Applied Mathematical Sciences, Vol. 2, 2008, no. 44, 2173 - 2186

Common Fixed Point Theorems with Applications in Dynamic Programming

Zeqing Liu

Department of Mathematics, Liaoning Normal University
P. O. Box 200, Dalian, Liaoning 116029, P.R. China
zeqingliu@sina.com.cn

Min Liu

Department of Mathematics, Liaoning Normal University
P. O. Box 200, Dalian, Liaoning 116029, P.R. China
min_liu@yeah.net

Chahn Yong Jung

Department of Business Administration
Gyeongsang National University
Jinju 660-701, Korea
bb5734@nongae.gsnu.ac.kr

Shin Min Kang*

Department of Mathematics
The Research Institute of Natural Science
Gyeongsang National University
Jinju 660-701, Korea
smkang@nongae.gsnu.ac.kr

Abstract

In this paper, the concepts of compatible mappings of type (A) and type (P) are introduced in an induced metric space, two common fixed point theorems for two pairs of compatible mappings of type (A) and type (P) in an induced complete metric space are established, and the

*Corresponding author

existence and uniqueness results of common solution for a system of functional equations arising in dynamic programming as applications of these common fixed point theorems presented are discussed.

Mathematics Subject Classification: 54H25, 49L20, 49L99

Keywords: Common fixed point, compatible mappings of type (A), compatible mappings of type (P), countable family of pseudometrics, induced metric space, system of functional equations, common solution

1 Introduction and Preliminaries

Contractive type mappings and corresponding fixed or common fixed point theorems and their applications have been studied and discussed during the past decades, see [1-17] and the references therein. In particular, Huang, Lee and Kang [4] established a common fixed point theorems for two pairs of compatible mappings A, S and B, T in a complete metric space (X, d) which satisfy the following condition

$$d(Ax, By) \leq \phi\Big(\max\Big\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{1}{2}[d(Sx, By) + d(Ty, Ax)]\Big\}\Big), \quad \forall x, y \in X, \tag{1.1}$$

where $\phi : [0, \infty) \to [0, \infty)$ is nondecreasing, upper semi-continuous and $\phi(t) < t$ for all $t > 0$. Moreover, Pathak, Cho, Kang and Lee [17] produced several common fixed point theorems for two pairs of compatible mappings of type (P) satisfying (1.1).

As proposed in Bellman and Lee [1], the essential form of the functional equation in dynamic programming is

$$f(x) = \operatorname*{opt}_{y \in D} H(x, y, f(T(x, y))), \quad \forall x \in S, \tag{1.2}$$

where x and y denote the state and decision vectors, respectively. T denotes the transformation of the process, $f(x)$ denotes the optimal return function with the initial state x, and the opt represents sup or inf.

Bhakta and Mitra [3], Huang, Lee and Kang [4], Liu [5-7], Liu, Agarwal and Kang [9], Liu and Kang [10-11], Liu and Kim [12], Liu and Ume [13], Liu, Ume and Kang [14], Liu, Xu, Ume and Kang [15], Pathak and Fisher [16], Pathak, Cho, Kang and Lee [17] and others established the existence and uniqueness of solutions or common solutions for several classes of functional equations or systems of functional equations arising in dynamic programming by means of various fixed and common fixed point theorems. Bhakta and

Choudhury [2] obtained two fixed point theorems by using countable family of pseudometrics and discussed also the existence of solutions for the following functional equation:

$$f(x) = \inf_{y \in D} G(x, y, f), \quad \forall x \in S. \tag{1.3}$$

Aroused and motivated by the above achievements in [1-17], we introduce the following contractive type mappings and system of functional equations arising in dynamic programming, respectively,

$$\begin{aligned} & d_k^{s+t}(fx, gy) \qquad (1.4)\\ \le \varphi\Big(& \max\Big\{\{p^s q^t : p, q \in \{d_k(lx, hy), d_k(fx, lx), d_k(gy, hy), \\ & d_k(fx, gy)\}\} \cup \Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(lx, hy), \\ & d_k(fx, lx), d_k(gy, hy), d_k(fx, gy)\}, \\ & q \in \{d_k(fx, hy), d_k(gy, lx)\}\Big\}\Big\}\Big), \quad \forall x, y \in X,\ k \ge 1, \end{aligned}$$

where s and t are some nonnegative numbers with $s + t > 0$ and $\varphi \in \Phi$, where $\Phi = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous nondecreasing function satisfying $\varphi(t) < t, \quad \forall t > 0\}$, and

$$\begin{aligned} f_1(x) &= \mathop{\mathrm{opt}}_{y \in D}\{u(x, y) + H_1(x, y, f_1(T(x, y)))\}, \quad \forall x \in S, \qquad (1.5)\\ f_2(x) &= \mathop{\mathrm{opt}}_{y \in D}\{u(x, y) + H_2(x, y, f_2(T(x, y)))\}, \quad \forall x \in S, \\ g_1(x) &= \mathop{\mathrm{opt}}_{y \in D}\{u(x, y) + G_1(x, y, g_1(T(x, y)))\}, \quad \forall x \in S, \\ g_2(x) &= \mathop{\mathrm{opt}}_{y \in D}\{u(x, y) + G_2(x, y, g_2(T(x, y)))\}, \quad \forall x \in S. \end{aligned}$$

The main aim in this paper is to study the existence and uniqueness of common fixed point for the contractive type mappings (1.4) in an induced complete metric space, which is generated by a countable family of pseudometrics $\{d_k\}_{k\ge1}$. Under certain conditions, we prove two common fixed point theorems for the contractive type mappings (1.4). As applications, we use the common fixed point theorems presented to establish the existence and uniqueness results of common solutions for the system of functional equations (1.5).

