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An Enlarged Analysis of the Asymptotic Compensation Problem for a Class of Disturbed Systems

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Abstract

This work concerns an enlarged problem of the asymptotic compensation for a class of disturbed systems. It consists to study, with respect to the output of the system and a given tolerance zone $\mathcal{C}$, the possibility to reduce asymptotically the effect of any disturbance by bringing back the observation in the zone $\mathcal{C}$. Under convenient hypothesis, we prove the existence and the unicity of the optimal control ensuring this compensation and we show how to find it.

Keywords: Enlarged asymptotic compensation, optimal control, observation, actuators, sensors

1 Problem statement

The notions of remediability and efficient actuators, ensuring the finite time compensation of any disturbance, have been introduced firstly in the finite time case [1, 2, 3, 4]. These notions are motivated by environmental and pollution problems. It is shown that the remediability is weaker and more flexible than the controllability [10, 11, 12, 15, 16], and hence that strategic actuators [10, 11, 12] are efficient. The converse is not true.
Extensions to the asymptotic case are given in [5, 6, 7] where it is shown that a system may be remediable without being stable or even stabilizable [8, 9, 12, 13, 14, 15].
In the two cases, characterization results are established and various situations are examined. Using the Hilbert Uniqueness Method (H.U.M.), it is shown how to find the optimal control ensuring the compensation of a disturbance.

In this work, we consider a class of linear disturbed systems and we examine a more general problem which consists to study the possibility to reduce asymptotically the effect of a disturbance by bringing the observation in a given tolerance zone $\mathcal{C}$.
This problem has been studied in the finite time case [3]. In the asymptotic case, the situation is more complex and need a special and adapted approach. Indeed, the enlarged asymptotic compensation problem is approximated by a family of finite time problems. Then, under convenient hypothesis and using a penalization method, the H.U.M. approach and convenient mathematical developments depending on time T and also on the other parameters, particularly those considered in the penalization criterion, we establish the convergence of the finite time results when $T \longrightarrow +\infty$ and we deduce the main result showing how to find the optimal control which ensures the enlarged asymptotic compensation of a known or unknown disturbance.

Without loss of generality, we consider a class of disturbed linear systems described by the following equation

$$(S)\begin{cases} \dot{z}(t) = Az(t) + Bu(t) + f(t) \\ \\ z(0) = 0 \end{cases} \tag{1}$$

where A generates a strongly continuous semigroup (s.c.s.g) $(S(t))_{t\geq 0}$ on the space X, $B \in \mathcal{L}(U, X)$, U is the control space, $u \in L^2(0, +\infty; U)$ and $f \in L^2(0, +\infty; X)$ is a term of a known or unknown disturbance. X and U are two real Hilbert spaces.

The system (1) is augmented by the output equation

$$(E) \quad y(t) = Cz(t) \tag{2}$$

where $C \in \mathcal{L}(X, Y)$, Y being the observation space (a Hilbert space).
In the usual case where the system is excited by p actuators $(D_i, g_i)_{1 \le i \le p}$ and where the observation is given by means of q sensors $(\Omega_i, h_i)_{1 \le i \le q}$ we have [10, 11, 12] $U = \mathbb{R}^p$; $Bu(t) = \sum_{i=1}^{p} g_i u_i(t)$ and $Y = \mathbb{R}^q$; $Cz = (\langle h_1, z \rangle, \cdots, \langle h_q, z \rangle)^{tr}$.

The solution of (1) is given by

$$z_{u,f}(t) = S(t)z_0 + \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)f(s)ds$$

and the corresponding observation is given by

$$y_{u,f}(t) = CS(t)z_0 + \int_0^t CS(t-s)Bu(s)ds + \int_0^t CS(t-s)f(s)ds \tag{3}$$

Let us note that in the normal case where $f = 0$ and $u = 0$, the observation is given by:

$$y_{0,0}(t) = CS(t)z_0 \tag{4}$$

But if the system is disturbed by a term f, the observation becomes

$$y_{0,f}(t) = CS(t)z_0 + \int_0^t CS(t-s)f(s)ds \tag{5}$$

Generally, $y_{0,f}(.) \neq CS(.)z_0$.
Then we introduce a control term Bu in order to reduce asymptotically the effect of this disturbance by bringing asymptotically the observation in a given zone of tolerance $\mathcal{C}$, i.e. such that

For any $f \in L^2(0, +\infty; X)$, there exists $u \in L^2(0, +\infty; U)$ satisfying:

$$y_{u,f}^{\infty} = \int_0^{+\infty} CS(t)Bu(t)dt + \int_0^{+\infty} CS(t)f(t)dt \in \mathcal{C} \tag{6}$$

where $\mathcal{C}$ is a convex, closed and nonempty subset of Y. This notion is called enlarged asymptotic remediability. Actuators ensuring the enlarged asymptotic remediability of any disturbance $f \in L^2(0,+\infty;X)$ are said to be $\mathcal{C}$-efficient actuators.

In the particular case where $\mathcal{C}$ is a closed boule, $B(0,\epsilon) \subset Y$, centered in 0 and with a radius $\epsilon > 0$, these actuators will be said ϵ-efficient actuators. If the operators

$$\begin{array}{rcl} H^\infty : L^2(0,+\infty;U) & \longrightarrow & X \\ u & \longrightarrow & H^\infty u = \displaystyle\int_0^{+\infty} S(t)Bu(t)dt \end{array} \tag{7}$$

and

$$\begin{array}{rcl} \overline{H}^\infty : L^2(0,+\infty;X) & \longrightarrow & X \\ f & \longrightarrow & \overline{H}^\infty f = \displaystyle\int_0^{+\infty} S(t)f(t)dt \end{array} \tag{8}$$

are well defined, then (6) is equivalent to

$$CH^\infty u + C\overline{H}^\infty f \in \mathcal{C} \tag{9}$$

Let us note that if

$$\exists\, k \in L^2(0,+\infty;\mathbb{R}^+) \text{ such that } \|S(t)\| \leq k(t);\ \forall t \geq 0 \tag{10}$$

then H^∞ and $\overline{H}^\infty$ are well defined.

Remark 1.1

1- *If* $(S(t))_{t\geq 0}$ *is exponentially stable, i.e. if*

$$\exists \beta > 0 \text{ and } \exists \alpha > 0 \text{ such that } \|S(t)\| \leq \beta e^{-\alpha t}\ ;\ \ \forall t \geq 0 \tag{11}$$

then (10) is satisfied with $k(t) = \beta e^{-\alpha t} \in L^2(0,+\infty;\mathbb{R}^+)$, *consequently* H^∞ *and* $\overline{H}^\infty$ *(and hence* CH^∞ *and* $C\overline{H}^\infty$*) are well defined.*

2- *In fact, the stability is not necessary and we need a weaker hypothesis than (10). Indeed, we consider the operators* K_C^∞ *and* R_C^∞ *defined by*

$$K_C^\infty u = \int_0^{+\infty} CS(t)Bu(t)dt \tag{12}$$

and

$$R_C^\infty f = \int_0^{+\infty} CS(t)f(t)dt \tag{13}$$

We assume that

$$\exists\ k \in L^2(0, +\infty; \mathbb{R}^+) \text{ such that } \|CS(t)\| \leq k(t)\ ;\ \forall t \geq 0 \tag{14}$$

In this case, K_C^∞ *and* R_C^∞ *are well defined and (9) becomes*

$$K_C^\infty u + R_C^\infty f \in \mathcal{C} \tag{15}$$

3- *Let us note that* K_C^∞ *and* R_C^∞ *may be well defined even if* H^∞ *and* $\overline{H}^\infty$ *are not well defined.*

2 Notion of asymptotic compensation

In this part, we recall the principal notions of exact and weak asymptotic remediability and asymptotic efficient actuators and their main characterization results. We consider the system described by (1) augmented by the output equation (2).

