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Dual Hybrid Boundary Node Method for Transient Eddy Current Problem

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Abstract

Dual hybrid boundary node method (DHBNM) is presented for solving transient eddy current problems in the paper. With difference to the traditional boundary element method (BEM), the DHBNM combines dual reciprocity method (DRM) and hybrid boundary node method (HBNM), which does not require the 'boundary element mesh', either for the purpose of interpolation of the solution variables, or for the integration of the 'energy'. In this method, the solution composes into two parts, i. e., the complementary solution and the particular solution. The complementary solution is solved by HBNM, and the particular one is obtained by DRM. Theoretical analysis in details is given and a transient eddy current example is also presented to prove the proposed theory.

Keywords: Hybrid boundary node method, Dual reciprocity method, Radial basis function, Eddy current problem

1. Introduction

Boundary element methods (BEMs) are attractive and important computational techniques for reducing the dimensionality of solving problems. The construction of interpolation functions and the discretization are two key steps of BEMs. Using a scattered set of points instead of using element, the complex mesh generation process can be alleviated. Several boundary-type meshless methods have been developed for many potential and elastic problems, such as the boundary node method(BNM) [1], the hybrid boundary node method (HBNM) [2,3], the boundary radial basis function method(BRBFM) [4], the boundary point interpolation method(BPIM) [5] and the local boundary integral equation method(LBIEM) [6]. They need no discretization of the boundary and are proven as robust numerical methods. But very few of them are used to solve electromagnetic problems, such as transient eddy current problems. This may be due to the fact that more modification is needed in transient analysis when compared with static analysis [7,8].

In this paper, the HBNM is applied for solving transient eddy current problems. However, an initial restriction of HBNM is that the fundamental solution for the original partial differential equation is required to obtain a local integral equation. On the other hand, as a time-dependent problem, the fundamental solution is difficult to obtain. The inhomogeneous terms accounting distributed loads and time dependent term are included in the formulae of the domain integrals. Thus the method loses the attraction of its 'boundary-only' and true meshless method characters. To solve the obstacle, the dual reciprocity method (DRM) [9,10] is combined with the HBNM, which named dual hybrid boundary node (DHBNM), to avoid the domain integral that comes out from the inhomogeneous terms of the equation. In DHBNM, the solution composes two parts: complementary solution and particular solution. For the first part, same as HBNM, the variables inside the domain are interpolated by the fundamental solution while the unknown boundary variables are approximated by the moving least square (MLS) [11] approximation. The modified variational formulation is applied to form the discrete equations of HBNM. For the second one, the radial basis functions are applied to interpolating the inhomogeneous part of the equations. In order to overcome the singular integration, the rigid body moving method has been applied. The method keeps its 'boundary-only' and true meshless method characters. The theoretical analysis is given in details, and an example is also illustrated to prove the proposed theory of the DHBNM.

The discussions of this method are arranged as following: DHBNM will be presented in Section 2; numerical example of transient eddy current problem will be shown in Section 3; finally, the paper will end with conclusions in Section 4.

2. Dual hybrid boundary node method

The full set of equations for a low-frequency electromagnetic field can be now written as

$$\nabla \times \mathbf{E} = \partial \mathbf{B} / \partial t, \qquad \mathbf{B} = \mu \mathbf{H}, \tag{1a}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_e + \mathbf{J}_s, \quad \mathbf{J}_e = \sigma \mathbf{E} \tag{1b}$$

where $\mathbf{J}_e$ and $\mathbf{J}_s$ are the eddy current and source current; σ and μ are magnetic conductivity and electric conductivity respectively. For a linear conductive medium, with Lorentz gauge, the equivalent form by using the magnetic potential vector of (1) can be written as

$$\nabla^2 \mathbf{A} = \sigma\mu \frac{\partial \mathbf{A}}{\partial t} - \mu \mathbf{J}_s \tag{2}$$

Therefore, for 2-D problems, (2) can be reduced to a scalar diffusion equation as

$$\nabla^2 u = \sigma\mu \frac{\partial u}{\partial t} - P \tag{3}$$

where u is one of the three components of the vector potential $\mathbf{A}$ and P is the supplied source. The initial and boundary problems for 2-D transient eddy current field are written as

$$\begin{cases} \nabla^2 u = \sigma\mu \dfrac{\partial u}{\partial t} - P & \text{in } \Omega \text{ at } 0<t<t_n \\ u = \bar{u} & \text{on } \Gamma_u \text{ at } 0<t<t_n \\ q = \bar{q} & \text{on } \Gamma_q \text{ at } 0<t<t_n \\ u = u_0 & \text{in } \Omega \text{ at } t=0 \end{cases} \tag{4}$$

where Ω is the solver domain with boundary $\Gamma = \Gamma_u + \Gamma_q$, in which Γ_u and Γ_q are the essential and natural boundaries, $\bar{u}$ and $\bar{q}$ are known boundary conditions.