Throughout this paper, let $\mathbb{R}^+ = [0, +\infty)$ and $\{d_k\}_{k\ge1}$ be a countable family of pseudometrics on a nonempty set X such that for any distinct $x, y \in X$, $d_k(x, y) \neq 0$ for some $k \ge 1$. Define

$$d(x, y) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(x, y)}{1 + d_k(x, y)}, \quad \forall x, y \in X.$$

It is clear that d is a metric on X and the metric d and the metric space (X, d) are called, respectively, *an induced metric and induced metric space by the countable pseudometircs family* $\{d_k\}_{k\geq 1}$, respectively. A sequence $\{x_n\}_{n\geq 1} \subseteq X$ converges to a point $x \in X$ if and only if $d_k(x_n, x) \to 0$ as $n \to \infty$ for each $k \geq 1$, and $\{x_n\}_{n\geq 1} \subseteq X$ is a Cauchy sequence if and only if $d_k(x_n, x_m) \to 0$ as $n, m \to \infty$ for each $k \geq 1$. The induced metric space (X, d) is called *complete* if each Cauchy sequence in X converges to some point in X. A self mapping f on (X, d) is said to be *continuous* in X if $\lim_{n\to\infty} fx_n = fx$ whenever $\{x_n\}_{n\geq 1} \subseteq X$ such that $\{x_n\}_{n\geq 1}$ converges to $x \in X$.

2 Unique Common Fixed Point Theorems

Throughout this section, we assume that (X, d) is the induced metric space by the countable pseudometrics family $\{d_k\}_{k\geq 1}$ such that for any distinct $x, y \in X, d_k(x, y) \neq 0$ for some $k \geq 1$.

Firstly, we introduce the concepts of compatible mappings of type (A) and type (P) in the induced metric space (X, d).

Definition 2.1. The mappings $g, h : (X, d) \to (X, d)$ are said to be *compatible of type* (A) if

$$\lim_{n\to\infty} d_k(ghx_n, hhx_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d_k(hgx_n, ggx_n) = 0, \quad \forall k \geq 1,$$

whenever $\{x_n\}_{n\geq 1}$ is a sequence in X such that $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} hx_n = z$ for some $z \in X$.

Definition 2.2. The mappings $g, h : (X, d) \to (X, d)$ are said to be *compatible of type* (P) if

$$\lim_{n\to\infty} d_k(ggx_n, hhx_n) = 0, \quad \forall k \geq 1,$$

whenever $\{x_n\}_{n\geq 1}$ is a sequence in X such that $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} hx_n = z$ for some $z \in X$.

Secondly, we establish two common fixed point theorems for contractive type mappings (1.4) in the induced complete metric spaces (X, d).

Theorem 2.3. *Let the induced metric space* (X, d) *be complete and* f, g, h *and* l *be four self mappings of* (X, d) *such that*

$(B1)$ *one of* f, g, h *and* l *is continuous;*

$(B2)$ *the pairs* f, l *and* g, h *are compatible of type* (A);

$(B3)$ $f(X) \subseteq h(X), g(X) \subseteq l(X)$;

$(B4)$ *there exist some* $\varphi \in \Phi, s, t \in \mathbb{R}^+$ *with* $s + t > 0$ *satisfying* (1.4),

then f, g, h *and* l *have a unique common fixed point in* X.

Proof. Let x_0 be an arbitrary point in X. In terms of $(B3)$ there exist two sequences $\{x_n\}_{n\geq 0} \subseteq X, \{y_n\}_{n\geq 1} \subseteq X$ such that $fx_{2n} = hx_{2n+1} = y_{2n+1}$, $gx_{2n+1} = lx_{2n+2} = y_{2n+2}$ for $n \geq 0$. Define $d_{kn} = d_k(y_n, y_{n+1})$ for $n \geq 1$ and $k \geq 1$. We firstly demonstrate that

$$d_{kn+1}^{s+t} \leq \varphi(d_{kn}^{s+t}), \quad \forall n \geq 1,\, k \geq 1. \tag{2.1}$$

