. There are x_{ij} in the number of mn decision variables that indicate the number of the transported units from O_i to D_j. The total cost of transportation program is obtained from decision variables as follow:

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \tag{2.0.1}$$

Balanced TP is a problem in which, the total of all demanded and supplied products are equal. Saving the problem generality, it can be assumed that, each TP is a balanced one or can be easily changed into a balanced problem, adding extra redundant column or row of supply or demand. As it mentioned before, the following equation is applicable to a balanced problem.

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j \tag{2.0.2}$$

Supposing the balance, problem-solving algorithms of TP solve the problems. Thus, it can be supposed that, there is a constraint for righthand-side parameters which is essential in every TP. The constraints of supply and demand in a TP are as follow:

$$\sum_{i=1}^{m} x_{ij} \leq s_i \quad , \quad i = 1, 2, ..., m \tag{2.0.3}$$

$$\sum_{i=1}^{n} x_{ij} \geq d_i \quad , \quad j = 1, 2, ..., n \tag{2.0.4}$$

The matrix shape of TP is as bellow:

$$\begin{aligned} \mathbf{min} \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\ s.t. \quad \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq 0 \end{aligned}$$

Where $\mathbf{b} = (s_1, s_2, ..., s_m, -d_1, -d_2, ..., -d_n)^T$ is the vector of righthand-side and $\mathbf{A}$ is the coefficient matrix of TP. It can be demonstrated that, matrix $\mathbf{A}$ is a unimodular one. It means that, determinant of each square sub matrix of $\mathbf{A}$ is ± 1 or zero [5]. Indices of basic variables are located in set s_B and basic variables coefficient matrix is indicated with $\mathbf{B}$ and includes $m+n-1$ rows ($Rank(\mathbf{A}) = m+n-1$). Thus, each basic solution of TP includes $m+n-1$ non-negative members. Therefore, one of the transportation constraints is redundant and concerning the unimodularity of coefficient matrix, the following equation is set: $|B| = \pm 1$.

2.1. **Arsham-Kahn Algorithm.** Using coefficient matrix characteristic, Arsham and Kahn represented a simple simplex algorithm in two phases for the TP. In the first phase, a feasible or an infeasible basic solution is obtained. Then in the second phase it is led to feasibility and turns into a basic optimal solution. These two phases utilize the row- column matrix by Gauss-Jordan [4]. This algorithm is utilized to obtain B^{-1} and also defines the redundant constraint. It is noteworthy that, Arsham- Kahn algorithm consequently eliminates non-basic columns from the simplex table (in order to decrease the computation). Since, there is not any artificial variable in Arsham-Kahn algorithm; the inverse of basic matrix, B^{-1}, is not available in their algorithm (and does not also exist in any available business software like Lindo). However, formulating righthand-side values based on changeable parameters and solving the obtained TP by Arsham-Kahn algorithm, such information can be obtained with a little computational cost [3].

3. Sensitivity Analysis of TP

To analyze sensitivity in linear programming, after obtaining the optimal solution, one of the righthand-side values or coefficients of objective function are changed, then, the changes in optimal solution and optimal value are examined. For instance, if the optimal solution is as follow:

$$z^* = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{b} \quad , \quad \mathbf{x}_B^* = \mathbf{B}^{-1} \mathbf{b} \tag{3.0.1}$$

Where $\mathbf{c}_B$ is the coefficient vector of objective function whose coefficients are related to the index of basic variables, $\mathbf{x}_B^*$ is basic optimal solution and z^* is the optimal value. Thus, the changes' ratio of optimal solutions and optimal values to the changes of b_i are as follow:

$$(3.0.2) \qquad \frac{d\mathbf{x}_B^*}{d\mathbf{b}} = \mathbf{B}^{-1} \quad , \quad \frac{dx_{B_i}^*}{db_k} = y_{i,k}^* \qquad k = 1, 2, ..., m$$

Where $y_{i,k}^*$ from the $(i,k)^{th}$ element of matrix $\mathbf{B}^{-1}$.

Concerning two reasons, the sensitivity analysis of righthand-side parameters in transportation problem can not be implemented by the mentioned methods. First, instead of one parameter, several parameters may be changed simultaneously, secondly, the changed parameters should also meet the following balanced equation($\sum s_i = \sum d_j$) (Since the transportation problem is balanced, at least, one more righthand-side parameter must be changed). Therefore, current examination of the parameters' changes is different from those in classical methods of sensitivity analysis.