2.1 Definitions and characterizations

Definition 2.1

i) *(S)+(E) is exactly remediable asymptotically, if for every* $f \in L^2(0, +\infty; X)$, *there exists a control* $u \in L^2(0, +\infty; U)$ *such that*

$$K_C^\infty u + R_C^\infty f = 0 \tag{16}$$

ii) *(S)+(E) is weakly remediable asymptotically, if for every* $f \in L^2(0, +\infty; X)$ *and every* $\epsilon > 0$, *there exists* $u \in L^2(0, +\infty; U)$ *such that*

$$\|K_C^\infty u + R_C^\infty f\| < \epsilon \tag{17}$$

Let B^*, $(R_C^\infty)^*$, $S^*(\cdot)$ and C^* be the adjoint operators of B, R_C^∞, $S(\cdot)$ and C respectively and let X', U' and Y' be the dual spaces of X, U and Y.

The operator $(R_C^\infty)^*$ is defined by

$$\begin{array}{cccc} (R_C^\infty)^* : & Y' & \longrightarrow & L^2(0,+\infty;X') \\ & \theta & \longrightarrow & (R_C^\infty)^*\theta = S^*(\cdot)C^*\theta \end{array} \tag{18}$$

Under hypothesis (14), we have the following general characterization results:

Proposition 2.2

i) *$(S)+(E)$ is exactly remediable asymptotically if and only if*

$$Im(R_C^\infty) = Im(K_C^\infty) \tag{19}$$

this is equivalent to

ii) $\exists \gamma > 0$ *such that*

$$\|S^*(\cdot)C^*\theta\|_{L^2(0,+\infty;X')} \leq \gamma \, \|B^*S^*(\cdot)C^*\theta\|_{L^2(0,+\infty;U')}$$

Proposition 2.3

i) *$(S)+(E)$ is weakly remediable asymptotically if and only if*

$$Im(R_C^\infty) \subset \overline{Im(K_C^\infty)}$$

or equivalently

ii) $Ker(B^*(R_C^\infty)^*) = Ker((R_C^\infty)^*)$.

2.2 Notion of efficient actuators

Definition 2.4 *Actuators $(\Omega_i, g_i)_{1\leq i\leq p}$ are said to be asymptotically efficient, or just efficient, if the corresponding system $(S)+(E)$ is weakly remediable asymptotically.*

Let us note that in the case of a system (S) with a dynamics A defined as follows:

$$Az = \sum_{n\geq 1} \lambda_n \sum_{j=1}^{r_n} \langle z, \varphi_{nj} \rangle \, \varphi_{nj} \tag{20}$$

where $\lambda_1, \lambda_2, \cdots$ are real parameters such that $\lambda_1 > \lambda_2 > \cdots$; $\{\varphi_{nj};\ n \geq 1,\ j = 1, \cdots, r_n\}$ is an orthonormal basis of X, r_n is the multiplicity of the eigenvalue λ_n, A generates a s.c.s.g. $(S(t))_{t\geq 0}$ defined by

$$S(t)z = \sum_{n\geq 1} e^{\lambda_n t} \sum_{j=1}^{r_n} \langle z, \varphi_{nj} \rangle \varphi_{nj}$$

We have the following characterization:

Proposition 2.5
Actuators $(\Omega_i, g_i)_{1\leq i\leq p}$ *are asymptotically efficient if and only if*

$$Ker(C^*) = \bigcap_{n\geq 1} Ker(M_n f_n) \tag{21}$$

where $M_n = (\langle g_i, \varphi_{nj} \rangle)_{\,1\leq i\leq p\,;\,1\leq j\leq r_n}$ *and*

$$\begin{array}{rcccl} f_n & : & Y' & \longrightarrow & \mathbb{R}^{r_n} \\ & & \theta & \longrightarrow & f_n(\theta) = (\langle C^*\theta, \varphi_{n1} \rangle, \cdots, \langle C^*\theta, \varphi_{nr_n} \rangle)^{tr} \end{array} \tag{22}$$

where generally M^{tr} *is the transposal matrix of* M.

In the case of sensors $(D_i, h_i)_{1\leq i\leq q}$ [10, 11, 12], the characterization becomes:

Proposition 2.6
Actuators $(\Omega_i, g_i)_{1\leq i\leq p}$ *are asymptotically efficient if and only if*

$$\bigcap_{n\geq 1} Ker(M_n G_n^{tr}) = \{0\} \tag{23}$$

where $G_n = (\langle h_i, \varphi_{nj} \rangle)_{\,1\leq i\leq p\,;\,1\leq j\leq r_n}$

Remark 2.7
1) *Let us note that in this case and using the analyticity property, the semi-norm*

$$\|\theta\|^2_{F_{T,0}} = \int_0^T \|B^* S^*(t) C^* \theta\|^2_U \, dt$$

is a norm on Y for any $T > 0$, and hence for $T \geq T_0$ (the notation $F_{T,0}$ will be specified later).
2) *In the case of q sensors, Y is a finite dimension space ($Y = \mathbb{R}^q$). Consequently, all the norms on Y are equivalent and the notions of weak and exact asymptotic remediability are equivalent. The weak and the strong convergence are also equivalent, and one talk only about convergence.*

2.3 Asymptotic compensation with minimal energy

For $f \in L^2(0, +\infty; X)$, does exists an optimal control $u \in L^2(0, +\infty; U)$ such that

$$K_C^\infty u + R_C^\infty f = 0$$

i.e. which minimizes the function $J(v) = \|v\|^2$ on the set

$$U_{ad} = \left\{ v \in L^2(0, +\infty; U) \ / \ K_C^\infty v + R_C^\infty f = 0 \right\}$$

This problem was resolved using an extension of the HUM approach. Indeed, for $\theta \in Y' \equiv Y$, we consider

$$\|\theta\|_{F_\infty} = \left(\int_0^{+\infty} \|B^* S^*(t) C^* \theta\|^2_{U'} \, dt \right)^{\frac{1}{2}} \tag{24}$$

where F_∞ is a space which will be specified.
$\|\cdot\|_{F_\infty}$ is a semi-norm on Y. If $Ker((R_C^\infty)^*) = \{0\}$, then $\|\cdot\|_{F_\infty}$ is a norm on Y if and only if $(S) + (E)$ is weakly remediable asymptotically.

Under this condition, we consider the space

$$F_\infty = \overline{Y}^{\|\cdot\|_{F_\infty}} \tag{25}$$

F_∞ is a Hilbert space with the inner product

$$\langle \theta, \sigma \rangle_{F_\infty} = \int_0^{+\infty} \langle B^* S^*(t) C^* \theta, B^* S^*(t) C^* \sigma \rangle \, dt \; ; \; \forall \, \theta, \; \sigma \in F_\infty \tag{26}$$

and the operator Λ_C^∞ defined on Y by

$$\Lambda_C^\infty \theta = \int_0^{+\infty} CS(t) BB^* S^*(t) C^* \theta dt = K_C^\infty (K_C^\infty)^* \theta \tag{27}$$

has a unique extension as an isomorphism from F_∞ to F'_∞ such that

$$\langle \Lambda_C^\infty \theta, \sigma \rangle_Y = \langle \theta, \sigma \rangle_{F_\infty} \; ; \; \forall \, \theta, \sigma \in F_\infty$$

and

$$\| \Lambda_C^\infty \theta \|_{F'_\infty} = \| \theta \|_{F_\infty} \; ; \; \forall \, \theta \in F_\infty$$

The following result shows how to find the optimal control ensuring the asymptotic compensation of a disturbance f.

Proposition 2.8 *If $R_C^\infty f \in F'_\infty$, then there exists a unique θ_f in F_∞ such that*

$$\Lambda_C^\infty \theta_f = -R_C^\infty f \tag{28}$$

and the control u_{θ_f} defined by

$$u_{\theta_f}(t) = B^* S^*(t) C^* \theta_f = (K_C^\infty)^* \theta_f \; ; \; t > 0 \tag{29}$$

verifies

$$K_C^\infty u_{\theta_f} + R_C^\infty f = 0$$

Moreover, u_{θ_f} is optimal with

$$\left\| u_{\theta_f} \right\|_{L^2(0,+\infty;U)} = \| \theta_f \|_{F_\infty} \tag{30}$$

3 Notion of enlarged asymptotic remediability

3.1 Enlarged asymptotic remediability

Under hypothesis (14), we have the following definition.