In view of (1.4), we acquire that, $\forall k \geq 1$

$$\begin{aligned} & d_{k2n+1}^{s+t} \qquad (2.2) \\ &= d_k^{s+t}(fx_{2n}, gx_{2n+1}) \\ &\leq \varphi\Big(\max \big\{ \{p^s q^t : p, q \in \{d_k(lx_{2n}, hx_{2n+1}), d_k(fx_{2n}, lx_{2n}), \\ &\quad d_k(gx_{2n+1}, hx_{2n+1}), d_k(fx_{2n}, gx_{2n+1})\}\} \\ &\quad \cup \Big\{p^s \Big(\frac{q}{2}\Big)^t, p^t \Big(\frac{q}{2}\Big)^s : p \in \{d_k(lx_{2n}, hx_{2n+1}), \\ &\quad d_k(fx_{2n}, lx_{2n}), d_k(gx_{2n+1}, hx_{2n+1}), d_k(fx_{2n}, gx_{2n+1})\}, \\ &\quad q \in \{d_k(fx_{2n}, hx_{2n+1}), d_k(gx_{2n+1}, lx_{2n})\}\Big\}\big\}\Big) \\ &\leq \varphi\Big(\max \big\{ \{p^s q^t : p, q \in \{d_{k2n}, d_{k2n+1}\}\} \cup \Big\{p^s \Big(\frac{q}{2}\Big)^t, p^t \Big(\frac{q}{2}\Big)^s : \\ &\quad p \in \{d_{k2n}, d_{k2n+1}\}, q \in \{0, d_{k2n} + d_{k2n+1}\}\Big\}\big\}\Big) \\ &= \varphi\bigg(\max \bigg\{ d_{k2n}^{s+t}, d_{k2n+1}^{s+t}, d_{k2n}^s d_{k2n+1}^t, d_{k2n}^t d_{k2n+1}^s, \\ &\quad d_{k2n}^s \Big(\frac{d_{k2n} + d_{k2n+1}}{2}\Big)^t, d_{k2n}^t \Big(\frac{d_{k2n} + d_{k2n+1}}{2}\Big)^s, \\ &\quad d_{k2n+1}^s \Big(\frac{d_{k2n} + d_{k2n+1}}{2}\Big)^t, d_{k2n+1}^t \Big(\frac{d_{k2n} + d_{k2n+1}}{2}\Big)^s \bigg\}\bigg). \end{aligned}$$

Suppose that $d_{k2n+1} > d_{k2n}$ for some $n \geq 1$ and $k \geq 1$. It follows from (2.2) that $d_{k2n+1}^{s+t} \leq \varphi(d_{k2n+1}^{s+t}) < d_{k2n+1}^{s+t}$, which is a contradiction. Hence $d_{k2n+1} \leq d_{k2n}$ for any $n \geq 1$ and $k \geq 1$. Thus (2.2) means that $d_{k2n+1}^{s+t} \leq \varphi(d_{k2n}^{s+t})$ for any $n \geq 1$ and $k \geq 1$. Similarly we conclude that $d_{k2n}^{s+t} \leq \varphi(d_{k2n-1}^{s+t})$ for each $n \geq 1$ and $k \geq 1$. It follows that (2.1) holds.

Now we demonstrate that

$$\lim_{n\to\infty} d_{kn} = 0, \quad \forall k \geq 1. \tag{2.3}$$

In view of (2.1) and the continuity of φ, we deduce that

$$\begin{aligned} & 0 \leq d_{kn+1}^{s+t} \leq \varphi(d_{kn}^{s+t}) \leq d_{kn}^{s+t} \leq \varphi(d_{kn-1}^{s+t}) \\ & \leq d_{kn-1}^{s+t} \leq \cdots \leq \varphi(d_{k1}^{s+t}) \leq d_{k1}^{s+t}, \quad \forall n \geq 1,\, k \geq 1, \end{aligned}$$

which deduces that $\lim_{n\to\infty} d_{kn}^{s+t} = a$ for some $a \in \mathbb{R}^+$. Thus $a = \lim_{n\to\infty} \varphi(d_{kn}^{s+t}) = \varphi(a)$, therefore $a = 0$. It follows that (2.3) holds.

For the sake of showing that $\{y_n\}_{n\geq 1}$ is a Cauchy sequence, in view of (2.3), it is sufficient to show that $\{y_{2n}\}_{n\geq 1}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}_{n\geq 1}$ is not a Cauchy sequence. Hence there exists a positive number ε and $k \geq 1$ on condition that for each even integer $2r$, there are even integers $2m(r)$ and $2n(r)$ such that $2m(r) > 2n(r) > 2r$, and

$$d_k(y_{2m(r)}, y_{2n(r)}) > \varepsilon.$$

For each even integer $2r$, let $2m(r)$ be the least even integer exceeding $2n(r)$ satisfying the above inequality, therefore we obtain that

$$d_k(y_{2m(r)-2}, y_{2n(r)}) \leq \varepsilon \quad \text{and} \quad d_k(y_{2m(r)}, y_{2n(r)}) > \varepsilon. \tag{2.4}$$

It follows that for each even integer $2r$,

$$d_k(y_{2m(r)}, y_{2n(r)}) \leq d_{k2m(r)-1} + d_{k2m(r)-2} + d_k(y_{2m(r)-2}, y_{2n(r)}).$$