To examine and measure these values, complete differential concept is utilized. It is supposed that, among righthand-side values, k parameters are changed. Thus, concerning the balance of transportation problem, $\sum s_i = \sum d_j$ and $\Delta d_j < d_j$ and $\Delta s_i < s_i$ must be set. Now, regarding complete differential concept, the following equation are set:

$$(3.0.3) \qquad dx_{B_i}^* = \frac{\partial x_{B_i}^*}{\partial b_1} db_1 + \frac{\partial x_{B_i}^*}{\partial b_2} db_2 + ... + \frac{\partial x_{B_i}^*}{\partial b_m} db_m$$

Concerning the concept of general changes in the complete differential case, it can be supposed that $db_1 = \Delta b_1$, thus $dx_{B_i}^* = \Delta x_{B_i}^*$ is set. Now, by replacing $\frac{dx_{B_i}^*}{db_k} = y_{i,k}^*$ in equation (3.0.3) and also regarding the changes in k parameters of righthand-side values, these equation are set:

$$(3.0.4) \qquad \Delta x_{B_i}^* = y_{i,1}^* \Delta b_1 + y_{i,2}^* \Delta b_2 + ... + y_{i,k}^* \Delta b_k$$

Since $y_{i,j}^* \quad , \quad j = 1, 2, ..., k$ are the factors of i^{th} row of matrix $\mathbf{B}^{-1}$ and also concerning the unimodularity state of $\mathbf{B}$, the following equation is set:

$$(3.0.5) \qquad \Delta x_{B_i}^* = \sum_{j=1}^{k} \alpha(\Delta b_i) \quad , \quad i = 1, 2, ..., m \quad , \quad \alpha = 1 or - 1 or 0$$

As it is obvious, in the sensitivity analysis of a problem, parameters must be changed in a way that optimal basis remains unchanged. Thus, after changing of righthand-side values, it is supposed to have:

$$x_{B_i}^* + \Delta x_{B_i}^* \geq 0 \quad , \quad i = 1, 2, ..., m$$

That is the ability of basis to remain feasible after changes. The feasible optimal solution and the unchanged basis through simultaneous changes of righthand-side parameters are examined as follow: It is necessary to obtain the bound for Δb_i that does not lead to infeasibility of optimal solution. For instance, the following equations are set for Δb_i:

$$x^*_{B_i} + \Delta x^*_{B_i} \geq 0 \quad , \quad i = 1, 2, ..., m$$

$$x^*_{B_i} + [y^*_{i,1} + y^*_{i,2}\frac{\Delta b_2}{\Delta b_1} + y^*_{i,k}\frac{\Delta b_2}{\Delta b_1}]\Delta b_1 \geq 0 \quad , \quad i = 1, 2, ..., m$$

$$x^*_{B_i} = \frac{dx^*_{B_i}}{db_1} \geq 0 \quad , \quad i = 1, 2, ..., m$$

If $\frac{dx^*_{B_i}}{db_1} < 0$, then $db_1 \leq \frac{-x^*_{B_i}}{\frac{dx^*_{B_i}}{db_1}}$ and in general, $\Delta b_1 \leq \min_{S_B}\{\frac{-x^*_{B_i}}{\frac{dx^*_{B_i}}{db_1}} | \frac{dx^*_{B_i}}{db_1} < 0\}$ is set that indicates the maximum changes of b_1, where S_B is index of basis variables. The borders of other parameters can be obtained this way.

4. Numerical Example

For better understanding of mentioned aspects, the proposed example by Ravi and Wendell [10] is demonstrated and the changes in values of supply and demand are examined. The transportation cost is figured in Table 1.

Table 1. Transportation Problem of Ravi and Wendell

	D_1	D_2	supply
O_1	20	30	200
O_2	10	40	100
demand	150	150	300

It obviously is a balanced problem solved by Arsham and Kahn using transportation simplex [3]. The results are as follow:

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{x}^*_B = (x_{11}, x_{21}, x_{12}) = (50, 100, 150) \quad z^* = 6500$$

Thus, in this problem, the third constraint is redundant. The values of supply and demand's changes in this problem are consequently shown as $\Delta s_1, \Delta s_2, \Delta d_1$ and Δd_2. To analyze the sensitivity, following changes are supposed for supply and demand's values:

$$\Delta s_1 = -40, \ \Delta s_2 = 40, \ \Delta d_1 = 45, \ \Delta d_2 = -45$$

Thus $\sum \Delta s = \sum \Delta d$. Implementing above changes in transportation problem, Arsham found that the basic solution is changed as follow:

$$\mathbf{x}^*_B = (x_{11}, x_{21}, x_{12}) = (55, 140, 105) \quad , \quad z^* = 5650$$