The left-hand side of Eq.(4) can be dealt with by HBNM for the Laplace equation, and the integrals corresponding to the right-hand side are taken to the boundary using RBF interpolation. In DHBNM, the solution variables u can be divided into the complementary solution u^c and the particular solution u^p, i.e.

$$u = u^c + u^p \tag{5}$$

For convenience, $P = 0$ is assumed. The particular solution u^p just needs to satisfy the inhomogeneous equation as following

$$\nabla^2 u^p = \sigma\mu \frac{\partial u}{\partial t} \tag{6}$$

2.1. HBNM for complementary solution

The complementary solution u^c must satisfy the Laplace equation and the modified boundary condition, so those are written as following

$$\nabla^2 u^c = 0 \tag{7}$$

$$u^c = \bar{u}^c = \bar{u} - u^p \tag{8}$$

$$q^c = \bar{q}^c = \bar{q} - q^p \tag{9}$$

The HBNM is based on a modified variational principle. The functions in the modified principle assumed to be independent are: displacement field within the domain u, boundary potential $\tilde{u}$ and boundary normal flux $\tilde{q}$. Consider a domain Ω enclosed by $\Gamma = \Gamma_u + \Gamma_q$ with prescribed potential $\bar{u}$ and normal flux $\bar{q}$ at the boundary portions Γ_u and Γ_q, respectively. The corresponding variational function Π_{AB} is defined as

$$\Pi_{AB} = \int_{\Omega} \frac{1}{2} u_{,i} u_{,i} d\Omega - \int_{\Gamma} \tilde{q} \left(u - \tilde{u} \right) d\Gamma - \int_{\Gamma_t} \overline{q}\, \tilde{u} d\Gamma \tag{10}$$

where, the boundary potential $\tilde{u}$ satisfies the essential boundary condition, i.e. $\overline{u} = \tilde{u}$, on Γ_u.

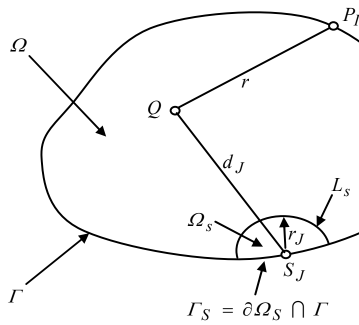

Figure 1. Local domain and source point of fundamental solution corresponding to S_J

The variational theory holds in any portion of the domain Ω, for example, a sub-domain Ω_s, defined as an intersection of a domain and a small circle centered at node S_I, and its boundary Γ_s and L_s. (see Fig.1). With the vanishing of $\delta\Pi_{AB}$ over the sub-domain and its boundary, the following equivalent integral can be obtained

$$\int_{\Gamma_s + L_s} \left(q - \tilde{q}_s \right) h d\Gamma - \int_{\Omega_s} u_{,ii} h d\Omega = 0 \tag{11}$$

$$\int_{\Gamma_s + L_s} \left(u - \tilde{u}_s \right) h d\Gamma = 0 \tag{12}$$

where h is a test function. We approximate $\tilde{u}_s$ and $\tilde{q}_s$ at the boundary Γ_s by the moving least square (MLS) approximation. In Eqs. (11) and (12), $\tilde{u}_s$ and $\tilde{q}_s$ at L_s has not been defined yet. To solve this problem, we select h so that all integrals vanish over L_s. This can be easily accomplished by using the weight function in the MLS approximation for h, with the half-length of the major axis d_I of the support of the weight function being replaced by the radius of the sub-domain Ω_s, i.e.

$$h_J(Q) = \begin{cases} \dfrac{\exp\left[-\left(d_J / c_J \right)^2 \right] - \exp\left[-\left(r_J / c_J \right)^2 \right]}{1 - \exp\left[-\left(r_J / c_J \right)^2 \right]}, & 0 \le d_J \le r_J \\ 0, & d_J \ge r_J \end{cases} \tag{13}$$

where d_J is the distance between point Q in the domain and the nodal point s_J. Therefore, $h_J(Q)$ vanishes on L_s.

Making use of fundamental solutions, we approximate u inside the domain by

$$u = \sum_{I=1}^{N} u_I^s x_I \tag{14}$$

where u_I^s is the fundamental solution; x_I are unknown parameters; N is the total number of boundary nodes. The fundamental solution is written as

$$u_I^s = \frac{1}{2\pi} \ln r(Q, s_I) \tag{15}$$

where Q and s_I are field point and source point, respectively.