According to (2.3), (2.4) and the above inequality we conclude that

$$\lim_{r\to\infty} d_k(y_{2m(r)}, y_{2n(r)}) = \varepsilon. \tag{2.5}$$

It is apparent that for any $r \geq 1$,

$$|d_k(y_{2m(r)}, y_{2n(r)+1}) - d_k(y_{2m(r)}, y_{2n(r)})| \leq d_{k2n(r)},$$

$$|d_k(y_{2m(r)+1}, y_{2n(r)+1}) - d_k(y_{2m(r)}, y_{2n(r)+1})| \leq d_{k2m(r)}$$

and

$$|d_k(y_{2m(r)+1}, y_{2n(r)+2}) - d_k(y_{2m(r)+1}, y_{2n(r)+1})| \leq d_{k2n(r)+1}.$$

In light of (2.3), (2.5) and the above inequalities we draw that

$$\begin{aligned}\varepsilon = \lim_{r\to\infty} d_k(y_{2m(r)}, y_{2n(r)+1}) &= \lim_{r\to\infty} d_k(y_{2m(r)+1}, y_{2n(r)+1}) \\ &= \lim_{r\to\infty} d_k(y_{2m(r)+1}, y_{2n(r)+2}).\end{aligned}$$

By (1.4), we arrive at

$$\begin{aligned}&d_k^{s+t}(fx_{2m(r)}, gx_{2n(r)+1}) \\ &\leq \varphi\Big(\max\Big\{\{p^s q^t : p, q \in \{d_k(lx_{2m(r)}, hx_{2n(r)+1}), d_k(fx_{2m(r)}, lx_{2m(r)}), \\ &\qquad d_k(gx_{2n(r)+1}, hx_{2n(r)+1}), d_k(fx_{2m(r)}, gx_{2n(r)+1})\}\} \\ &\qquad \cup\Big\{p^s \Big(\frac{q}{2}\Big)^t, p^t \Big(\frac{q}{2}\Big)^s : p \in \{d_k(lx_{2m(r)}, hx_{2n(r)+1}),\end{aligned}$$

$$d_k(fx_{2m(r)}, lx_{2m(r)}), d_k(gx_{2n(r)+1}, hx_{2n(r)+1}), d_k(fx_{2m(r)}, gx_{2n(r)+1})\},$$
$$q \in \{d_k(fx_{2m(r)}, hx_{2n(r)+1}), d_k(gx_{2n(r)+1}, lx_{2m(r)})\}\}\}\Big).$$

Letting $r \to \infty$, we get that

$$\varepsilon^{s+t} \leq \varphi\Big(\max\Big\{\varepsilon^{s+t}, 0, \varepsilon^s\Big(\frac{\varepsilon}{2}\Big)^t, \varepsilon^t\Big(\frac{\varepsilon}{2}\Big)^s\Big\}\Big) = \varphi(\varepsilon^{s+t}) < \varepsilon^{s+t},$$

which is a contradiction. Thus $\{y_n\}_{n\geq 1}$ is a Cauchy sequence and it converges to a point $u \in X$ by completeness of X. Suppose that l is continuous. Consequently, we have that

$$\lim_{n\to\infty} llx_{2n} = \lim_{n\to\infty} ly_{2n} = lu.$$

In light of $(B2)$, we obtain that

$$\lim_{n\to\infty} fy_{2n} = \lim_{n\to\infty} flx_{2n} = lu.$$

We subsequently produce that u is a common fixed point of f, g, h and l. It follows from (1.4) that

$$d_k^{s+t}(fy_{2n}, gx_{2n+1})$$
$$\leq \varphi\Big(\max\Big\{\{p^s q^t : p, q \in \{d_k(ly_{2n}, hx_{2n+1}), d_k(fy_{2n}, ly_{2n}), d_k(gx_{2n+1}, hx_{2n+1}),$$
$$d_k(fy_{2n}, gx_{2n+1})\}\} \cup \Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(ly_{2n}, hx_{2n+1}),$$
$$d_k(fy_{2n}, ly_{2n}), d_k(gx_{2n+1}, hx_{2n+1}), d_k(fy_{2n}, gx_{2n+1})\},$$
$$q \in \{d_k(fy_{2n}, hx_{2n+1}), d_k(gx_{2n+1}, ly_{2n})\}\Big\}\Big\}\Big), \quad \forall k \geq 1.$$

As $n \to \infty$ in above inequality, we infer that

$$d_k^{s+t}(lu, u) \leq \varphi(d_k^{s+t}(lu, u)) \leq d_k^{s+t}(lu, u), \quad \forall k \geq 1,$$

which signifies that $d_k(lu, u) = 0$ for any $k \geq 1$, that is, $lu = u$. By (1.4) we get that

$$d_k^{s+t}(fu, gx_{2n+1})$$
$$\leq \varphi\Big(\max\Big\{\{p^s q^t : p, q \in \{d_k(lu, hx_{2n+1}), d_k(fu, lu), d_k(gx_{2n+1}, hx_{2n+1}),$$
$$d_k(fu, gx_{2n+1})\}\} \cup \Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(lu, hx_{2n+1}),$$
$$d_k(fu, lu), d_k(gx_{2n+1}, hx_{2n+1}), d_k(fu, gx_{2n+1})\},$$
$$q \in \{d_k(fu, hx_{2n+1}), d_k(gx_{2n+1}, lu)\}\Big\}\Big\}\Big), \quad \forall k \geq 1.$$