Definition 3.1 *A disturbance $f \in L^2(0, +\infty; X)$ is said to be $\mathcal{C}$-remediable asymptotically if there exists $u \in L^2(0, +\infty; U)$ such that*

$$K_C^\infty u + R_C^\infty f \in \mathcal{C}$$

where $\mathcal{C}$ is a convex, closed and non nonempty subset of Y, K_C^∞ and R_C^∞ are respectively defined by (12) and (13).

We have the following characterization.

Proposition 3.2 *The following properties are equivalent:*

i) *f is $\mathcal{C}$-remediable asymptotically*

ii) *$Im(K_C^\infty) \cap \mathcal{C}_1 \neq \emptyset$ where $\mathcal{C}_1 = \mathcal{C} - R_C^\infty f$*

Remark 3.3 *If $\mathcal{C} = \{0\}$, then we have a problem of exact asymptotic remediability.*

3.2 Case where $\mathcal{C}$ is the boule $B(0, \epsilon)$

Definition 3.4 *We say that f is ϵ-remediable asymptotically if there exists $u \in L^2(0, +\infty; U)$ such that*

$$\|K_C^\infty u + R_C^\infty f\| \leq \epsilon$$

We have the following result which is analogous to that established in the finite time case [3].

Proposition 3.5 *The following properties are equivalent:*

i) *f is ϵ-remediable asymptotically*
ii) *$Im(K_C^\infty) \cap B(R_C^\infty f, \epsilon) \neq \emptyset$*
iii) *$\left\| P_{Ker(B^*(R_C^\infty)^*)}(R_C^\infty f) \right\| < \epsilon$*

where P_F is the orthogonal projection on F, F is a closed subset of Y.

Remark 3.6
i) If f is ϵ-remediable asymptotically, then it's ϵ'-remediable asymptotically, for any $\epsilon' > \epsilon$. The converse is not true.
ii) The cost increases when ϵ decreases.

4 Enlarged asymptotic remediability with minimum energy

4.1 Problem statement

Let $\mathcal{C}$ be a convex, closed and nonempty subset of Y and $f \in L^2(0,+\infty;X)$. Under the weak asymptotic remediability hypothesis, we consider the following problem (P) of enlarged asymptotic compensation with minimum energy

$$(P)\left\{\begin{array}{c} \min J(u) \qquad \text{with } J(u) = \|u\|^2 \\ \text{subject to } K_C^{\infty} u + R_C^{\infty} f \in \mathcal{C} \end{array}\right. \tag{31}$$

We suppose that the disturbance f is $\mathcal{C}$-remediable asymptotically, the problem (P) is well defined and has a unique solution v^* in the set of admissible controls defined by

$$U_{ad} = \left\{ u \in L^2(0,+\infty;U) / K_C^{\infty} u + R_C^{\infty} f \in \mathcal{C} \right\}$$

The solution v^* of (P) is characterized by

$$J(v^*)(v - v^*) \geq 0 \ ; \ \forall v \in U_{ad}$$

i.e.

$$\langle v^*, v - v^* \rangle \geq 0 \ ; \ \forall v \in U_{ad}$$

Remark 4.1

1- *The problem (P) is a generalization of the following problem (P_0) of exact asymptotic compensation*

$$(P_0)\left\{\begin{array}{c} \min \|u\|^2 \\ \textit{subject to } K_C^{\infty} u + R_C^{\infty} f = 0 \end{array}\right. \tag{32}$$

It is sufficient to consider $\mathcal{C} = \{0\}$.

2- *If u^* is the solution of the problem (P_0), we have*

$$\|v^*\| \leq \|u^*\|$$

Hence the optimal cost corresponding to (P) is reduced with respect to that corresponding to (P_0).

3- *The problem (P) is also a generalization of the ϵ-remediability one, it sufficient to consider $\mathcal{C} = B(0, \epsilon)$.*

4- *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two nonempty, closed and convex subsets of Y such that $\mathcal{C}_1 \subset \mathcal{C}_2$, then the $\mathcal{C}_1$-remediability implies the $\mathcal{C}_2$-remediability, and the cost is decreasing when $\mathcal{C}$ is increasing.*

4.2 The main result

We give hereafter the main result where we show the existence and the unicity of the optimal control ensuring the $\mathcal{C}$-asymptotic compensation of a disturbance, and we give its characterization as well as the minimum cost.
We consider, without loss of generality, the case where the dynamics A of the system is given by (20) and the usual situation where the system is excited by efficient actuators and where the observation (measures) is given by a finite number of sensors. For the proof, the considered approach is not specific to the considered case and can be extended to other systems and situations.

Theorem 4.2
If the convex and closed subset $\mathcal{C}$ of Y and $f \in L^2(0, +\infty; X)$ are such that

$$\overset{\circ}{\mathcal{C}} \cap (R_C^\infty f + F'_\infty) \neq \emptyset$$

then

i) *there exists a unique θ_f in F_∞ such that*

$$\Lambda_C^\infty \theta_f + R_C^\infty f \in \mathcal{C} \tag{33}$$

and

$$\langle \theta_f, y - \Lambda_C^\infty \theta_f - R_C^\infty f \rangle \geq 0 \ ; \ \forall y \in \mathcal{C} \cap (R_C^\infty f + F'_\infty) \tag{34}$$

ii) *The control u_{θ_f} defined by*

$$u_{\theta_f}(t) = B^* S^*(t) C^* \theta_f \quad ; \forall t > 0 \tag{35}$$

is the unique solution of the problem (P). Moreover, u_{θ_f} is optimal with

$$\left\| u_{\theta_f} \right\|^2_{L^2(0,+\infty;U)} = \|\theta_f\|^2_{F_\infty}$$

4.3 Proof: Existence, unicity and characterization of the optimal control

The unicity is easy and is proved using an extension of HUM. But the proof of the existence is heavy and more complex, it is based on a combination of HUM and a penalization method. The problem (P) of asymptotic compensation is firstly approximated by a finite time problem (P_T) conveniently chosen. Then, for T sufficiently large, we consider an appropriate criterion depending on another parameter $\alpha > 0$. We establish intermediary and convergence results respectively when $\alpha \longrightarrow 0$ and $T \longrightarrow +\infty$, which lead to the solution of the initial problem (P).

First, let us give the following remark.

Remark 4.3 *The condition* $\overset{\circ}{\mathcal{C}} \cap (R_C^\infty f + F'_\infty) \neq \emptyset$ *implies that there exists a unique* θ *in* F_∞ *such that*

$$\Lambda_C^\infty \theta + R_C^\infty f \in \overset{\circ}{\mathcal{C}} \tag{36}$$

Since

$$\begin{aligned} \Lambda_C^\infty \theta &= \int_0^{+\infty} CS(t)BB^*S^*(t)C^*\theta dt \\ &= \int_0^{+\infty} CS(t)Bu_\theta(t)dt \\ &= K_C^\infty u_\theta(t) \end{aligned}$$

where $u_\theta(t) = B^*S^*(t)C^*\theta$, *then (36) becomes* $K_C^\infty u_\theta + R_C^\infty f \in \overset{\circ}{\mathcal{C}}$.

If we note $u_\theta^T = u_{\theta_{|[0,T]}}$, *i.e. the restriction of* u_θ *to* $[0,T]$ *and*

$$v_\theta^T = \begin{cases} u_\theta^T & on\,[0,T] \\ 0 & otherwise \end{cases}$$

then v_θ^T *converges to* u_θ *in* $L^2(0,+\infty;U)$.

Moreover, there exists $T_0 > 0$ *such that for every* $T \geq T_0$, *the control* $v_\theta^T \in L^2(0,+\infty;U)$ *verifies*

$$\int_0^T CS(t)Bv_\theta^T(t)dt + \int_0^T CS(t)f(t)dt \in \overset{\circ}{\mathcal{C}}$$

In other words, there exists $T_0 > 0$ such that the convex $\mathcal{C}$ is reached at any time $T \geq T_0$.

