Application of an Approximate Analytical

Method to Nonlinear Troesch's Problem

S. H. Mirmoradi[a,1]**, I. Hosseinpour**[a]**, S. Ghanbarpour**[b]**, A. Barari**[a]

[a]Department of Civil Engineering
Babol University of Technology, Babol, Iran

[b] Departments of Civil Engineering
University of Science & Technology of Mazandaran
Babol, Iran

Abstract

In this paper, the solution of the nonlinear Troesch's problem is obtained by means of homotopy perturbation method (HPM). Troesch's problem arises in the investigation of the confinement of a plasma column by radiation pressure. The results reveal that the homotopy perturbation method (HPM) is very effective, convenient and quite accurate to nonlinear differential equations. It is predicted that the HPM can be found widely applicable in engineering.

Keywords: nonlinear Troesch's problem; plasma column; Homotopy-perturbation method.

1   Introduction

In this paper, we consider the nonlinear Troesch's problem:

$$u'' = \gamma \sinh(\gamma u) \tag{1}$$

Troesch's problem arises in the investigation of the confinement of a plasma

[1]Author for correspondence, sh.mirmoradi@gmail.com, Tel:+989111156621

column by radiation pressure. Troesch's problem was described and solved by Weibel [1].

Linear and Nonlinear phenomena are of fundamental importance in various fields of science and engineering. Most models of real – life problems are still very difficult to solve. Therefore, approximate analytical solutions such as Homotopy-perturbation method (HPM) [2-11] were introduced. This method is the most effective and convenient ones for both linear and nonlinear equations.

Perturbation method is based on assuming a small parameter. The majority of nonlinear problems, especially those having strong nonlinearity, have no small parameters at all and the approximate solutions obtained by the perturbation methods, in most cases, are valid only for small values of the small parameter.

Generally, the perturbation solutions are uniformly valid as long as a scientific system parameter is small. However, we cannot rely fully on the approximations, because there is no criterion on which the small parameter should exists. Thus, it is essential to check the validity of the approximations numerically and/or experimentally. To overcome these difficulties, HPM have been proposed recently In this paper we will apply homotopy perturbation method (HPM) to the nonlinear Troesch's problem.

2   Basic idea of homotopy-perturbation method

To explain this method, let us consider the following function:

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{2}$$

With the boundary conditions of:

$$B(u, \frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma, \tag{3}$$

Where A, B, $f(r)$ and Γ are a general differential operator, a boundary operator, a known analytical function and the boundary of the domain Ω, respectively.

Generally speaking the operator A can be divided in to a linear part L and a nonlinear part $N(u)$. Eq. (2) can therefore, be written as:

$$L(u) + N(u) - f(r) = 0, \tag{4}$$

By the homotopy technique, we construct a homotopy
$v(r, p) : \Omega \times [0,1] \to R$ Which satisfies:

$$H(v, p) = (1 - p)\left[L(v) - L(u_0) \right] + p\left[A(v) - f(r) \right] = 0, \tag{5}$$

$$p \in [0,1], r \in \Omega,$$

Or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p\left[N(v) - f(r) \right] = 0, \tag{6}$$

where $p \in [0,1]$ is an embedding parameter, while u_0 is an initial approximation of Eq. (2), which satisfies the boundary conditions. Obviously, from Eqs. (5) and (6) we will have:

$$H(v,0) = L(v) - L(u_0) = 0, \tag{7}$$

$$H(v,1) = A(v) - f(r) = 0, \tag{8}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from u_0 to $u(r)$. In topology, this is called deformation, while $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopy.

According to the HPM, we can first use the embedding parameter p as a "small parameter", and assume that the solutions of Eqs. (5) and (6) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + ..., \tag{9}$$

Setting $p = 1$ yields in the approximate solution of Eq. (2) to:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots, \tag{10}$$

The combination of the perturbation method and the homotopy method is called the HPM, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantage.

The series (10) is convergent for most cases. However, the convergent rate depends on the nonlinear operator $A(v)$. Moreover, He made the following suggestions [10]:

- The second derivative of $N(v)$ with respect to v must be small because the parameter may be relatively large, i.e. $p \to 1$.

- The norm of $L^{-1} \dfrac{\partial N}{\partial v}$ must be smaller than one so that the series converges.

3 Example

We consider the second-order non-linear Troesch's problem. The equation of the form:

$$u'' = \gamma \sinh(\gamma u) \tag{11}$$

On the finite x-interval $[0,1]$ with boundary condition:

$$u(0) = 0 \quad \text{and} \quad u(1) = 1 \tag{12}$$

The closed form solution of (11)–(12) is given in [1] by:

$$u(x) = \frac{2}{\gamma} \sinh^{-1}\left[\frac{u'(0)}{2} sc(\gamma x | 1 - \frac{1}{4}(u'(0))^2)\right] \tag{13}$$

3.1   Application of homotopy-perturbation method

To solve Eq. (11) by means of HPM, we consider the following process after separating the linear and nonlinear parts of the equation:

$$H(v, p) = (1 - p)\left(\frac{d^2}{dx^2}v(x) - \frac{d^2}{dx^2}v_0(x)\right) +$$

$$p\left(\frac{d^2}{dx^2}v(x) - (\frac{\gamma}{2})v(x) - \frac{\gamma^4 v^3(x)}{6}\right) = 0, \tag{14}$$