As $n \to \infty$ in the above inequality, we gain that

$$d_k^{s+t}(fu,u) \le \varphi(d_k^{s+t}(fu,u)) \le d_k^{s+t}(fu,u), \quad \forall k \ge 1,$$

which implies that $d_k(fu,u) = 0$ for any $k \ge 1$. Therefore $fu = u$. In view of $(B3)$, there exists a point $w \in X$ such that $u = fu = hw$. Using (1.4), we conclude that

$$\begin{aligned} d_k^{s+t}(u,gw) &= d_k^{s+t}(fu,gw) \\ &\le \varphi\Big(\max\Big\{\{p^s q^t : p,q \in \{d_k(lu,hw), d_k(fu,lu), d_k(gw,hw), \\ &\qquad d_k(fu,gw)\}\} \cup \Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(lu,hw), \\ &\qquad d_k(fu,lu), d_k(gw,hw), d_k(fu,gw)\}, \\ &\qquad q \in \{d_k(fu,hw), d_k(gw,lu)\}\Big\}\Big\}\Big) \\ &= \varphi(d_k^{s+t}(u,gw)) \le d_k^{s+t}(u,gw), \quad \forall k \ge 1, \end{aligned}$$

which implies that $d_k(u,gw) = 0$ for each $k \ge 1$. Thereupon $u = gw$. By means of $(B2)$ and $gw = hw = u$, we come into $d_k(gu,hu) = d_k(ghw,hhw) = 0$ for any $k \ge 1$, which means that $gu = hu$. In a similar manner, by (1.4) we know that

$$\begin{aligned} d_k^{s+t}(fu,gu) &\le \varphi\Big(\max\Big\{\{p^s q^t : p,q \in \{d_k(lu,hu), d_k(fu,lu), d_k(gu,hu), \\ &\qquad d_k(fu,gu)\}\} \cup \Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(lu,hu), \\ &\qquad d_k(fu,lu), d_k(gu,hu), d_k(fu,gu)\}, \\ &\qquad q \in \{d_k(fu,hu), d_k(gu,lu)\}\Big\}\Big\}\Big), \quad \forall k \ge 1, \end{aligned}$$

which deduces that

$$d_k^{s+t}(fu,gu) \le \varphi(d_k^{s+t}(fu,gu)) \le d_k^{s+t}(fu,gu), \quad \forall k \ge 1,$$

which implies that $d_k(fu,gu) = 0$ for each $k \ge 1$. Therefore, $fu = gu$. Hence we come to a conclusion that $fu = gu = hu = lu = u$. Similarly, we can obtain this result in the case of the continuity of f or g or h.

We finally demonstrate that u is a unique common fixed point of f, g, h and l. If $v \in X \setminus \{u\}$ is another common fixed point of f, g, h and l, it is easy to see that from (1.4)

$$d_k^{s+t}(u,v) = d_k^{s+t}(fu,gv) \le \varphi(d_k^{s+t}(u,v)) \le d_k^{s+t}(u,v), \quad \forall k \ge 1,$$

which implies that $d_k(u,v) = 0$ for all $k \ge 1$. That is, $u = v$, which is a contradiction. This completes the proof. □

Theorem 2.4. *Let the induced metric space (X,d) be complete and f, g, h and l be four self mappings of (X,d) such that $(B1),(B3),(B4)$ and*

$(B5)$ the pairs f,l and g,h are compatible mappings of type (P),

then f, g, h and l have a unique common fixed point in X.

Proof. As in the proof of Theorem 2.3, we arrive at $\{y_n\}_{n\geq 1}$ is a Cauchy sequence and it converges to a point $u \in X$ by completeness of X. Suppose that l is continuous. It follows that

$$\lim_{n\to\infty} llx_{2n} = \lim_{n\to\infty} ly_{2n} = lu.$$

In view of $(B5)$, we obtain that

$$\lim_{n\to\infty} fy_{2n+1} = \lim_{n\to\infty} ffx_{2n} = lu.$$

The rest of the proof is the similar as that of Theorem 2.3. This completes the proof. □

Remark 2.5. The conditions $(B1)$ and $(B4)$ in Theorem 2.3 and 2.4 weaken and improve the corresponding conditions in Theorem 3.1 of Pathak, Cho, Kang and Lee [17].