By substituting Eqs.(14) and (15) into Eqs. (11) and (12), and using the MLS, we have

$$\sum_{I=1}^{N} \int_{\Gamma_s} \frac{\partial u_I^s}{\partial n} h_J(Q) x_I d\Gamma = \sum_{I=1}^{N} \int_{\Gamma_s} \Phi_I(s) h_J(Q) \hat{q}_I d\Gamma \tag{16}$$

$$\sum_{I=1}^{N} \int_{\Gamma_s} u_I^s h_J(Q) x_I d\Gamma = \sum_{I=1}^{N} \int_{\Gamma_s} \Phi_I(s) h_J(Q) \hat{u}_I d\Gamma \tag{17}$$

where $\Phi_I(s)$ is the shape function of MLS.

Using the above equations for all nodes, we can get the system equations

$$\mathbf{Tx} = \mathbf{H}\hat{\mathbf{q}}^c \tag{18}$$

$$\mathbf{Ux} = \mathbf{H}\hat{\mathbf{u}}^c \tag{19}$$

where

$$U_{IJ} = \int_{\Gamma_s} u_I^s h_J(Q) d\Gamma \tag{20}$$

$$T_{IJ} = \int_{\Gamma_s} \frac{\partial u_I^s}{\partial n} h_J(Q) d\Gamma \tag{21}$$

$$H_{IJ} = \int_{\Gamma_s} \Phi_I(s) h_J(Q) d\Gamma \tag{22}$$

2.2. RBF interpolation for particular solution

The DRM can be used in transient eddy current problem to transform the domain integral arising from the application of inhomogeneous into equivalent boundary integrals. Applying interpolation for inhomogeneous term, the following approximation can be proposed for the term $\sigma\mu \frac{\partial u}{\partial t}$:

$$\sigma\mu(\partial u / \partial t) = \sum_{J=1}^{N+L} f^J \alpha^J \tag{23}$$

where α^J are a set of initially unknown coefficients; the f^J are approximation functions; N and L are the total number of boundary nodes and total number of interior nodes, respectively.

As the same of the Eq.(23), the particular solution can be approximated by the basis form of the particular solutions. It can be written as following

$$u^p = \sum_{J=1}^{N+L} \bar{\bar{u}}^J \alpha^J \tag{24}$$

If u^p satisfies Eq.(20), the following equation can be obtained

$$\nabla^2 \bar{\bar{u}}^J = f^J \tag{25}$$

The approximation function f^J can be chosen as $f^J = 1 + r + r^2$. Obviously, the basis form of the particular solution $\bar{\bar{u}}$ satisfying Eq.(25) can be obtained as

$$\bar{\bar{u}} = \frac{r^2}{4} + \frac{r^3}{9} + \frac{r^4}{16} \tag{26}$$

The corresponding expression for the flux $\bar{\bar{q}}$ is

$$\bar{\bar{q}} = (r_{,x}\frac{\partial x}{\partial n} + r_{,y}\frac{\partial y}{\partial n})(\frac{1}{2} + \frac{r}{3} + \frac{r^2}{4}) \tag{27}$$

Solving Eq. (24), (26) and (27), the particular solution can be written in matrix form as following

$$\mathbf{u}^p = \sigma\mu\bar{\bar{\mathbf{u}}}\mathbf{F}^{-1}\dot{\mathbf{u}} \tag{28}$$

$$\mathbf{q}^p = \sigma\mu\bar{\bar{\mathbf{q}}}\mathbf{F}^{-1}\dot{\mathbf{u}} \tag{29}$$

where each column of $\mathbf{F}$ consists of a vector f^J containing the values of the function f^J at the DRM collocation nodes; $\bar{\bar{\mathbf{u}}}$ and $\bar{\bar{\mathbf{q}}}$ are the matrix forms of the basis type of particular solution.

