Applied Mathematical Sciences, Vol. 3, 2009, no. 31, 1513 - 1517

# A Numerical Computaion of Non-linear Hydrodynamic Model

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**Abstract**

Water pollution assessment problems arise frequently in environmental science. The hydrodynamic model is used to simulated current in the reservoir that provides the velocity and height of the water. In this research, the numerical computation of velocity and water elevation in two-dimensional non-linear hydrodynamic model is considered. The simulating processes, the Lax-Wendroff method is used to solve the model. The accuracy of the models is tested by the illustrative example.

**Mathematics Subject Classification:** 65M06, 62P12

**Keywords:** Non-linear hydrodynamic model, Lax-Wendroff method

## 1 Introduction

The increases in an industrial occupation is the principal reason for the growth of pollution. The methods to detect the amount of pollutant both in the air and water mostly are conducted by a field measurement and a mathematical simulation. For the shallow water mass transport problems that presented in [1], the method of characteristics has been reported as being applied with success, but it presents in real cases some difficulties. In [4] and [6], the finite element method for solving the water pollution models in one- and two-dimensional water areas are presented, respectively. The most of mathematical model require data concerning with velocity of the current at any point in the domain. The hydrodynamic model provides the velocity field and tidal elevation of the water. Those results are data for the dispersion model. In [5], they used the finite difference method to the hydrodynamic model with constant coefficients in the uniform reservoir.

Averaging the equation over the depth, discarding the term due to Coriolis force, shearing stresses and surface wind, it follows that the two-dimensional non-linear shallow water equation is applicable [3]. In this research, we use the Lax-Wendroff method to approximate the velocity and the tidal elevation with non-linear terms.

## 2 The Hydrodynamic Model

The continuity and momentum equations are govern the hydrodynamic behavior of the reservoir. Averaging the equations over the depth, discarding the term due to Coriolis parameter, shearing stresses and surface wind. The well-known two-dimensional shallow water equations

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}[(h+\zeta)u] + \frac{\partial}{\partial y}[(h+\zeta)v] = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + g\frac{\partial \zeta}{\partial x} = 0, \tag{2}$$

$$\frac{\partial v}{\partial t} + g\frac{\partial \zeta}{\partial y} = 0. \tag{3}$$

where $h(x, y)$ be the depth measured from the mean water level to the bed of the reservoir, $\zeta(x, y, t)$ is the elevation from the mean water level to the temporary water surface or the tidal elevation, $g$ is the acceleration due to gravity, and $u(x, y, t)$ and $v(x, y, t)$ are the velocity components, for all $(x, y) \in [0, l] \times [0, l]$. We now introduce the two-dimensional non-linear shallow water equations with dimensionless form by letting $U = u/\sqrt{gh}, V = v/\sqrt{gh}, X = x/l, Y = y/l, Z = \zeta/h$ and $T = t\sqrt{gh}/l$,

$$\frac{\partial Z}{\partial T} + \frac{\partial}{\partial X}[(1+XY)U] + \frac{\partial}{\partial Y}[(1+XY)V] = 0, \tag{4}$$

$$\frac{\partial U}{\partial T} + \frac{\partial Z}{\partial X} = 0, \tag{5}$$

$$\frac{\partial V}{\partial T} + \frac{\partial Z}{\partial Y} = 0 \tag{6}$$

in $\Omega \times [0, \mathcal{T}]$ where $\Omega = (0, 1) \times (0, 1)$ with the initial conditions $\frac{\partial Z}{\partial T} = 0$ and $Z(X, Y, 0) = f(X, Y)$. The boundary conditions $Z(0, Y, T) = Z(1, Y, T) = Z(X, 0, T) = Z(X, 1, T) = 0$ at $\partial\Omega$.

## 3 The Numerical Technique

The equations (4)-(6) can be written in the matrix form

$$\frac{\partial W}{\partial T} = A\frac{\partial W}{\partial X} + B\frac{\partial W}{\partial Y}, \tag{7}$$

where

$$W = \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix}, A = \begin{bmatrix} 0 & -1 & 0 \\ -(1+XY) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -(1+XY) & 0 & 0 \end{bmatrix}, \quad (8)$$