3 Applications

In this section, we assume that $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_1)$ be both real Banach spaces, $S \subseteq X$ be the state space, and $D \subseteq Y$ be the decision space. Denote by $BB(S)$ the set of all real-value mappings on S that are bounded on bounded subsets of S. It is easy to verify that $BB(S)$ is a linear space over $\mathbb{R}$ under usual definitions of addition and multiplication by scalars. For $k \geq 1$ and $f, g \in BB(S)$, let

$$d_k(f,g) = \sup\{|f(x) - g(x)| : x \in \overline{B}(0,k)\},$$

$$d(f,g) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(f,g)}{1 + d_k(f,g)},$$

where $\overline{B}(0,k) = \{x : x \in S \text{ and } \|x\| \leq k\}$. Clearly, $\{d_k\}_{k\geq 1}$ is a countable family of pseudometrics on $BB(S)$ and $(BB(S),d)$ is a complete metric space.

Now we study the existence and uniqueness of common solutions for the systems of functional equations (1.5) in the deduced complete metric space $(BB(S),d)$.

Theorem 3.1. *Let* $u : S \times D \to S, T : S \times D \to S$ *and* $H_1, H_2, G_1, G_2 : S \times D \times \mathbb{R} \to \mathbb{R}$ *satisfy the following conditions:*

$(C1)$ *for given* $k \geq 1$ *and* $a \in BB(S)$, *there exists* $r(k, a) > 0$ *such that*

$$|u(x,y)| + \max\{|H_i(x,y,a(T(x,y)))|, |G_i(x,y,a(T(x,y)))| : i \in \{1,2\}\}$$

$$\leq r(k,a), \quad \forall (x,y) \in \overline{B}(0,k) \times D;$$

$(C2)$ *there exists* $\varphi \in \Phi, s, t \in \mathbb{R}^+$ *and* $s + t \geq 1$ *such that*

$$|H_1(x,y,a(w)) - H_2(x,y,b(w))|^{s+t} \tag{3.2}$$

$$\leq \varphi\Big(\max\Big\{\{p^s q^t : p, q \in \{d_k(J_1a, J_2b), d_k(A_1a, J_1a), d_k(A_2b, J_2b), d_k(A_1a, A_2b)\}\} \cup \Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(J_1a, J_2b), d_k(A_1a, J_1a), d_k(A_2b, J_2b), d_k(A_1a, A_2b)\}, q \in \{d_k(A_1a, J_2b), d_k(A_2b, J_1a)\}\Big\}\Big\}\Big), \quad \forall (x,y,w) \in \overline{B}(0,k) \times D \times S,\ k \geq 1,$$

where A_1, A_2, J_1 *and* J_2 *are defined as follows:*

$$A_i a(x) = \operatorname*{opt}_{y \in D}\{u(x,y) + H_i(x,y,a(T(x,y)))\},$$

$$J_i a(x) = \operatorname*{opt}_{y \in D}\{u(x,y) + G_i(x,y,a(T(x,y)))\},$$

for all $(x,a) \in S \times BB(S)$, $i \in \{1,2\}$; $(C3)$ *either there exists* $A_i \in \{A_1, A_2\}$ *such that for any sequence* $\{a_n\}_{n\geq 1} \subset BB(S), a \in BB(S)$ *and* $k \geq 1$,

$$\lim_{n\to\infty} \sup_{x \in \overline{B}(0,k)} |a_n(x) - a(x)| = 0 \Rightarrow \lim_{n\to\infty} \sup_{x \in \overline{B}(0,k)} |A_i a_n(x) - A_i a(x)| = 0$$

or there exists $J_i \in \{J_1, J_2\}$ *such that for any sequence* $\{a_n\}_{n\geq 1} \subset BB(S), a \in BB(S)$ *and* $k \geq 1$,

$$\lim_{n\to\infty} \sup_{x \in \overline{B}(0,k)} |a_n(x) - a(x)| = 0 \Rightarrow \lim_{n\to\infty} \sup_{x \in \overline{B}(0,k)} |J_i a_n(x) - J_i a(x)| = 0;$$

$(C4)$ $A_1(BB(S)) \subseteq J_2(BB(S))$ *and* $A_2(BB(S)) \subseteq J_1(BB(S))$;

$(C5)$ *for any* $k \geq 1$ *and* $i \in \{1,2\}$

$$\lim_{n\to\infty} \sup_{x \in \overline{B}(0,k)} |A_i J_i a_n(x) - J_i J_i a_n(x)|$$

$$= \lim_{n\to\infty} \sup_{x \in \overline{B}(0,k)} |J_i A_i a_n(x) - A_i A_i a_n(x)| = 0$$

whenever $\{a_n\}_{n\geq 1} \subset BB(S)$ *is a sequence in* $BB(S)$ *such that*

$$\lim_{n\to\infty} \sup_{x\in\overline{B}(0,k)} |A_i a_n(x) - a(x)| = \lim_{n\to\infty} \sup_{x\in\overline{B}(0,k)} |J_i a_n(x) - a(x)| = 0$$

for some $a \in BB(S)$,

then the system of functional equations (1.5) *possesses a unique common solution in* $BB(S)$.

Proof. It follows from $(C1)$ and $(C2)$ that A_1, A_2 J_1 and J_2 are self mappings in $BB(S)$. Clearly, $(C3)$ implies that one of A_1, A_2, J_1 and J_2 is continuous, and $(C5)$ means that A_1, J_1 and A_2, J_2 are compatible mappings of type (A).