Substituting Eq. (9) in to Eq. (14) and rearranging the resultant equation based on powers of p-terms, one has:

$$p^0 : \frac{d^2}{dx^2}v_0(x) = 0 \tag{15}$$

$$p^1 : \left(\frac{d^2}{dx^2}v_1(x)\right) - \frac{1}{6}\gamma^4 v_0(x)^3 - \gamma^2 v_0(x) = 0 \tag{16}$$

$$p^2 : -\gamma^2 v_1(x) + \left(\frac{d^2}{dx^2}v_2(x)\right) - \frac{1}{2}\gamma^4 v_0(x)v_1(x) = 0 \tag{17}$$

With the following conditions:

$$\begin{aligned} v_0(0) &= 0 \ , \ v_0(1) = 1 \\ v_i(0) &= 0 \ , \ v_i(1) = 0 \qquad i = 1, 2, ... \end{aligned} \tag{18}$$

With the effective initial approximation for v_0 from the conditions (18) and solutions of Eqs. (15-17) may be written as follows:

$$v_0(x) = x \tag{19}$$

$$v_1(x) = \frac{1}{120}\gamma^4 x^5 + \frac{1}{6}\gamma^2 x^3 + \left(-\frac{1}{120}\gamma^4 - \frac{1}{6}\gamma^2\right)x \tag{20}$$

$$v_2(x) = \frac{1}{17280}\gamma^8 x^9 + \frac{11}{5040}\gamma^6 x^6 + \frac{1}{120}\gamma^4 x^5 - \frac{1}{4800}\gamma^8 x^5 - \frac{1}{240}\gamma^6 x^5$$

$$- \frac{1}{720}\gamma^3 x^6 - \frac{1}{36}\gamma^4 x^3 + \left(\frac{13}{86400}\gamma^8 + \frac{17}{5040}\gamma^6 + \frac{7}{360}\gamma^4\right)x \tag{21}$$

In the same manner, the rest of components were obtained using the Maple package.

According to the HPM, we can conclude that:

$$u(x) = \lim_{p \to 1} v(x) = v_0(x) + v_1(x) + \dots \tag{22}$$

Therefore, substituting the values of $v_0(x)$, $v_1(x)$ and $v_2(x)$ from Eqs. (19- 21) into Eq.(22) yields:

$$u(x) = x + \frac{1}{60}\gamma^4 x^5 + \frac{1}{6}\gamma^2 x^3 + \left(-\frac{1}{120}\gamma^4 \frac{1}{6}\gamma^2\right)x + \frac{1}{17280}\gamma^8 x^9$$
$$+ \frac{11}{5040}\gamma^6 x^7 - \frac{1}{4800}\gamma^8 x^5 - \frac{1}{240}\gamma^6 x^5 - \frac{1}{720}\gamma^3 x^6 - \frac{1}{36}\gamma^4 x^3 \tag{23}$$

4   Numerical examples

In this section, the HPM method is carried out to two special cases of Troesch's problem. The first case is when $\gamma = 0.5$. In Table I, the exact solution for the case $\gamma = 0.5$ derived from Eq. (13), is compared with the HPM method.

Table I shows that the absolute errors, i.e. the difference between the approximate analytical solution and the exact solution, is less 0.0031 for HPM.

The second case is when $\gamma = 1$. Table II shows that the absolute error HPM technique is less than 0.0142 for HPM method.

Table 1: Comparison between different results for the Case $\gamma = 0.5$

x	Exact solution	HPM solution	HPM error
0.1	0.095176902	0.095948026	0.000771124
0.2	0.190633869	0.192135797	0.001501928
0.3	0.286653403	0.288804238	0.002150835
0.4	0.383522929	0.386196642	0.002673713
0.5	0.481537385	0.4845599	0.003022515
0.6	0.581001975	0.584145785	0.00314381
0.7	0.682235133	0.685212297	0.002977164
0.8	0.785571787	0.788025104	0.002453317
0.9	0.891366988	0.892859085	0.001492098

Table 2: Comparison between different results for the Case $\gamma = 1$

x	Exact Solution	HPM Solution	HPM Error
0.1	0.081796997	0.084934415	0.003137419
0.2	0.164530871	0.170697546	0.006166675
0.3	0.249167361	0.258133224	0.008965863
0.4	0.336732209	0.348116627	0.011384418
0.5	0.428347161	0.44157274	0.013225579
0.6	0.52527403	0.539498234	0.014224205
0.7	0.628971143	0.642987984	0.01401684
0.8	0.741168378	0.753267551	0.012099173
0.9	0.863970021	0.871733059	0.007763039

5 Conclusion

The homotopy perturbation method has been successfully used to study non-linear Troesch's problem. Troesch's problem arises in the investigation of the confinement of a plasma column by radiation pressure. The results obtained here were compared with the exact solutions. The results revealed that the homotopy perturbation method is powerful mathematical tool for solution of nonlinear differential equations in terms of accuracy and efficiency.

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