Unicity:

Let θ_f and $\sigma_f \in F'_\infty$ be such that for every $y \in \mathcal{C} \cap (R_C^\infty f + F'_\infty)$, we have:

$$\langle \theta_f, y - \Lambda_C^\infty \theta_f - R_C^\infty f \rangle \geq 0 \; and \; \langle \sigma_f, y - \Lambda_C^\infty \sigma_f - R_C^\infty f \rangle \geq 0$$

Then for $y = \Lambda_C^\infty \sigma_f + R_C^\infty f$ in the first inequality and $y = \Lambda_C^\infty \theta_f + R_C^\infty f$ in the second one, we obtain:

$$\langle \theta_f, \Lambda_C^\infty \sigma_f - \Lambda_C^\infty \theta_f \rangle \geq 0 \; and \; \langle \sigma_f, \Lambda_C^\infty \theta_f - \Lambda_C^\infty \sigma_f \rangle \geq 0$$

hence

$$\begin{aligned} & \langle \theta_f, \Lambda_C^\infty \sigma_f - \Lambda_C^\infty \theta_f \rangle + \langle \sigma_f, \Lambda_C^\infty \theta_f - \Lambda_C^\infty \sigma_f \rangle \geq 0 \\ \Longleftrightarrow & \langle \theta_f - \sigma_f, \Lambda_C^\infty \sigma_f - \Lambda_C^\infty \theta_f \rangle \geq 0 \\ \Longleftrightarrow & - \|\theta_f - \sigma_f\|_{F_\infty}^2 \geq 0 \\ \Longrightarrow & \|\theta_f - \sigma_f\|_{F_\infty} = 0 \\ \Longrightarrow & \theta_f = \sigma_f. \end{aligned}$$

Existence:

For $T \geq T_0$, we consider the following minimization problem

$$(P_T) \begin{cases} \min \|u\|^2 \\ \text{subject to } y_{u,f}^T \in \mathcal{C} \end{cases} \tag{37}$$

with

$$y_{u,f}^T = \int_0^T CS(t)Bu(t)dt + \int_0^T CS(t)f(t)dt \tag{38}$$

If we consider the operators K_C^T and R_C^T defined by

$$K_C^T u = \int_0^T CS(t)Bu(t)dt \text{ and } R_C^T f = \int_0^T CS(t)f(t)dt$$

then (38) becomes

$$y_{u,f}^T = K_C^T u + R_C^T f$$

For $\theta \in Y' \equiv Y$, let

$$\|\theta\|^2_{F_{T,\alpha}} = \int_0^T \|B^* S^*(t) C^* \theta\|^2_U \, dt + \frac{\alpha^2}{T} \|\theta\|^2_Y \tag{39}$$

$\|\cdot\|_{F_{T,\alpha}}$ is a norm; $\forall\, T > 0$ and $\forall\, \alpha > 0$.

Hence, in the general case, the choice of such a norm doesn't need the hypothesis of remediability of the system on $[0, T]$ and is adapted to the developed approach which is not specific to the considered case and can be extended to other systems and situations. Let us consider the space

$$F_{T,\alpha} = \overline{Y}^{\|\cdot\|_{F_{T,\alpha}}}$$

$F_{T,\alpha}$ is a Hilbert space with the inner product

$$\langle \theta, \sigma \rangle_{F_{T,\alpha}} = \int_0^T \langle B^* S^*(t) C^* \theta, B^* S^*(t) C^* \sigma \rangle_U \, dt + \frac{\alpha^2}{T} \langle \theta, \sigma \rangle_Y$$

and the operator Λ_C^T defined on Y by

$$\Lambda_C^T \theta = \int_0^T C S(t) B B^* S^*(t) C^* \theta \; dt + \frac{\alpha^2}{T} \theta$$

has a unique extension as an isomorphism from $F_{T,\alpha} \longrightarrow F'_{T,\alpha}$ such that

$$\left\langle \Lambda_C^T \theta, \sigma \right\rangle_Y = \langle \theta, \sigma \rangle_{F_{T,\alpha}} \;\; ; \; \forall\, \theta, \sigma \in F_{T,\alpha}$$

Remark 4.4

1- *For every $\alpha > 0$, we have $\|\cdot\|_{F_{T,\alpha}} \longrightarrow \|\cdot\|_{F_\infty}$ when $T \longrightarrow +\infty$.*

2- *For every $\theta \in Y$, we have $\Lambda_C^T \theta \;\longrightarrow\; \Lambda_C^\infty \theta$ when $T \longrightarrow +\infty$.*

3- *The norm (39) can be generalized to*

$$\|\theta\|^2_{F_{T,\alpha}} = \int_0^T \|B^* S^*(t) C^* \theta\|^2_U \, dt + \frac{\alpha^p}{T^q} \|\theta\|^2_Y \tag{40}$$

with $p \geq 2$ and $q \geq 1$.

4.3.1 Penalization criterion $J_{T,\alpha}$

For $T \geq T_0$, we consider the following criterion

$$J_{T,\alpha}(y,v) = \frac{1}{\alpha}\left\|y_{v,f}^T - y\right\|^2 + \|v\|^2 \tag{41}$$

with $y \in Y$ and $v \in L^2(0,T;U)$. We consider the following minimization problem

$$(P_{T,\alpha})\left\{\begin{array}{c} \min J_{T,\alpha}(y,v) \\ y \in \mathcal{C} \text{ and } v \in L^2(0,T;U) \end{array}\right. \tag{42}$$

Proposition 4.5 *The minimization problem $(P_{T,\alpha})$ with two variables admits a solution $(y_{T,\alpha}, v_{T,\alpha})$.*

Proof:
Let $(y_{T,\alpha}^{(k)}, v_{T,\alpha}^{(k)})_{k\geq 0}$ be a minimizing sequence.

$$J_{T,\alpha}(y_{T,\alpha}^{(k)}, v_{T,\alpha}^{(k)}) \searrow \inf_{(y,v)\in\mathcal{C}\times L^2(0,T;U)} J_{T,\alpha}(y,v)$$

when $k \longrightarrow +\infty$.
The sequence $(J_{T,\alpha}(y_{T,\alpha}^{(k)}, v_{T,\alpha}^{(k)}))_{k\geq 0}$ is bounded because it is convergent. Consequently, there exists $c_1 > 0$ such that $J_{T,\alpha}(y_{T,\alpha}^{(k)}, v_{T,\alpha}^{(k)}) \leq c_1;\ \forall k \geq 0$.
Since
$$\begin{aligned}\left\|v_{T,\alpha}^{(k)}\right\|^2 &= J_{T,\alpha}(y_{T,\alpha}^{(k)}, v_{T,\alpha}^{(k)}) - \frac{1}{\alpha}\left\|y_{v_{T,\alpha}^{(k)},f}^T - y_{T,\alpha}^{(k)}\right\|^2 \\ &\leq J_{T,\alpha}(y_{T,\alpha}^{(k)}, v_{T,\alpha}^{(k)})\end{aligned}$$

then $\left\|v_{T,\alpha}^{(k)}\right\|^2 \leq c_1;\ \forall k \geq 0$, i.e. $(v_{T,\alpha}^{(k)})_{k\geq 0}$ is bounded in $L^2(0,T;U)$.

The map $v \in L^2(0,T;U) \longrightarrow K_C^T v = \int_0^T CS(t)Bv(t)dt \in Y$ is linear and continuous, then there exists $c_2 > 0$ such that

$$\left\|K_C^T v\right\| = \left\|y_{v,f}^T - y_{0,f}^T\right\| \leq c_2\|v\| \ ;\ \forall v \in L^2(0,T;U)$$

Consequently, $(y_{v_{T,\alpha}^{(k)},f}^T)_{k\geq 0}$ is bounded in Y.