2.3. Dual hybrid boundary node method

For a well-posed problem, either $\tilde{u}$ or $\tilde{q}$ is known at each node on the boundary. However, transformation between $\hat{u}_I$ and $\tilde{u}_I$ or $\hat{q}_I$ and $\tilde{q}_I$ is necessary because the MLS approximation lacks the delta function property. For the panels where $\tilde{u}_I$ is prescribed, $\tilde{u}_I$ is related to $\hat{u}_I$ by

$$\hat{u}_I = \sum_{J=1}^{N_t} R_{IJ}\tilde{u}_J = \sum_{J=1}^{N_t} R_{IJ}\bar{u}_J \tag{30}$$

and for the panels where $\tilde{q}_I$ is prescribed, $\tilde{q}_I$ is related to $\hat{q}_I$ by

$$\hat{q}_I = \sum_{J=1}^{N_t} R_{IJ}\tilde{q}_J = \sum_{J=1}^{N_t} R_{IJ}\bar{q}_J \tag{31}$$

where $R_{IJ} = \left[\Phi_J(s_I)\right]^{-1}$; N_t is the total number on a piece of the edge ; $\bar{u}_J$ and $\bar{q}_J$ are the related nodal values.

Substituting Eqs.(28), (29), (30) and (31) into Eq. (5), then substituting the result into Eqs. (18) and (19), we can obtain

$$\mathbf{Ux} + \sigma\mu\mathbf{HR}\bar{\bar{\mathbf{u}}}\mathbf{F}^{-1}\dot{\mathbf{u}} = \mathbf{HR}\bar{\mathbf{u}} \tag{32}$$

$$\mathbf{Tx} + \sigma\mu\mathbf{HR}\bar{\bar{\mathbf{q}}}\mathbf{F}^{-1}\dot{\mathbf{u}} = \mathbf{HR}\bar{\mathbf{q}} \tag{33}$$

Eqs. (32) and (33) are the system equations of the DHBNM for transient eddy current problem. Assuming that N nodes are located on the boundary, we can get N unknown variables on the boundary from Eqs. (32) and (33). However, the Eqs. above include the displacement of the L internal nodes, so the additional equations are needed.

2.4. Additional Equations

The Eqs. (32) and (33) can not be solved for the variables of the internal node, and additional equations for transient eddy current problem will be developed in this section.

The unknown variables of the internal nodes can be expressed as following

$$\mathbf{u}^* = \mathbf{u}^c + \mathbf{u}^p = \mathbf{u}^s\mathbf{x} + \sigma\mu\bar{\bar{\mathbf{u}}}\mathbf{F}^{-1}\dot{\mathbf{u}} \tag{34}$$

where $\mathbf{u}^*$ is the displacement of the internal nodes; $\mathbf{u}^s$ is the matrix of the fundamental solution on each internal nodes; $\bar{\bar{\mathbf{u}}}$ is the matrix of values of basis type of particular solution.

Solving out the coefficient vector $\mathbf{x}$ in Eq. (32), we can obtain

$$\mathbf{x} = \mathbf{U}^{-1}\mathbf{HR}\left(\bar{\mathbf{u}} - \sigma\mu\,\bar{\bar{\mathbf{u}}}\,\mathbf{F}^{-1}\dot{\mathbf{u}}\right) \tag{35}$$

Substituting Eq. (35) into Eqs. (34) and (33), we can obtain

$$\hat{\mathbf{C}}\dot{\mathbf{u}} + \hat{\mathbf{H}}\bar{\mathbf{u}} = \hat{\mathbf{G}}\bar{\mathbf{q}} \tag{36}$$

where

$$\hat{\boldsymbol{C}} = \begin{bmatrix} \sigma\mu\mathbf{HR}\bar{\bar{\mathbf{q}}}\,\mathbf{F}^{-1} & -\sigma\mu\mathbf{TU}^{-1}\mathbf{HR}\bar{\bar{\mathbf{u}}}\mathbf{F}^{-1} \\ \sigma\mu\bar{\bar{\mathbf{u}}}\mathbf{F}^{-1} & -\sigma\mu\mathbf{u}^s\mathbf{U}^{-1}\mathbf{HR}\bar{\bar{\mathbf{u}}}\mathbf{F}^{-1} \end{bmatrix} \tag{37}$$

$$\hat{\boldsymbol{H}} = \begin{bmatrix} \mathbf{TU^{-1}HR} & \mathbf{0} \\ \mathbf{u^sU^{-1}HR} & \mathbf{-I} \end{bmatrix} \tag{38}$$

$$\hat{\boldsymbol{G}} = \begin{bmatrix} \mathbf{HR} \\ 0 \end{bmatrix} \tag{39}$$

Like other hybrid models (for example, the hybrid boundary element method), the present method has a drawback of 'boundary layer effect' (the accuracy of the results in the vicinity of the boundary is very sensitive to the proximity of the interior points to the boundary). To avoid this drawback, an adaptive integration scheme has been proposed in [3]. As demonstrated, the DHBNM is a boundary-only meshless approach. No boundary elements are used for both interpolation and integration purpose. The nodes in the domain are needed just for interpolation for the particular solution, which can not influence the present method as a boundary-type method.