$W_1 = Z, W_2 = (1+XY)U$ and $W_3 = (1+XY)V$. We now discretize Eq.(7) by dividing the interval $[0,1]$ into $L$ and $M$ subintervals such that $L\Delta X = 1$ and $M\Delta Y = 1$, and the interval $[0,\mathcal{T}]$ into $N$ subintervals such that $N\Delta T = \mathcal{T}$. We can then approximate $W_1(X_l, Y_m, T_n)$ by $W^n_{1_{l,m}}$, value of the difference approximation of $W_1(X,Y,T)$ at point $X = l\Delta X$, $Y = m\Delta Y$ and $T = n\Delta T$, where $0 \le l \le L$, $0 \le m \le M$ and $0 \le n \le N$, and similarly defined for $W^n_{2_{l,m}}$ and $W^n_{3_{l,m}}$. The grid point $(X_l, Y_m, T_n)$ are defined by $X_l = l\Delta X$ for all $l = 0,1,2,\dots,L$, $Y_m = m\Delta Y$ for all $m = 0,1,2,\dots,M$ and $T_n = n\Delta T$ for all $n = 0,1,2,\dots,N$ in which $L, M$ and $N$ are positive integers. Using the Lax-Wendroff method [2] to Eq.(7), we can simplified the following finite difference equation

$$\begin{aligned} W^{n+1}_{l,m} = {} & W^n_{l,m} + \frac{1}{2}pA_{l,m}(W^n_{l+1,m} - W^n_{l-1,m}) + \frac{1}{2}pB_{l,m}(W^n_{l,m+1} - W^n_{l,m-1}) + \\ & \frac{1}{4}p^2A_{l,m}[(A_{l+1,m}W^n_{l+1,m} - A_{l,m}W^n_{l,m}) - (A_{l+1,m}W^n_{l,m} - A_{l,m}W^n_{l-1,m})] + \\ & \frac{1}{4}p^2B_{l,m}[(B_{l,m+1}W^n_{l,m+1} - B_{l,m}W^n_{l,m}) - (B_{l,m+1}W^n_{l,m} - B_{l,m}W^n_{l,m-1})] + \\ & \frac{1}{8}p^2(A_{l,m}B_{l,m} + B_{l,m}A_{l,m})[W^n_{l+1,m+1} - W^n_{l-1,m+1} - W^n_{l+1,m-1} + W^n_{l-1,m-1}] \quad (9) \end{aligned}$$

where

$$W^n_{l,m} = \begin{pmatrix} W^n_{1_{l,m}} \\ W^n_{2_{l,m}} \\ W^n_{3_{l,m}} \end{pmatrix}, \Delta_X W^n_i = W^n_{i+1} - W^n_i, \nabla_X W^n_i = W^n_i - W^n_{i-1}$$

and $p = \Delta t/\Delta x$. A stability analysis of Lax-Wendroff scheme (9) with matrices $A$ and $B$ has shown in [2]. The Lax-Wendroff scheme is stable if $p|\lambda_0| \le \frac{1}{2\sqrt{2}}$ where $|\lambda_0| = \max\{|\lambda_A|, |\lambda_B|\}$ where $\lambda_A, \lambda_B$ are eigenvalues of $A$ and $B$ respectively.

# 4 The Numerical Experiment

We consider the uniform reservoir with dimension 3.2×3.2 km and the constant depth $h = 1$ m. The reservoir is meshed with 6,400 grids points with $\Delta x = \Delta y = 40$ m and taking time interval $\Delta t = 2.5$ sec. Initially the water in the reservoir is assumed to be motionless $U = 0$, $V = 0$ and the water elevation is specified $Z(X,Y,0) = X(1-X)Y(1-Y)$. Using the equation (9), the water elevation and velocity in x-direction and y-direction are shown in Tables 1-3 respectively.

# 5 Conclusions

In this research the model for approximation of the the velocity of the reservoir is constructed. This model can be applied to the real cases for current in the reservoir by the industry, which we can change the inputs elevation $f(x,y)$.

**ACKNOWLEDGEMENTS.** The author would like to thanks: S. Tangmanee, L.J. Crane and J.J.H. Miller for their valuable comments.