On the other hand

$$\begin{aligned}\left\|y_{v_{T,\alpha}^{(k)},f}^T - y_{T,\alpha}^{(k)}\right\|^2 &= \alpha J_{T,\alpha}(y_{T,\alpha}^{(k)}, v_{T,\alpha}^{(k)}) - \alpha\left\|v_{T,\alpha}^{(k)}\right\|^2 \\ &\leq \alpha J_{T,\alpha}(y_{T,\alpha}^{(k)}, v_{T,\alpha}^{(k)}) \leq \alpha c_1;\ \forall k \geq 0\end{aligned}$$

then $(y^{(k)}_{T,\alpha})_{k\geq 0}$ is bounded in Y.

Finally, the sequence $(y^{(k)}_{T,\alpha}, v^{(k)}_{T,\alpha})_{k\geq 0}$ is bounded in $Y \times L^2(0,T;U)$.

Hence, one can extract a convergent subsequence which converges to a limit $(y_{T,\alpha}, v_{T,\alpha})$.

$J_{T,\alpha}$ is continuous on $Y \times L^2(0,T;U)$, then

$$J_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha}) \leq \lim_k \inf J_{T,\alpha}(y^{(k)}_{T,\alpha}, v^{(k)}_{T,\alpha}) = \inf_{(y,v)\in\mathcal{C}\times L^2(0,T;U)} J_{T,\alpha}(y,v)$$

Since $\mathcal{C}$ is closed, $(y_{T,\alpha}, v_{T,\alpha}) \in \mathcal{C} \times L^2(0,T;U)$ and

$$\inf_{(y,v)\in\mathcal{C}\times L^2(0,T;U)} J_{T,\alpha}(y,v) \leq J_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha})$$

then $J_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha}) = \inf_{(y,v)\in\mathcal{C}\times L^2(0,T;U)} J_{T,\alpha}(y,v)$

i.e. $(y_{T,\alpha}, v_{T,\alpha})$ is a solution of $(P_{T,\alpha})$.

4.3.2 Characterization of the minimum of $\mathbf{J}_{T,\alpha}$

We have the following results.

Proposition 4.6 *Let $(y_{T,\alpha}, v_{T,\alpha})$ be a solution of $(P_{T,\alpha})$.*

$(y_{T,\alpha}, v_{T,\alpha})$ verifies the inequalities

$$\frac{1}{\alpha}\left\langle y_{T,\alpha} - y^T_{v_{T,\alpha},f}, y - y_{T,\alpha}\right\rangle \geq 0 \ ; \forall y \in \mathcal{C} \tag{43}$$

and

$$-\frac{1}{\alpha}\left\langle y_{T,\alpha} - y^T_{v_{T,\alpha},f}, y^T_{v,f} - y^T_{v_{T,\alpha},f}\right\rangle + \langle v_{T,\alpha}, v - v_{T,\alpha}\rangle \geq 0 \ ; \forall v \in L^2(0,T;U) \tag{44}$$

Proof:

$(y_{T,\alpha}, v_{T,\alpha})$ satisfies the following necessary condition

$$J'_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha})(y - y_{T,\alpha}, v - v_{T,\alpha}) \geq 0 \ ; \ \forall y \in \mathcal{C} \times L^2(0,T;U) \tag{45}$$

Since

$$J'_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha})(y,v) = \frac{2}{\alpha}\left\langle y^T_{v_{T,\alpha},f} - y, y^T_{v,f} - R^T_C f\right\rangle - \frac{2}{\alpha}\left\langle y^T_{v_{T,\alpha},f} - y_{T,\alpha}, y\right\rangle + 2\left\langle v_{T,\alpha}, v\right\rangle$$

then

$$J'_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha})(y - y_{T,\alpha}, v - v_{T,\alpha}) = \frac{2}{\alpha}\left\langle y^T_{v_{T,\alpha}} - y_{T,\alpha}, y^T_{v,f} - y^T_{v_{T,\alpha},f}\right\rangle - \frac{2}{\alpha}\left\langle y^T_{v_{T,\alpha},f} - y_{T,\alpha}, y - y_{T,\alpha}\right\rangle + 2\left\langle v_{T,\alpha}, v - v_{T,\alpha}\right\rangle$$

The necessary condition (45) becomes

$$\frac{1}{\alpha}\left\langle y^T_{v_{T,\alpha},f} - y_{T,\alpha}, y^T_{v,f} - y^T_{v_{T,\alpha},f}\right\rangle - \frac{1}{\alpha}\left\langle y^T_{v_{T,\alpha},f} - y_{T,\alpha}, y - y_{T,\alpha}\right\rangle + \left\langle v_{T,\alpha}, v - v_{T,\alpha}\right\rangle \geq 0 \tag{46}$$

By replacing $v = v_{T,\alpha}$ in (46), we obtain (43).
By replacing $y = y_{T,\alpha}$ in (46), we obtain (44).

Proposition 4.7 $(y_{T,\alpha}, v_{T,\alpha})$ *is characterized by*

$$\left\langle d_{T,\alpha}, y - y_{T,\alpha}\right\rangle \geq 0 \ ; \forall y \in \mathcal{C} \tag{47}$$

$$v_{T,\alpha}(t) = B^* p_{T,\alpha}(t) \tag{48}$$

where $d_{T,\alpha}$ *is given by*

$$d_{T,\alpha} = \frac{1}{\alpha}(y_{T,\alpha} - y^T_{v_{T,\alpha},f}) \tag{49}$$

and $p_{T,\alpha}(\cdot)$ *is defined by*

$$p_{T,\alpha}(t) = S^*(t) C^* d_{T,\alpha} \tag{50}$$

i.e. the solution of the adjoint equation

$$\begin{cases} \dot{p}_{T,\alpha}(t) = A^* p_{T,\alpha}(t) \\ p_{T,\alpha}(0) = C^* d_{T,\alpha} \end{cases} \tag{51}$$

Proof:
In inequality (43) of proposition 4.6, the element $d_{T,\alpha}$ given by (49) appears.

If we replace $v_{T,\alpha}(\cdot)$ by its value, the expression given by (44) becomes

$$-\frac{1}{\alpha}\left\langle y_{T,\alpha}-y^T_{v_{T,\alpha},f}, y^T_{v,f}-y^T_{v_{T,\alpha},f}\right\rangle+\int_0^T\langle B^*S^*(t)C^*d_{T,\alpha}, v(t)-v_{T,\alpha}(t)\rangle\,dt$$
$$=-\frac{1}{\alpha}\left\langle y_{T,\alpha}-y^T_{v_{T,\alpha},f}, y^T_{v,f}-y^T_{v_{T,\alpha},f}\right\rangle+\left\langle d_{T,\alpha},\int_0^T CS(t)B(v(t)-v_{T,\alpha}(t))dt\right\rangle$$
$$=-\frac{1}{\alpha}\left\langle y_{T,\alpha}-y^T_{v_{T,\alpha},f}, y^T_{v,f}-y^T_{v_{T,\alpha},f}\right\rangle+\left\langle d_{T,\alpha}, y^T_{v,f}-y^T_{v_{T,\alpha},f}\right\rangle = 0.$$

4.3.3 The convergence when $\alpha \longrightarrow 0$

We have the following result.

Proposition 4.8 *For every $T \geq T_0$ and when α decreases to 0:*

1- *The sequence $(y^T_{v_{T,\alpha},f}-y_{T,\alpha})$ converges strongly in Y to 0.*
2- *The sequence $J_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha})$ is increasing and bounded and then convergent.*
3- *The sequence $(y_{T,\alpha}, v_{T,\alpha}, d_{T,\alpha})$ is bounded in $Y \times L^2(0,T;U) \times F_{T,0}$.*

Proof:
1- For $T \geq T_0$ and using remark 4.3, there exists a control $\hat{u}_T$ in $L^2(0,T;U)$ such that $y^T_{\hat{u}_T,f} \in \mathcal{C}$. By noting $\hat{y}_T = y^T_{\hat{u}_T,f}$, we have $(\hat{y}_T, \hat{u}_T) \in \mathcal{C} \times L^2(0,T;U)$, and hence

$$J_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha}) \leq J_{T,\alpha}(\hat{y}_T, \hat{u}_T) = \|\hat{u}_T\|^2 \tag{52}$$

Then for every $\alpha > 0$, we have

$$\left\|y^T_{v_{T,\alpha},f}-y_{T,\alpha}\right\|^2 = \alpha J_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha}) - \alpha\|v_{T,\alpha}\|^2 \leq \alpha J_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha}) \leq \alpha\|\hat{u}_T\|^2$$

and hence $\lim\limits_{\alpha\longrightarrow 0}\left\|y^T_{v_{T,\alpha},f}-y_{T,\alpha}\right\| = 0$.