Presume that $\text{opt}_{y\in D} = \inf_{y\in D}$. Let $a, b \in BB(S)$, $k \geq 1, x \in \overline{B}(0,k)$ and $\varepsilon > 0$. Using (3.2), we deduce that there exist $y, z \in D$ such that

$$A_1 a(x) > u(x,y) + H_1(x,y,a(T(x,y))) - \varepsilon, \tag{3.3}$$

$$A_2 b(x) > u(x,z) + H_2(x,z,b(T(x,z))) - \varepsilon. \tag{3.4}$$

It is easy to see that

$$A_1 a(x) \leq u(x,z) + H_1(x,z,a(T(x,z))), \tag{3.5}$$

$$A_2 b(x) \leq u(x,y) + H_2(x,y,b(T(x,y))). \tag{3.6}$$

By virtue of (3.4) and (3.5), we infer that

$$\begin{aligned} A_1 a(x) - A_2 b(x) &< H_1(x,z,a(T(x,z))) - H_2(x,z,b(T(x,z))) + \varepsilon \qquad (3.7) \\ &\leq |H_1(x,z,a(T(x,z))) - H_2(x,z,b(T(x,z)))| + \varepsilon. \end{aligned}$$

From (3.3) and (3.6) we conclude that

$$\begin{aligned} A_1 a(x) - A_2 b(x) &> H_1(x,y,a(T(x,y))) - H_2(x,y,b(T(x,y))) - \varepsilon \qquad (3.8) \\ &\geq -|H_1(x,y,a(T(x,y))) - H_2(x,y,b(T(x,y)))| - \varepsilon. \end{aligned}$$

It follows from (3.7) and (3.8) that

$$\begin{aligned} |A_1 a(x) - A_2 b(x)| \leq \max\{&|H_1(x,y,a(T(x,y))) - H_2(x,y,b(T(x,y)))|, \\ &|H_1(x,z,a(T(x,z))) - H_2(x,z,b(T(x,z)))|\} + \varepsilon. \end{aligned}$$

(3.1), the above inequality and the mean value theorem lead to

$$\begin{aligned} &|A_1 a(x) - A_2 b(x)|^{s+t} \\ &\leq (\max\{|H_1(x,y,a(T(x,y))) - H_2(x,y,b(T(x,y)))|, \\ &\quad |H_1(x,z,a(T(x,z))) - H_2(x,z,b(T(x,z)))|\} + \varepsilon)^{s+t} \end{aligned}$$

$$\begin{aligned}
&\leq (\max\{|H_1(x,y,a(T(x,y))) - H_2(x,y,b(T(x,y)))|,\\
&\quad |H_1(x,z,a(T(x,z))) - H_2(x,z,b(T(x,z)))|\})^{s+t}\\
&\quad +(s+t)(2r(k,a)+\varepsilon)^{s+t-1}\varepsilon\\
&\leq \varphi\Big(\max\Big\{\{p^s q^t : p,q \in \{d_k(J_1a,J_2b), d_k(A_1a,J_1a), d_k(A_2b,J_2b),\\
&\quad d_k(A_1a,A_2b)\}\} \cup \Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(J_1a,J_2b),\\
&\quad d_k(A_1a,J_1a), d_k(A_2b,J_2b), d_k(A_1a,A_2b)\},\\
&\quad q \in \{d_k(A_1a,J_2b), d_k(A_2b,J_1a)\}\Big\}\Big\}\Big)\\
&\quad +(s+t)(2r(k,a)+\varepsilon)^{s+t-1}\varepsilon,
\end{aligned}$$

which yields that

$$\begin{aligned}
&d_k^{s+t}(A_1a,A_2b) \qquad (3.9)\\
&\leq \varphi\Big(\max\Big\{\{p^s q^t : p,q \in \{d_k(J_1a,J_2b), d_k(A_1a,J_1a), d_k(A_2b,J_2b),\\
&\quad d_k(A_1a,A_2b)\}\} \cup \Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(J_1a,J_2b),\\
&\quad d_k(A_1a,J_1a), d_k(A_2b,J_2b), d_k(A_1a,A_2b)\},\\
&\quad q \in \{d_k(A_1a,J_2b), d_k(A_2b,J_1a)\}\Big\}\Big\}\Big)\\
&\quad +(s+t)(2r(k,a)+\varepsilon)^{s+t-1}\varepsilon.
\end{aligned}$$

Similarly we infer that (3.9) also holds for $\mathrm{opt}_{y\in D} = \sup_{y\in D}$. Letting $\varepsilon \to 0$ in (3.9), we deduce that

$$\begin{aligned}
&d_k^{s+t}(A_1a,A_2b)\\
&\leq \varphi\Big(\max\Big\{\{p^s q^t : p,q \in \{d_k(J_1a,J_2b), d_k(A_1a,J_1a), d_k(A_2b,J_2b),\\
&\quad d_k(A_1a,A_2b)\}\} \cup \Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(J_1a,J_2b),\\
&\quad d_k(A_1a,J_1a), d_k(A_2b,J_2b), d_k(A_1a,A_2b)\},\\
&\quad q \in \{d_k(A_1a,J_2b), d_k(A_2b,J_1a)\}\Big\}\Big\}\Big).
\end{aligned}$$