It is clear that, such changes in supply and demand decreases transportation costs to 850. Now, using the mentioned method in this paper, according to equation (3.0.4) the following equations are set (the redundant third constraint has been omitted):

$$\Delta x^*_{B_i} = [y^*_{i,1}\Delta s_1 + y^*_{i,2}\Delta s_2 + y^*_{i,3}\Delta d_2]$$

$$\begin{aligned}
\Delta x^*_{1,1} &= [y^*_{1,1}\Delta s_1 + y^*_{1,2}\Delta s_2 + y^*_{1,3}\Delta d_2] = 1(-40) + 0 - 1(-45) = 5 \\
\Delta x^*_{2,1} &= [y^*_{2,1}\Delta s_1 + y^*_{2,2}\Delta s_2 + y^*_{2,3}\Delta d_2] = 0 + 1(40) + 0 = 40 \\
\Delta x^*_{1,2} &= [y^*_{3,1}\Delta s_1 + y^*_{3,2}\Delta s_2 + y^*_{3,3}\Delta d_2] = 0 + 0 + 1(-45) = -45
\end{aligned}$$

That result in:

$$\mathbf{x}^*_B = (x_{11}, x_{21}, x_{12}) = (50+5, 100+40, 150-45) = (55, 140, 105)$$
$$z^* = 5650$$

That is the same as Arsham's results [4].

Now, the permitted changes for s_1 and the bound of its changes are examined.

$$\frac{dx^*_{B_i}}{ds_1} = y^*_{i,1} + y^*_{i,2}\frac{\Delta s_2}{\Delta s_1} + y^*_{i,3}\frac{\Delta d_2}{\Delta d_1}$$
$$\frac{dx^*_{11}}{ds_1} = \frac{-1}{8} \quad , \quad \frac{dx^*_{21}}{ds_1} = -1 \quad , \quad \frac{dx^*_{12}}{ds_1} = \frac{9}{8}$$
$$\Delta s_1 \leq min\{\frac{-100}{-1}, \frac{-50}{\frac{-1}{8}}\} = 100$$

It can be seen that, $\Delta s_1 = 40$ is applied to the above condition. Thus, these changes do not lead the problem to infeasibility and similarly, $\Delta d_2 \leq \frac{100}{8}, \Delta s_2 \leq \frac{1200}{9}$ is set.

The next example is a 3×2 problem that is presented by Davis and colleagues [4]. Table 2 demonstrates the transportation costs of this problem.

Table 2. Transportation Problem of Davis and Colleagues

	D_1	D_2	supply
O_1	3	6	400
O_2	4	5	300
O_3	7	3	400
demand	450	350	

This is not a balanced problem but it can be turned into a balanced one adding a redundant column of 300 units of demand (Table 3).

Table 3. Balanced Transportation Problem of Davis and Colleagues

	D_1	D_2	D_3	supply
O_1	3	6	0	400
O_2	4	5	0	300
O_3	7	3	0	400
demand	450	350	300	1100

Solving this problem with Arsham's algorithm, the following results are obtained:

$$\mathbf{B}^{-1} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{x}_B^* = (x_{21}, x_{23}, x_{33}, x_{11}, x_{32}) = (50, 250, 50, 400, 350) \quad , \quad z^* = 2450.$$

And the sixth constraint is defined as the redundant one. Now, the following changes are done in supply and demand.

$$\Delta s_1 = 10, \quad \Delta s_2 = 20, \quad \Delta s_3 = -20, \quad \Delta d_1 = 5, \quad \Delta d_2 = -20, \quad \Delta d_3 = 25$$

Using Arsham's approach, the obtained results are as follow:

$$(x_{21}, x_{23}, x_{33}, x_{11}, x_{32}) = (45, 275, 50, 410, 330)$$

Now, using the proposed method, this equation is set:

$$\Delta x^*_{B_i} = [\frac{\partial x^*_{B_i}}{\partial s_1}\Delta s_1 + \frac{\partial x^*_{B_i}}{\partial s_2}\Delta s_2 + \frac{\partial x^*_{B_i}}{\partial s_3}\Delta s_3 + \frac{\partial x^*_{B_i}}{\partial d_1}\Delta d_1 + \frac{\partial x^*_{B_i}}{\partial d_2}\Delta d_2]$$

$$\Delta x^*_{21} = (-1)10 + 0 + 0 + 1(5) + 0 = -5 \quad \Delta x^*_{23} = 1(10) + 1(20) + 0 - 1(5) + 0 = 25$$
$$\Delta x^*_{33} = 0 + 0 + 1(-20) + 0 - 1(-20) = 0 \quad \Delta x^*_{11} = 1(10) + 0 + 0 + 0 + 0 = 10$$
$$\Delta x^*_{32} = 0 + 0 + 0 + 0 + 1(-20) = -20$$

Thus:

$$\begin{aligned}(x_{21}, x_{23}, x_{33}, x_{11}, x_{32}) &= (50 - 5, 250 + 25, 50 + 0, 400 + 10, 350 - 20) \\ &= (45, 275, 50, 410, 330)\end{aligned}$$

That is the same as Arsham's again.