2- i) Using (52), $(J_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha}))_{\alpha>0}$ is bounded.
ii) If $\delta \leq \alpha$, we have

$$J_{T,\alpha}(y,v) \leq J_{T,\delta}(y,v)\ ;\ \forall (y,v) \in \mathcal{C} \times L^2(0,T;U)$$

consequently

$$J_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha}) \leq J_{T,\alpha}(y_{T,\delta}, v_{T,\delta}) \leq J_{T,\delta}(y_{T,\delta}, v_{T,\delta})$$

then $J_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha})$ increases when $\alpha \longrightarrow 0$.

3- i) $(v_{T,\alpha})_{\alpha>0}$ is bounded because

$$\|v_{T,\alpha}\|^2 \leq J_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha}) \leq J_{T,\alpha}(\hat{y}_T, \hat{u}_T) = \|\hat{u}_T\|^2$$

ii) The map $v \in L^2(0,T;U) \longrightarrow y^T_{v,f} \in Y$ is affine and continuous, then there exists $c_3 > 0$ such that

$$\left\|y^T_{v,f} - R^T_C f\right\| \leq c_3 \|v\| \ ; \ \forall v \in L^2(0,T;U)$$

then $(y^T_{v_{T,\alpha},f})_{\alpha>0}$ is bounded in Y because $(v_{T,\alpha})_{\alpha>0}$ is bounded.
This shows that $y_{T,\alpha} = (y_{T,\alpha} - y^T_{v_{T,\alpha},f}) + y^T_{v_{T,\alpha},f}$ is bounded in Y.

iii- We have

$$\begin{aligned} \|d_{T,\alpha}\|^2_{F_{T,\alpha}} &= \int_0^T \|B^* S^*(t) C^* d_{T,\alpha}\|^2_U \, dt + \frac{\alpha^2}{T} \|d_{T,\alpha}\|^2_Y \\ &= \int_0^T \|v_{T,\alpha}\|^2_U \, dt + \frac{\alpha^2}{T} \|d_{T,\alpha}\|^2_Y \\ &= \|v_{T,\alpha}\|^2_{L^2(0,T;U)} + \frac{1}{T} \|\alpha d_{T,\alpha}\|^2_Y \end{aligned}$$

Since $(\alpha d_{T,\alpha})_{\alpha>0}$ is convergent, then it is bounded and $(v_{T,\alpha})_{\alpha>0}$ is also bounded. Consequently, $(\|d_{T,\alpha}\|_{F_{T,\alpha}})_{\alpha>0}$ is bounded and hence $(d_{T,\alpha})_{\alpha>0}$ is bounded in $F_{T,0}$.

Proposition 4.9 *We consider a convergent subsequence of* $(y_{T,\alpha}, v_{T,\alpha}, d_{T,\alpha})_{\alpha>0}$*, with the same notation. Its limit* (y^*_T, v^*_T, d_T) *is characterized by*

1- $y^*_T = y^T_{v^*_T,f}$

2- v^*_T *is a solution of the problem* (P_T) *given by (37).*

3- *The sequence* $J_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha})$ *is increasing with a limit* $\|v^*_T\|^2$ *and* $(v_{T,\alpha})_{\alpha>0}$ *converges strongly to* v^*_T *in* $L^2(0,T;U)$.

4- *The control* v^*_T *is given by*

$$v_T^*(t) = B^* p_T(t) \tag{53}$$

where $p_T(t) = S^*(t)C^*d_T$, *i.e. the solution of the adjoint equation*

$$\begin{cases} \dot{p}_T(t) & = & A^* p_T(t) \\ p_T(0) & = & C^* d_T \end{cases} \tag{54}$$

5- *The element* d_T *is characterized by*

$$\left\langle d_T, y - y^T_{v_T^*,f} \right\rangle \geq 0 \ ; \forall y \in \mathcal{C} \cap (R_C^T f + F'_{T,0}) \tag{55}$$

Proof:

1- The sequence $(v_{T,\alpha})_{\alpha>0}$ converges weakly to v_T^* and the map $v \in L^2(0,T;U) \longrightarrow y^T_{v,f} - R_C^T f \in Y$ is linear and continuous, then $y^T_{v_{T,\alpha},f}$ converges to $y^T_{v_T^*,f}$ and hence

$$\begin{aligned} y_T^* &= \lim_{\alpha \longrightarrow 0} y_{T,\alpha} \\ &= \lim_{\alpha \longrightarrow 0} (y_{T,\alpha} - y^T_{v_{T,\alpha},f}) + \lim_{\alpha \longrightarrow 0} y^T_{v_{T,\alpha},f} \\ &= 0 + \lim_{\alpha \longrightarrow 0} y^T_{v_{T,\alpha},f} \\ &= y^T_{v_T^*,f} \end{aligned}$$

2- The set $\mathcal{C}$ is closed, then $y^T_{v_T^*,f} = \lim\limits_{\alpha \longrightarrow 0} y_{T,\alpha} \in \mathcal{C}$.
Moreover, if $v \in L^2(0,T;U)$ is such that $y^T_{v,f} \in \mathcal{C}$, then

$$\begin{aligned} \|v_{T,\alpha}\|^2 &= J_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha}) - \frac{1}{\alpha} \left\| y^T_{v_{T,\alpha},f} - y_{T,\alpha} \right\|^2 \\ &\leq J_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha}) \leq J_{T,\alpha}(y^T_{v,f}, v) = \|v\|^2 \end{aligned} \tag{56}$$

i.e. $\|v_{T,\alpha}\|^2 \leq \|v\|^2$, consequently

$$\|v_T^*\|^2 \leq \liminf_{\alpha} \|v_{T,\alpha}\|^2 \leq \|v\|^2$$

then (v_T^*) is a solution of (P_T).

3.i- We show in proposition 4.8 that the sequence $J_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha})$ is increasing when α is decreasing to 0.

ii- For any $v \in L^2(0,T;U)$ such that $y^T_{v,f} \in \mathcal{C}$, we have using (56),

$$\|v_{T,\alpha}\|^2 \leq J_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha}) \leq \|v\|^2$$

Particularly for $v = v^*_T$, we have

$$\|v_{T,\alpha}\|^2 \leq J_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha}) \leq \|v^*_T\|^2 \tag{57}$$

The weak convergence of $(v_{T,\alpha})_{\alpha>0}$ to v^*_T implies

$$\|v^*_T\|^2 \leq \liminf_{\alpha} \|v_{T,\alpha}\|^2 \leq \lim_{\alpha} J_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha}) \leq \|v^*_T\|^2$$

then

$$\lim_{\alpha \longrightarrow 0} J_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha}) = \|v^*_T\|^2$$

iii- Using inequality (57), we have

$$\|v^*_T\|^2 \leq \liminf_{\alpha} \|v_{T,\alpha}\|^2 \leq \limsup_{\alpha} \|v_{T,\alpha}\|^2 \leq \|v^*_T\|^2$$

and then $\|v_{T,\alpha}\|^2$ converges to $\|v^*_T\|^2$. Then, using the weak convergence of $(v_{T,\alpha})_{\alpha>0}$ to v^*_T, $(v_{T,\alpha})_{\alpha>0}$ converges strongly to v^*_T in $L^2(0,T;U)$.