3. Numerical Example

In order to verify the proposed method, a metal column with infinite length is magnetized here. Its cross section of one quadrant is shown as Fig.2.

The parameters of the size and the medium type are $OA = 0.4\text{m}$, $OB = 0.2\text{m}$, $\mu_r = 5000$ and $\sigma = 5200\text{S/m}$. At time $t = 0$, a step magnetic field H_0 with direction $-z$ is posed on the outer surface of the metal column. In the present calculation, the boundary of the column is divided into four piecewise smooth segments. The regular meshes of 30 nodes (10 nodes on OA and BC each, 5 node on OB and AC each) on the boundary ($N = 30$) and 20 internal nodes ($L = 20$) are used. Points $O(0,0)$,

$P(0.1, 0.1)$ and $Q(0.3, 0)$ are investigated here to compare the analytical solution and the numerical one by using the DHBNM. The corresponding initial-boundary problem is

$$\begin{cases} \nabla^2 H_Z = \sigma\mu \dfrac{\partial H_z}{\partial t} & \text{in } \Omega \text{ at } t > 0 \\ H_z = H_0 & \text{on } AC, BC \text{ at t>0} \\ \dfrac{\partial H_z}{\partial n} = 0 & \text{on } OB, OA \text{ at t>0} \\ H_z = 0 & \text{in } \Omega \text{ at } t = 0 \end{cases} \tag{40}$$

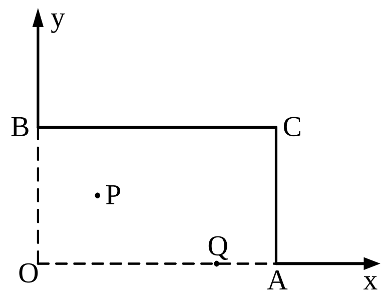

Fig.2. Cross section of one quadrant of the metal column

Fig.3 shows the numerical solution by DHBNM and the analytical one. The results yielded by using the proposed method are found to be in good agreement with the analytical solution. The eddy current density on symmetry axis OA is also given by using the proposed method in Fig. 4. Those prove that DHBNM is an effective technique to analyze and solve transient eddy current problems.

Numerical example shows that the number of internal node $L = \frac{N}{2}$, where N is the number of boundary nodes, provides solutions which are satisfactory for the transient eddy current problem.

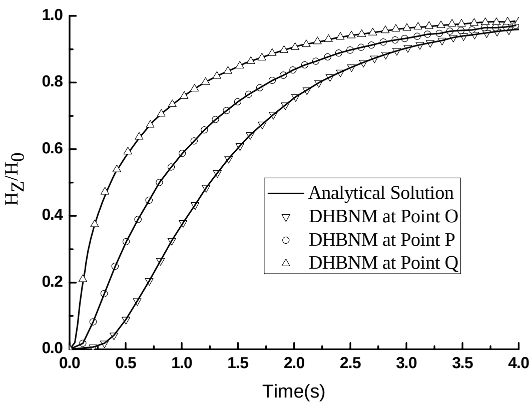

Fig.3. Normalized field intensity by analytical and DHBNM

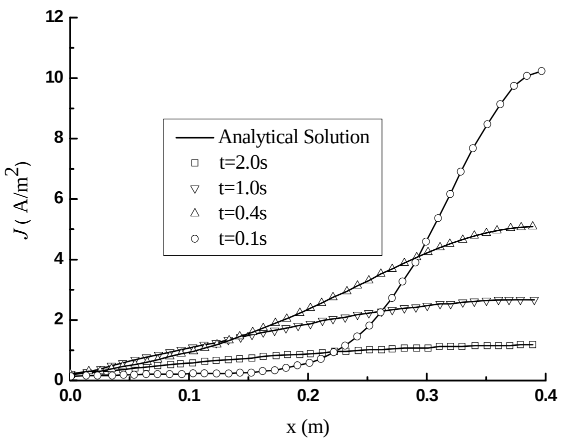

Fig.4. Eddy current intensity on symmetry axis OA

4. Conclusion

A truly meshless method for solving the transient eddy current problem, which called DHBNM, has been presented in this paper. This method combines the DRM and the HBNM. The HBNM is used to solve the homogeneous equations, and the DRM is employed to solve the inhomogeneous terms. No cells are needed either for the interpolation purposes or for integration process, only discrete nodes are constructed on the boundary of a domain, several nodes in the domain are needed just for the RBF interpolation. Further work is in progress to test the method in more realistic problems such as large geometrical deformation and the propagating cracks in nondestructive evaluation.

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