Theorem 2.3 ensures that A_1, A_2, J_1 and J_2 have a unique common fixed point $j \in BB(S)$. That is, j is a unique common solution of the system of functional equations (1.5). This completes the proof. □

Theorem 3.2. *Let* $u : S \times D \to S, T : S \times D \to S$ *and* $H_1, H_2, G_1, G_2 : S \times D \times \mathbb{R} \to \mathbb{R}$ *satisfy* $(C1), (C2), (C3), (C4)$ *and*

$(C6)$ *For any* $k \geq 1$ *and* $i \in \{1, 2\}$

$$\lim_{n\to\infty} \sup_{x\in \overline{B}(0,k)} |A_i A_i a_n(x) - J_i J_i a_n(x)| = 0$$

whenever $\{a_n\}_{n\geq 1} \subset BB(S)$ *is a sequence in* $BB(S)$ *such that*

$$\lim_{n\to\infty} \sup_{x\in \overline{B}(0,k)} |A_i a_n(x) - a(x)| = \lim_{n\to\infty} \sup_{x\in \overline{B}(0,k)} |J_i a_n(x) - a(x)| = 0$$

for some $a \in BB(S)$,

then the system of functional equations in (1.5) *possesses a unique common solution in* $BB(S)$.

Proof. Note that $(C6)$ means that A_1, J_1 and A_2, J_2 are compatible mappings of type (P). The rest of the proof is similar to that of Theorem 3.1. This completes the proof. □

Acknowledgement. This work was supported by the Science Research Foundation of Educational Department of Liaoning Province (20060467).

References

[1] R. Bellman and E. S. Lee, *Functional equations arising in dynamic programming,* Aequationes Math. **17** (1978), 1–18.

[2] P. C. Bhakta and S. R. Choudhury, *Some existence theorems for functional equations arising in dynamic programming II,* J. Math. Anal. Appl. **131** (1988), 217–231.

[3] P. C. Bhakta and S. Mitra, *Some existence theorems for functional equations arising in dynamic programming,* J. Math. Anal. Appl. **98** (1984), 348–362.

[4] N. J. Huang, B. S. Lee and M. K. Kang, *Common fixed point theorems for compatible mappings with applications to the solutions of functional equations arising in dynamic programming,* Int. J. Math. Math. Sci. **20** (1997), 673–680.

[5] Z. Liu, *Existence theorems of solutions for certain classes of functional equations arising in dynamic programming,* J. Math. Anal. Appl. **262** (2001), 529–553.

[6] Z. Liu, *Coincidence theorems for expansive mappings with applications to the solutions of functional equations arising in dynamic programming,* Acta Sci. Math. (Szeged) **65** (1999), 359–369.

[7] Z. Liu, *Compatible mappings and fixed points,* Acta Sci. Math. (Szeged) **65** (1999), 371–383.

[8] Z. Liu, *A note on unique common fixed point,* Bull. Cal. Math. Soc. **85** (1993), 469–472.

[9] Z. Liu, R. P. Agarwal and S. M. Kang, *On solvability of functional equations and system of functional equations arising in dynamic programming,* J. Math. Anal. Appl. **297** (2004), 111–130.

[10] Z. Liu and S. M. Kang, *Properties of solutions for certain functional equations arising in dynamic programming,* J. Global Optim. **34** (2006), 273–292.

[11] Z. Liu and S. M. Kang, *Existence and uniqueness of solutions for two classes of functional equations arising in dynamic programming,* Acta Math. Appl. Sini. **23** (2007), 195–208.

[12] Z. Liu and J. K. Kim, *A common fixed point theorem with applications in dynamic programming,* Nonlinear Funct. Anal. Appl. **1** (2006), 11–19.

[13] Z. Liu and J. S. Ume, *On properties of solutions for a class of functional equations arising in dynamic programming,* J. Optim. Theory Appl. **117** (2003), 533–551.

[14] Z. Liu, J. S. Ume and S. M. Kang , *Some existence theorems for functional equations arising in dynamic programming,* J. Korean Math. Soc. **43** (2006), 11–28.

[15] Z. Liu, Y. Xu, J. S. Ume and S. M. Kang, *Solutions to two functional equations arising in dynamic programming,* J. Comput. Appl. Math. **192** (2006), 251–269.

[16] H. K. Pathak and B. Fisher, *Common fixed point theorems with applications in dynamic programming,* Glasnik Mate. **31** (1996), 321–328.

[17] H. K. Pathak, Y. J. Cho, S. M. Kang and B. S. Lee, *Fixed points theorems for compatible mappings of type (P) and applications to dynamic programming,* Le Matematiche. **L** (1995), 15–33.

Received: January 7, 2008