4- Since $v_{T,\alpha}(\cdot) = B^*S^*(\cdot)C^*d_{T,\alpha}$, then for $w \in L^2(0,T;U)$, we have

$$\begin{aligned}
\langle v_{T,\alpha}, w\rangle &= \int_0^T \langle B^*S^*(t)C^*d_{T,\alpha}, w(t)\rangle \, dt \\
&= \left\langle d_{T,\alpha}, \int_0^T CS(t)Bw(t)dt \right\rangle \\
&= \left\langle d_{T,\alpha}, y^T_{w,f} - R^T_C f \right\rangle
\end{aligned}$$

Since $y^T_{w,f} - R^T_C f \in F'_{T,0}$, from the weak convergence of $(v_{T,\alpha})_{\alpha>0}$ to v^*_T and the convergence of $(d_{T,\alpha})_{\alpha>0}$ to d_T, we deduce that

$$\begin{aligned} \langle v_T^*, w\rangle &= \left\langle d_T, y_{w,f}^T - R_C^T f\right\rangle \\ &= \left\langle d_T, \int_0^T CS(t)Bw(t)dt\right\rangle \\ &= \int_0^T \langle B^* S^*(t) C^* d_T, w(t)\rangle\, dt \end{aligned}$$

and hence $v_T^*(\cdot) = B^* S^*(\cdot) C^* d_T$.

5- The inequality (47) can be written

$$\langle d_{T,\alpha}, y_{T,\alpha}\rangle \le \langle d_{T,\alpha}, y\rangle \ ; \ \forall y \in \mathcal{C}$$

i.e.

$$\left\langle d_{T,\alpha}, y_{T,\alpha} - R_C^T f\right\rangle \le \left\langle d_{T,\alpha}, y - R_C^T f\right\rangle \ ; \ \forall y \in \mathcal{C} \tag{58}$$

We have

$$\begin{aligned} J_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha}) &= \frac{1}{\alpha}\left\| y_{v_{T,\alpha},f}^T - y_{T,\alpha}\right\|^2 + \|v_{T,\alpha}\|^2 \\ &= \frac{1}{\alpha}\left\langle y_{v_{T,\alpha},f}^T - y_{T,\alpha}, y_{v_{T,\alpha},f}^T - y_{T,\alpha}\right\rangle + \|v_{T,\alpha}\|^2 \\ &= \left\langle d_{T,\alpha}, y_{T,\alpha} - y_{v_{T,\alpha},f}^T\right\rangle + \|v_{T,\alpha}\|^2 \\ &= \left\langle d_{T,\alpha}, y_{T,\alpha} - R_C^T f\right\rangle - \left\langle d_{T,\alpha}, y_{v_{T,\alpha},f}^T - R_C^T f\right\rangle + \|v_{T,\alpha}\|^2 \\ &= \left\langle d_{T,\alpha}, y_{T,\alpha} - R_C^T f\right\rangle - \left\langle d_{T,\alpha}, \int_0^T CS(t)Bv_{T,\alpha}(t)dt\right\rangle + \|v_{T,\alpha}\|^2 \\ &= \left\langle d_{T,\alpha}, y_{T,\alpha} - R_C^T f\right\rangle - \int_0^T \langle B^* S^*(t) C^* d_{T,\alpha}, v_{T,\alpha}(t)\rangle\ dt + \|v_{T,\alpha}\|^2 \\ &= \left\langle d_{T,\alpha}, y_{T,\alpha} - R_C^T f\right\rangle - \|v_{T,\alpha}\|^2 + \|v_{T,\alpha}\|^2 \\ &= \left\langle d_{T,\alpha}, y_{T,\alpha} - R_C^T f\right\rangle \end{aligned}$$

then

$$\left\langle d_{T,\alpha}, y_{T,\alpha} - R_C^T f \right\rangle = J_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha})$$

Consequently

$$\begin{aligned} \lim_{\alpha \longrightarrow 0} \left\langle d_{T,\alpha}, y_{T,\alpha} - R_C^T f \right\rangle &= \lim_{\alpha \longrightarrow 0} J_{T,\alpha}(y_{T,\alpha}, v_{T,\alpha}) \\ &= \|v_T^*\|^2 \\ &= \int_0^T \langle B^* S^*(t) C^* d_T, v_T^*(t) \rangle \ dt \\ &= \left\langle d_T, \int_0^T CS(t) B v_T^*(t) dt \right\rangle \\ &= \left\langle d_T, y_{v_T^*,f}^T - R_C^T f \right\rangle \end{aligned}$$

For $y - R_C^T f \in F'_{T,0}$ and when $\alpha \longrightarrow 0$ in (58), we obtain

$$\left\langle d_T, y_{v_T^*,f}^T - R_C^T f \right\rangle \leq \left\langle d_T, y - R_C^T f \right\rangle \ ; \ \forall y \in \mathcal{C} \cap (R_C^T f + F'_{T,0})$$

i.e.

$$\left\langle d_T, y - y_{v_T^*,f}^T \right\rangle \geq 0 \ ; \ \forall y \in \mathcal{C} \cap (R_C^T f + F'_{T,0})$$

4.3.4 The convergence when $T \longrightarrow +\infty$

First, let us show the following results.

Proposition 4.10 *The sequence $(y_T^*, v_T^*)_{T \geq T_0}$ is bounded in $Y \times L^2(0, +\infty; U)$.*

Proof:

i- Using remark 4.3, there exists $u \in L^2(0, +\infty; U)$ such that for $T \geq T_0$, we have $y_{u_T,f}^T \in \mathcal{C}$ with $u_T = u_{|[0,T]}$ and

$$v_T = \begin{cases} u_T & \text{on } [0, T] \\ 0 & \text{otherwise} \end{cases} \tag{59}$$

converges to u in $L^2(0,+\infty;U)$ when $T \longrightarrow +\infty$.

Then, since (v_T^*) is a solution of (P_T), we have

$$\|v_T^*\| \leq \|u_T\|_{L^2(0,T;U)} = \|v_T\|_{L^2(0,+\infty;U)} \tag{60}$$

and $(v_T)_T$ is convergent and then bounded. Consequently, $(v_T^*)_T$ is bounded. One can also remark that

$$\|u_T\|_{L^2(0,T;U)} = \|v_T\|_{L^2(0,+\infty;U)} \leq \|u\|_{L^2(0,+\infty;U)} \tag{61}$$

ii- We have $\left\|y_T^* - R_C^T f\right\| = \left\|K_C^T v_T^*\right\| = \|K_C^\infty w_T\|$, where w_T is given by

$$w_T = \begin{cases} v_T^* & \text{on } [0,T] \\ 0 & \text{otherwise} \end{cases} \tag{62}$$

Since the map $v \longrightarrow K_C^\infty v$ is linear and continuous, there exists $c > 0$ such that

$$\left\|y_T^* - R_C^T f\right\| \leq c\,\|w_T\| = c\,\|v_T^*\|$$

then $(y_T^*)_T$ is bounded because $(R_C^T f)_T$ is and using (60) and (61).

Proposition 4.11

1- *The sequence $(d_T)_{T\geq T_0}$ is bounded in $F_{T_0,0}$.*

2- *We consider a convergent subsequence of $(d_T)_{T\geq T_0}$ with the same notation. Its limit d is independent on T_0, i.e. constant and is in F_∞.*

Proof:

1- For $T \geq T_0$, we have

$$\|d_T\|^2_{F_{T_0,0}} \leq \|d_T\|^2_{F_{T,0}} = \int_0^T \|B^*S^*(t)C^*d_T\|^2\,dt = \|v_T^*\|^2_{L^2(0,T;U)}$$

Then $(d_T)_{T\geq T_0}$ is bounded in $F_{T_0,0}$ because $(v_T^*)_{T\geq T_0}$ is and using (60) and (61).

2- Let $(T_n)_{n\geq 0}$ be an increasing sequence such that $T_n \geq T_0$ and $T_n \in N$. The sequence $(d_{T_n})_{T_n\geq T_0}$ is bounded in $F_{T_1,0}$, hence, we can extract a subsequence $(d^1_{T_n})_{T_n\geq T_0}$ which converges in $F_{T_1,0}$ to a limit d_1.