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**Received: November, 2008**

Table 1: The elevation $\zeta(x, y, t)$ m for $0 \leq x, y \leq 3.2$ km at $t = 1$ hr 24 min

| $y, x$(m) | 0 | 400 | 800 | 1,200 | 1,600 | 2,000 | 2,400 | 2,800 | 3,200 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 400 | 0.0000 | 0.0220 | 0.0293 | 0.0332 | 0.0295 | 0.0186 | 0.0162 | 0.0091 | 0.0000 |
| 800 | 0.0000 | 0.0293 | 0.0491 | 0.0534 | 0.0462 | 0.0375 | 0.0269 | 0.0151 | 0.0000 |
| 1,200 | 0.0000 | 0.0336 | 0.0536 | 0.0586 | 0.0545 | 0.0447 | 0.0316 | 0.0162 | 0.0000 |
| 1,600 | 0.0000 | 0.0294 | 0.0465 | 0.0547 | 0.0517 | 0.0434 | 0.0300 | 0.0152 | 0.0000 |
| 2,000 | 0.0000 | 0.0190 | 0.0380 | 0.0449 | 0.0434 | 0.0358 | 0.0244 | 0.0117 | 0.0000 |
| 2,400 | 0.0000 | 0.0160 | 0.0272 | 0.0317 | 0.0301 | 0.0244 | 0.0158 | 0.0080 | 0.0000 |
| 2,800 | 0.0000 | 0.0084 | 0.0141 | 0.0155 | 0.0148 | 0.0117 | 0.0084 | 0.0041 | 0.0000 |
| 3,200 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

Table 2: The velocity $u(x, y, t)$ m/s for $0 \leq x, y \leq 3.2$ km at $t = 1$ hr 24 min

| $y, x$(m) | 0 | 400 | 800 | 1,200 | 1,600 | 2,000 | 2,400 | 2,800 | 3,200 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 400 | 0.0071 | 0.0308 | 0.0555 | 0.0763 | 0.0830 | 0.0872 | 0.0807 | 0.0501 | 0.0100 |
| 800 | 0.0020 | 0.0127 | 0.0294 | 0.0532 | 0.0656 | 0.0645 | 0.0542 | 0.0297 | 0.0064 |
| 1,200 | 0.0027 | 0.0062 | 0.0205 | 0.0324 | 0.0396 | 0.0399 | 0.0293 | 0.0100 | 0.0011 |
| 1,600 | -0.0010 | -0.0002 | 0.0035 | 0.0013 | 0.0027 | 0.0029 | -0.0045 | -0.0031 | 0.0007 |
| 2,000 | -0.0015 | -0.0107 | -0.0214 | -0.0279 | -0.0362 | -0.0429 | -0.0426 | -0.0276 | -0.0035 |
| 2,400 | -0.0064 | -0.0233 | -0.0469 | -0.0734 | -0.0904 | -0.0963 | -0.0779 | -0.0564 | -0.0024 |
| 2,800 | -0.0121 | -0.0516 | -0.0955 | -0.1262 | -0.1415 | -0.1350 | -0.1097 | 0.0069 | -0.0359 |
| 3,200 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

Table 3: The velocity $v(x, y, t)$ m/s for $0 \leq x, y \leq 3.2$ km at $t = 1$ hr 24 min

| $y, x$(m) | 0 | 400 | 800 | 1,200 | 1,600 | 2,000 | 2,400 | 2,800 | 3,200 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.0000 | 0.0073 | 0.0024 | 0.0029 | -0.0008 | -0.0012 | -0.0066 | -0.0108 | 0.0000 |
| 400 | 0.0000 | 0.0303 | 0.0125 | 0.0063 | 0.0003 | -0.0096 | -0.0227 | -0.0489 | 0.0000 |
| 800 | 0.0000 | 0.0562 | 0.0287 | 0.0196 | 0.0036 | -0.0201 | -0.0455 | -0.0912 | 0.0000 |
| 1,200 | 0.0000 | 0.0772 | 0.0526 | 0.0313 | 0.0015 | -0.0268 | -0.0712 | -0.1219 | 0.0000 |
| 1,600 | 0.0000 | 0.0839 | 0.0644 | 0.0389 | 0.0031 | -0.0345 | -0.0871 | -0.1350 | 0.0000 |
| 2,000 | 0.0000 | 0.0880 | 0.0645 | 0.0388 | 0.0029 | -0.0414 | -0.0915 | -0.1249 | 0.0000 |
| 2,400 | 0.0000 | 0.0797 | 0.0530 | 0.0278 | -0.0051 | -0.0401 | -0.0658 | -0.0942 | 0.0000 |
| 2,800 | 0.0000 | 0.0506 | 0.0286 | 0.0076 | -0.0062 | -0.0282 | -0.0348 | -0.0141 | 0.0000 |
| 3,200 | 0.0000 | 0.0212 | 0.0435 | 0.0614 | 0.0758 | 0.0751 | 0.0611 | 0.0455 | 0.0000 |