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A Family of Multivariate Abel Series Distributions of Order *k*

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Abstract

In this paper an attempt is made to define the multivariate abel series distributions (MASDs) of order *k*. From MASD of order *k*, a new distribution called the quasi multivariate logarithmic series distribution (QMLSD) of order *k* is derived. Some well *k*nown distributions are also obtained by a new method of derivation. Limiting distribution of QNMD of order *k* are studied.

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1. Introduction

In this study, considering the multivariate Abel series distribution of order *k*, we have defined multivariate Abel series distributions of order *k*. From the MASDs of order *k*, a new distribution called Quasi Multivariate logarithmic series distribution of order *k* is obtained. Also a variant of quasi negative multinomial distribution of order *k* is studied. Moreover, on using a new method of derivation some well *k*nown distributions, viz. quasi multinomial distribution of type-I of order *k* (QMD-I (*k*)), quasi multinomial distribution of type-II of order *k* (QMD-II (*k*)), multiple generalized poisson distribution of order *k* (MGPD (*k*)) etc. are obtained. A property, i.e. the limiting distribution of the QNMD of order *k* has been found out.

2. Multivariate Abel series distribution of order *k* and its special cases

Let us consider a finite and positive function $f(a)$ of $a=(a_{11},.....,a_{mk})$, where each a_{ij} (i = 1 (1) m), j = 1 (1) *k* is a non-negative integer. For any real z, we have the multinomial abel series expansion of order *k* as,

$$f(a)\equiv f(a,z)=\sum_{\sum_j jx_{ij}=x_i}\left[\prod_{i=1}^{m}\prod_{j=1}^{k}a_{ij}\left(a_{ij}-x_{ij}z\right)^{x_{ij}-1}/\,x_{ij}\,!\right]\left[d^{(k)}f(a)|_{a=xz}\right] \quad .....(1)$$

where the summation is over $x_1,x_2,x_3,....,x_k$ such that $\sum_j jx_{ij}=x_i$ and each $x_i(i=1(1)m)$

being a non-negative integer and the factor $\left[d^{(k)}f(a)|_{a=xz}\right]/\left(\prod_i\prod_j x_{ij}\,!\right)$ is denoted by $\beta(x,z)$ is independent of ‘a’, which is always greater than zero. The domain of $a=(a_{11},.....,a_{mk})$ is a subspace of an mk-dimensional parameter space subject to restrictions $a_{ij}\ge 0$, if $z\le 0$ and $\left(a_{ij}-x_{ij}z\right)\ge 0$, if $z\ge 0$, z belonging to a suitable subject of real numbers. Thus, (2.1) can be written,

$$f(a)\equiv f(a,z)=\sum_{\sum_j jx_{ij}=x_i}\beta(x,z)\prod_i\prod_j a_{ij}\left(a_{ij}-x_{ij}z\right)^{x_{ij}-1} \quad .....(2)$$

using the formal series expansion (2.1), we suggest the following definition for the multivariate abel series distribution of order *k* (MASD (*k*))

Definition

A multivariate discrete distribution of order *k*, $p_k(x)$ is said to be a MASD (*k*) family, if it has the following probability function (p.f.),

$$p_k(x)\equiv p_k(x;a,z)=\sum_{\sum_j jx_{ij}=x_i}\left[\prod_i\prod_j a_{ij}\left(a_{ij}-x_{ij}z\right)^{x_{ij}-1}/\left(k_{ij}\right)!\right]\left[d^{(k)}f(a)\right]_{a=xz}/f(a) \quad ....(3)$$

where $\sum_j jx_{ij}=x_i$, each x_i being a non-negative integer, the a_{ij} ($i=1(1)m;\ j=1(1)k$) and z are parameters and f(a) are stated in (2.1).

For z = 0, the probability function (2.3) becomes multivariate power series distribution of order *k*.

For z = 0 and k = 1, the probability function (2.3) becomes multivariate Abel series distribution (Nandi and Das, 1996) and the z = 0, it becomes usual multivariate power series distribution (Patil, 1965).

Derivation of some distributions

I. Here we derive a new distribution from MASD (*k*), called the multivariate logarithmic series distribution of order *k*.

Let us consider Abel series expansion of order k of the logarithmic series function f(a) given by,

$$-\log\left(1-\sum_i\sum_j a_{ij}\right)=\sum_{\sum_j x_{ij}=x_i}\frac{\left(\sum_i\sum_j x_{ij}-1\right)!}{\prod_i\prod_j\left(x_{ij}\right)!}\prod_i\prod_j a_{ij}^{x_{ij}}\ ;\ \ldots\ldots\ldots\ldots(4)$$

;

$$x_i=0,1,\ldots.;\ for 1\le i\le m,\sum_i x_i>0,0<a_{ij}<1(\ for 1\le i\le m\,\&\,1\le\ j\le k);\sum_i\sum_j a_{ij}<1$$

Thus the associated MASD-family of order k has the following probability function,

$$p_k(x)=-\frac{1}{\log\left(1-\sum_i\sum_j a_{ij}\right)}\sum_{\sum_j x_{ij}=x_i}\frac{\left(\sum_i\sum_j x_{ij}-1\right)!}{\prod_i\prod_j\left(x_{ij}\right)!}\prod_i\prod_j a_{ij}^{x_{ij}}\ ;\ \ \ldots\ldots\ldots\ldots(5)$$

$$;\ x_i=0,1,\ldots.;\ for 1\le i\le m,\sum_i x_i>0,0<a_{ij}<1(\ for 1\le i\le m\,\&\,1\le\ j\le k);\sum_i\sum_j a_{ij}<1$$

$\sim MLSD_k(a_{11},\ldots..,a_{mk})$ (A*k*i, Kubo*k*i and Hirano (1984))

For k = 1, the probability function (2.5) becomes usual MLSD with parameters $a_1,a_2,a_3,\ldots.,a_m$, i.e.,

$$p(x_1,\ldots.,x_m)=\frac{\left(\sum_{i=1}^{m}x_i-1\right)!}{\left(\prod_{i=1}^{m}x_i!\right)\left\{-\log\left(1-\sum_{i=1}^{m}a_i\right)\right\}}\prod_{i=1}^{m}a_i^{x_i};\,x\ge 0;\sum_{i=1}^{m}x_i\ge 0\quad\ldots\ldots\ldots(6)$$

For m = 1, the p.f. (2.5) becomes the multiparameter logarithmic series distribution of order k (Philippou, 1988).

