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Homotopy Perturbation Method for Computing Eigenelements of Sturm-Liouville Two Point Boundary Value Problems

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Abstract

In this work we consider the application of the Homotopy Perturbation Method (HPM) to compute the eigenvalues and the corresponding normalized eigenfunction of Sturm-Liouville problem. To illustrate the method some experiments are provided. The results show the efficiency and accuracy of the HPM.

Mathematics Subject Classification: 65L15

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1 Introduction

Recently a great deal of interest has been focused on the application of HPM for the solution of many different problems. The technique has been applied with great success to obtain the solution of a large variety of nonlinear problems in both ordinary and partial differential equations and integro-differential equations[1-10]. In this work we apply HPM to approximate eigenvalues and eigenfunctions of a Sturm-Liouville problem. It will be shown that the method is easy to use and to compute. Numerical experiments are presented to show the efficiency of the method. A Sturm-Liouville (SL) boundary value problem is a second order linear ordinary differential equation and can be written as:

$$Ly = -\frac{1}{w(x)}[\frac{d}{dx}(P(x)\frac{dy}{dx}) + Q(x)y] = \lambda y, \tag{1}$$

with the conditions:

$$\alpha_1 y(a) + \alpha_2 y^{'}(a) = 0, (\alpha_1, \alpha_2) \neq (0, 0),$$

$$\beta_1 y(a) + \beta_2 y^{'}(a) = 0, (\beta_1, \beta_2) \neq (0, 0),$$

where y is a eigenfunction and λ is a eigenvalue. In this problem we assume $P(x)$, $P^{'}(x)$, $Q(x)$ and $w(x)$ are continuous functions on $[a, b]$. In addition we assumed that $P(x)$ and $w(x)$ are positive functions on $[a, b]$. The concept of an eigenvalue problem is rather important both in pure and applied mathematics. We refer to the books [11,12] and [13] where the subject is studied in more details. Several authors have considered the numerical computation of the eigenvalues and the corresponding normalized eigenfunctions [14-18]. In section 2 we explain the HPM. Numerical experiments and their results are presented in section 3. In section 4 conclusion is expressed.

2 Homotopy Perturbation Method [19]

To describe HPM, we consider the following non-linear differential equation:

$$A(u) - f(r) = 0, r \in \Omega, \tag{2}$$

with the boundary conditions:

$$B(u, \frac{\partial u}{\partial n}) = 0, r \in \Omega,$$

where A is differential operator, B is boundary operator, $f(r)$ is a known analytic function and Γ is the boundary of the domain Ω. The operator A can be divided into two parts L and N, where L is a linear and N is non-linear operators. Therefore equation $A(u) - f(r) = 0$ can be rewritten :

$$L(u) + N(u) - f(r) = 0.$$

Now we construct a homotopy $\upsilon(r, p) : \Omega \times [0, 1] \longrightarrow R$ which satisfies

$$H(\upsilon, p) = (1 - p)[L(u) - L(u_0)] + p[A(\upsilon) - f(r)] = 0, p \in [0, 1], r \in \Omega, \tag{3}$$

where $p \in [0, 1]$ is embedding parameter, u_0 is an initial approximation, which satisfies the boundary conditions. Obviously we have:

$$H(\upsilon, 0) = L(\upsilon) - L(u_0),$$

$$H(\upsilon, 1) = A(\upsilon) - f(r),$$

Changing process of p from zero to unity is just that of $\upsilon(r, p)$ from u_0 to $u(r)$. We assume that the solution of equation:

$$H(\upsilon, p) = (1 - p)[L(u) - L(u_0)] + p[A(\upsilon) - f(r)] = 0, p \in [0, 1], r \in \Omega, \tag{4}$$

can be written as a power series in p, $\upsilon = \upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + \cdots$. By setting $p = 1$, the approximation solution of $A(u) - f(r) = 0$ is obtained.

3 Numerical experiments

In this section the numerical results of three Sturm-Liouville problems are presented. We describe the method outlined in the previous section for these problems.

Problem 3.1 Consider the boundary value problem:

$$-y'' + y = \lambda y; x \in (0, 1), \tag{5}$$

with the boundary conditions:$y(0) = y(1) = 0, x \in (0, 1)$.