For the another example, We demonstrates the sensitivity analysis by applying our approach to the problem in Arsham [4]. The cost matrix is shown in Table 4.

TABLE 4. Transportation Problem of Arsham

	D_1	D_2	D_3	D_4	D_4	supply
O_1	14	15	6	13	14	7
O_2	16	9	22	13	16	18
O_3	8	5	11	4	5	6
O_4	12	4	18	9	10	15
demand	4	11	12	8	11	46

It obviously is a balanced problem ($\sum s_i = \sum d_j = 46$) solved by Arsham and Kahn using transportation simplex [3]. The results are as follow:

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$\mathbf{x}_B^* = (x_{13}, x_{22}, x_{42}, x_{21}, x_{24}, x_{33}, x_{45}, x_{35}) = (7, 6, 5, 4, 8, 5, 10, 1) \quad , \quad z^* = 444.$$

And the third row (Origin 3) is defined as the redundant one. Now, the following changes are done in supply and demand.

$$\Delta s_1 = 3 \quad , \quad \Delta s_2 = -2 \quad , \quad \Delta s_4 = -2 \quad , \quad \Delta d_1 = 2 \quad , \quad \Delta d_5 = -1$$

Using Arsham's approach, the obtained results are as follow:

$$(x_{13}, x_{22}, x_{42}, x_{21}, x_{24}, x_{33}, x_{45}, x_{35}) = (10, 2, 9, 6, 8, 2, 4, 6) \quad , \quad z^* = 406$$

Now, using the proposed method, this equation is set:

$$\Delta x_{B_i}^* = [\frac{\partial x_{B_i}^*}{\partial s_1}\Delta s_1 + \frac{\partial x_{B_i}^*}{\partial s_2}\Delta s_2 + \frac{\partial x_{B_i}^*}{\partial s_4}\Delta s_4 + \frac{\partial x_{B_i}^*}{\partial d_1}\Delta d_1 + \frac{\partial x_{B_i}^*}{\partial d_2}\Delta d_2 + \frac{\partial x_{B_i}^*}{\partial d_3}\Delta d_3 + \frac{\partial x_{B_i}^*}{\partial d_4}\Delta d_4 + \frac{\partial x_{B_i}^*}{\partial d_5}\Delta d_5]$$

$$\Delta x_{13}^* = 1(3) + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 3$$
$$\Delta x_{22}^* = 0 + 1(-2) + 0 - 1(2) + 0 + 0 - 1(0) + 0 = -4$$
$$\Delta x_{42}^* = 0 - 1(-2) + 0 + 1(2) + 1(0) + 0 + 1 + 0 = 4$$
$$\Delta x_{21}^* = 0 + 0 + 0 + 1(2) + 0 + 0 + 0 + 0 = 2$$
$$\Delta x_{24}^* = 0 + 0 + 0 + 0 + 0 + 0 + 1(0) + 0 = 0$$
$$\Delta x_{33}^* = -1(3) + 0 + 0 + 0 + 0 + 1(0) + 0 + 0 = -3$$
$$\Delta x_{45}^* = 0 + 1(-2) + 1(-2) - 1(2) - 1(0) + 0 - 1(0) + 0 = -6$$
$$\Delta x_{35}^* = 0 - 1(-2) - 1(-2) + 1(2) + 1(0) + 0 + 1(0) + 1(-1) = 5$$

Thus:

$$\begin{aligned}(x_{13}, x_{22}, x_{42}, x_{21}, x_{24}, x_{33}, x_{45}, x_{35}) &= (3 + 7, 6 - 4, 5 + 4, 4 + 2, 8 + 0, 5 - 3, 10 - 6, 1 + 5)\\ &= (10, 2, 9, 6, 8, 2, 4, 6)\\ z^* &= 406\end{aligned}$$

That is the same as Arsham's again

5. Conclusion

Supposing the balance in the algorithms of transportation problems, the sensitivity analysis of transportation problems is, in one hand, a simultaneous analysis of righthand-side parameters and on the other hand, the implementation of the balanced equation. Like a constraint, this balanced equation directly affects the parameters whose changes are important. Thus, in the sensitivity analysis of supply and demand in transportation problem, because of the functional relation between supply and demand parameters, we always face the simultaneous changes of parameters. In this paper, a particular method for evaluating the sensitivity analysis of supply and demand values was presented. Implementing Arsham's sensitivity analysis in few well known transportation problems, the validity of the mentioned method was examined.

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