The sequence $(d^1_{T_n})_{T_n \geq T_0}$ is also bounded in $F_{T_2,0}$, then, we can extract a subsequence $(d^2_{T_n})_{T_n \geq T_0}$ which converges in $F_{T_2,0}$ to a limit d_2.
Consequently, and from the unicity of the limit, we have $d_1 = d_2$.
We show by the same that at a step $k \geq 3$, there exists a subsequence $(d^k_{T_n})_{T_n \geq T_0}$ which converges in $F_{T_k,0}$ to a limit d_k with $d_1 = d_2 = ... = d_k = d$ and $d \in F_{T_k,0}$; $\forall k \geq 1$, i.e.

$$d \in \bigcap_{k \geq 1} F_{T_{k,0}} = F_\infty.$$

Now, we have the following convergence result.

Proposition 4.12 *We consider a convergent subsequence of (y^*_T, v^*_T) with the same notation. Its limit (y^*, v^*) is characterized by*

1- *The control v^* is given by*

$$v^*(.) = B^* p(.) \tag{63}$$

where $p(t) = S^*(t)C^*d$, i.e. the solution of the adjoint equation

$$\begin{cases} \dot{p}(t) & = & A^* p(t) \\ p(0) & = & C^* d \end{cases} \tag{64}$$

2- *v^* is the solution of the problem (P).*
3- *$y^* = y^\infty_{v^*,f}$*
4- *The element d is characterized by*

$$\left\langle d, y - y^\infty_{v^*,f} \right\rangle \geq 0 \ ; \forall y \in \mathcal{C} \cap (R^\infty_C f + F'_\infty) \tag{65}$$

Proof:
1- For $u \in L^2(0, +\infty; U)$ and w_T given by (62), we have

$$\begin{aligned}
\langle w_T, u\rangle_{L^2(0,+\infty;U)} &= \int_0^T \langle v_T^*(t), u(t)\rangle\, dt \\
&= \int_0^T \langle B^*S^*(t)C^*d_T, u(t)\rangle\, dt \\
&= \left\langle d_T, \int_0^T CS(t)Bu(t)dt\right\rangle \\
&= \left\langle d_T, \int_0^{+\infty} CS(t)Bu(t)dt\right\rangle - \left\langle d_T, \int_T^{+\infty} CS(t)Bu(t)dt\right\rangle \\
&= \left\langle d_T, y_{u,f}^{\infty} - R_C^{\infty} f\right\rangle - \left\langle d_T, \int_T^{+\infty} CS(t)Bu(t)dt\right\rangle
\end{aligned}$$

Since $y_{u,f}^{\infty} - R_C^{\infty} f \in F'_{\infty}$, the convergence of $(d_T)_T$ to d implies that $\left\langle d_T, y_{u,f}^{\infty} - R_C^{\infty} f\right\rangle$ converge to $\left\langle d, y_{u,f}^{\infty} - R_C^{\infty} f\right\rangle$.

Moreover $(d_T)_T$ is bounded and $\int_T^{+\infty} CS(t)Bu(t)dt \longrightarrow 0$ when $T \longrightarrow +\infty$, then $\left\langle d_T, \int_T^{+\infty} CS(t)Bu(t)dt\right\rangle \longrightarrow 0$ when $T \longrightarrow +\infty$.

Hence, the weak convergence of $(w_T)_T$ to v^* implies that

$$\begin{aligned}
\langle v^*, u\rangle &= \left\langle d, y_{u,f}^{\infty} - R_C^{\infty} f\right\rangle \\
&= \left\langle d, \int_0^{+\infty} CS(t)Bu(t)dt\right\rangle \\
&= \int_0^{+\infty} \langle B^*S^*(t)C^*d, u(t)\rangle\, dt
\end{aligned}$$

Consequently $v^*(\cdot) = B^*S^*(\cdot)C^*d$ in $L^2(0,+\infty;U)$.

2- We have $y^* = \lim_{T\longrightarrow+\infty} y_T^* \in \mathcal{C}$ because $\mathcal{C}$ is closed.
Moreover, for every $u \in L^2(0,+\infty;U)$ satisfying $y_{u,f}^{\infty} \in \mathcal{C}$, there exists T_0 such that for $T \geq T_0$, we have $y_{u_T,f}^T \in \mathcal{C}$ with $u_T = u_{|[0,T]}$ and v_T, given by (59), converges to u in $L^2(0,+\infty;U)$ when $T \longrightarrow +\infty$. Then

$$\|v_T^*\| \leq \|u_T\|_{L^2(0,T;U)} = \|v_T\|_{L^2(0,+\infty;U)}$$

and hence

$$\|v^*\| \leq \liminf_T \|v_T^*\| \leq \|u\|$$

consequently v^* is the solution of (P).

3- We have

$$\begin{aligned} y_T^* = y_{v_T^*,f}^T &= K_C^T v_T^* + R_C^T f \\ &= K_C^\infty w_T + R_C^\infty f - \int_T^{+\infty} CS(t)f(t)dt \\ &= y_{w_T,f}^\infty - \int_T^{+\infty} CS(t)f(t)dt \end{aligned}$$

where w_T is given by (62). The weak convergence of (w_T) to v^* implies the convergence of $y_{w_T,f}^\infty$ to $y_{v^*,f}^\infty$. On the other hand

$$\lim_{T \longrightarrow +\infty} \int_T^{+\infty} CS(t)f(t)dt = 0$$

then, $y^* = \lim_{T \longrightarrow +\infty} y_T^* = y_{v^*,f}^\infty$.

4- We have seen in (55) that

$$\left\langle d_T, y - y_{v_T^*,f}^T \right\rangle \geq 0 \ ; \forall y \in \mathcal{C} \cap (R_C^T f + F'_{T,0}) \ \ and\ T \geq T_0 \tag{66}$$

which can be written

$$\left\langle d_T, y_{v_T^*,f}^T - R_C^T f \right\rangle \leq \left\langle d_T, y - R_C^T f \right\rangle \ ; \ \forall y \in \mathcal{C} \cap (R_C^T f + F'_{T,0}) \ \ and\ T \geq T_0 \tag{67}$$

Since $(d_T)_T$ converges to d and $y_{v_T^*,f}^T - R_C^T f$ converges to $y_{v^*,f}^\infty - R_C^\infty f$, then $\left\langle d_T, y_{v_T^*,f}^T - R_C^T f \right\rangle$ converges to $\left\langle d, y_{v^*,f}^\infty - R_C^\infty f \right\rangle$.

On the other hand

$$\left\langle d_T, y - R_C^T f \right\rangle = \left\langle d_T, y - R_C^\infty f \right\rangle + \left\langle d_T, \int_T^{+\infty} CS(t)f(t)dt \right\rangle$$

Then for $y - R_C^\infty f \in F'_\infty$ and when $T \longrightarrow +\infty$, we have

$$\lim_{T \longrightarrow +\infty} \left\langle d_T, y - R_C^T f \right\rangle = \langle d, y - R_C^\infty f \rangle$$

Finally, when $T \longrightarrow +\infty$ in (67) and for $y - R_C^\infty f \in F'_\infty$, we have

$$\left\langle d, y_{v^\star,f}^\infty - R_C^\infty f \right\rangle \leq \langle d, y - R_C^\infty f \rangle \ ; \ \forall y \in \mathcal{C} \cap (R_C^\infty f + F'_\infty)$$

i.e.

$$\left\langle d, y - y_{v^\star,f}^\infty \right\rangle \geq 0 \ ; \ \ \forall y \in \mathcal{C} \cap (R_C^\infty f + F'_\infty)$$

In fact, in the considered case, this is not necessary and one can consider directly the limit in (66) when $T \longrightarrow +\infty$.

Conclusion

In this work, we defined and characterized the notion of enlarged asymptotic remediability which is a generalization of the exact asymptotic one. Using an appropriate combination of an extension of the HUM approach with a convenient penalization method, we showed the existence and the unicity of the optimal control ensuring enlarged asymptotic remediability and also how to find it. It is equally established that in the enlarged case, the cost is reduced with respect to the problem of exact asymptotic remediability.

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