Let $L(y) = y''$ and $N(y) = y - \lambda y$. We also assume that:

$$Y(x, p) = y_0(x) + py_1(x) + p^2 y_2(x) + p^3 y_3(x) + \cdots. \tag{6}$$

By substituting (6) in (4) and equating the coefficients of p we obtain:

coefficient of $p^0 : -y_0'' = 0, y_0(0) = 0,$

coefficient of $p^1 : -y_1'' + (\lambda - 1)y_0 = 0, y_1'(0) = y_1(0) = 0,$

coefficient of $p^2 : -y_2'' + (\lambda - 1)y_1 = 0, y_2'(0) = y_2(0) = 0,$

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coefficient of $p^n : -y_n'' + (\lambda - 1)y_{n-1} = 0, y_n'(0) = y_n(0) = 0.$

Therefore the solution of the problem is:

$$y(x, \lambda) = a \sum_{k=0}^{\infty} (-1)^k (\lambda - 1)^k \frac{x^{2k+1}}{(2k + 1)!} = \frac{a}{\sqrt{\lambda - 1}} \sum_{k=0}^{\infty} (-1)^k \frac{(\sqrt{\lambda - 1}\ t)^{2k+1}}{(2k + 1)!} \tag{7}$$

The infinite series converges to

$$y(x, \lambda) = \frac{a}{\sqrt{\lambda - 1}} sin(\sqrt{\lambda - 1} t), a \neq 0. \tag{8}$$

To satisfy the other boundary condition we have $y(1) = 0$, which implies the eigenvalues are $\lambda_n = 1 + n^2\pi^2, n = 0, 1, 2, \cdots$. By this method in this case we get the exact solution for both eigenvalues and eigenfunctions.

Problem 3.2 Consider the boundary value problem:

$$y'' = \lambda y; x \in (0, \pi), \tag{9}$$

with the boundary conditions:$y'(0) = y(\pi) = 0, x \in (0, \pi)$.

Let $L(y) = y''$ and $N(y) = -\lambda y$. We also assume that:

$$Y(x, p) = y_0(x) + py_1(x) + p^2 y_2(x) + p^3 y_3(x) + \cdots \tag{10}$$

By substituting (10) in (4) and equating the coefficients of p we obtain:

coefficient of $p^0 : y_0'' = 0, y_0'(0) = 0,$

coefficient of $p^1 : y_1'' - \lambda y_0 = 0, y_1'(0) = y_1(0) = 0,$

coefficient of p^2 : $y_2'' - \lambda y_1 = 0, y_2'(0) = y_2(0) = 0,$
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coefficient of p^n : $y_n'' - \lambda y_{n-1} = 0, y_n'(0) = y_n(0) = 0.$
Therefore the solution of the problem is:

$$y(x, \lambda) = acos(\sqrt{\lambda}x), a \neq 0. \tag{11}$$

To satisfy the other boundary condition we have $y(\pi) = 0$, which implies the eigenvalues are $\lambda_n = (\frac{2n+1}{2})^2, n = 0, 1, 2, 3, ...$ By this method in this case we get the exact solution for both eigenvalues and eigenfunctions.

Problem 3.3 Consider the boundary value problem:

$$y'' = \lambda y; x \in (0, \pi), \tag{12}$$

with the boundary conditions:$y(0) = y'(\pi) = 0, x \in (0, \pi)$. Let $L(y) = y''$ and $N(y) = -\lambda y$. We also assume that:

$$Y(x, p) = y_0(x) + py_1(x) + p^2y_2(x) + p^3y_3(x) + \cdots \tag{13}$$

By substituting (13) in (4) and equating the coefficients of p we obtain:
coefficient of p^0 : $y_0'' = 0, y_0'(0) = 0,$
coefficient of p^1 : $y_1'' - \lambda y_0 = 0, y_1'(0) = y_1(0) = 0,$
coefficient of p^2 : $y_2'' - \lambda y_1 = 0, y_2'(0) = y_2(0) = 0,$
.
.
coefficient of p^n : $y_n'' - \lambda y_{n-1} = 0, y_n'(0) = y_n(0) = 0.$
Therefore the solution of the problem is:

$$y(x, \lambda) = asin(\sqrt{\lambda}x), a \neq 0. \tag{14}$$

To satisfy the other boundary condition we have $y'(\pi) = 0$, which implies the eigenvalues are $\lambda_n = (\frac{2n+1}{2})^2, n = 0, 1, 2, 3, ...$ By this method in this case we get the exact solution for both eigenvalues and eigenfunctions.

4 Conclusion

According to the results we have obtained, we infer that HPM is a powerful tool for solving the eigenvalues and the corresponding normalized eigenfunction of the Sturm-Liouville problem. We have experienced that, this method is easy to implement and produces exact solution in these problems. Also, it competes very well with Adomian decomposition method [17].

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