II. Here we derive a new distribution from MASD (k), called the quasi-multivariate logarithmic series distribution of order k (QMLSD (k)) as follows.

Consider the logarithmic series function $f(a) = -\log(1-a)$, where $a = a_{11} + ... + a_{mk}$ and $a = (a_{11}, ..., a_{mk})$. Then the multivariate Abel series expansion of $-\log(1-a)$ of order *k* is,

$$-\log(1-a) = \sum_{\sum_j x_{ij} = x_i} \frac{\left(\sum_i \sum_j x_{ij} - 1\right)!}{\prod_i \prod_j (x_{ij})!} \prod_i \prod_j a_{ij} \left(a_{ij} - x_{ij} z\right)^{x_{ij}-1} \left(1 - \sum_i \sum_j x_{ij} z\right)^{-x_{ij}}$$

…………….(7)

where $\sum_{\sum_j j x_{ij} = x_i}$ is defined in (2.1) and $0 < a = a_{11} + ... + a_{mk} < 1$.

Thus the associated MASD family of order *k* has the following p. f.

$$p_k(x) = \sum_{\sum_j x_{ij} = x_i} \frac{\left(\sum_i \sum_j x_{ij} - 1\right)!}{\left(\prod_i \prod_j (x_{ij})!\right)[-\log(1-a)]} \prod_i \prod_j a_{ij} \left(a_{ij} - x_{ij} z\right)^{x_{ij}-1} \left(1 - \sum_i \sum_j x_{ij} z\right)^{-x_{ij}}$$

……….(8)

$; x_{ij} \geq 0, i = 1(1)m, j = 1(1)k \;\&\; 0 < a < 1$

The p.f. (2.8) is called the QMLSD (*k*).

If $z \to 0$, then the p. f. (2.8) becomes the multinomial logarithmic series distribution of order *k*.

For *k* = 1, the p.f. (2.8) becomes QMLSD (Nandi & Das, 1996) and then as $z \to 0$, it becomes the common multinomial logarithmic series distribution (Johnson & *K*otz, 1969, p.303).

III. Next we obtain quasi multinomial distribution of type-I of order *k* (QMD-I (*k*)).

Let us consider the simple series function $f(a) = (a+b)^n$, where $a = (a_{11}, ..., a_{mk})$ and $a = a_{11} + ... + a_{mk}$. Then the multivariate Abel series expansion of $(a+b)^n$ of order *k* is,

$$(a+b)^n = \sum_{\sum_j j x_{ij} = x_i} \binom{n}{x} \prod_i \prod_j a_{ij} \left(a_{ij} - x_{ij} z\right)^{x_{ij}-1} (b + xz)^{n-x}$$

Hence the corresponding MASD family finds the p.f. of QMD-I (*k*),

$$p_k(x) = \binom{n}{x} \prod_i \prod_j a_{ij} \left(a_{ij} - x_{ij} z\right)^{x_{ij}-1} (b + xz)^{n-x} / (a+b)^n \qquad \text{………..…(9)}$$

where, $0 \le \sum_j jx_{ij} = x_i \le n$, each x_{ij} being a non- negative integer and

$$\binom{n}{x} = \frac{n!}{(n-x)!\left(\prod_i \prod_j x_{ij}\right)!}$$

Suppose, $p_{ij} = \dfrac{a_{ij}}{(a+b)}; i = 1(1)m \;\&\; j = 1(1)k;$

$$p_0 = \frac{a}{(a+b)} \text{ and } \phi = \frac{z}{(a+b)} \qquad \text{…………(10)}$$

Using (2.10) in (2.9), we get

$$p_k(x) = \binom{n}{x}(p_0 + x\phi)^{n-x} \prod_{i=1}^{m}\prod_{j=1}^{k} p_{ij}\left(p_{ij} - x_{ij}\phi\right)^{x_{ij}-1} \qquad \text{………….(11)}$$

where, $0 \le x \le n, \sum_{i=1}^{m}\sum_{j=1}^{k} p_{ij} = 1$ and

For k = 1, we get the p.f. (2.9) as QMD-I (Janardan, 1975).

IV. Now, we derive the multiple generalized poisson distribution of order k.

Let us consider the exponential series function $f(a) = e^a$, where $a = (a_{11}, ..., a_{mk})$ and $a = a_{11} + ... + a_{mk}$. Then the multivariate Abel series expansion of $f(a) = e^a$ is,

$$e^a = \sum_{\sum_j jx_{ij} = x_i} \left[\prod_i \prod_j a_{ij}(a_{ij} - x_{ij}z)^{x_{ij}-1} / \left(x_{ij}\right)! \right] e^{xz} \qquad \text{……….....(12)}$$

where $\sum_j jx_{ij} = x_i$ is stated in (2.1)

Thus the corresponding p.f. of the MASD family of order k is,

$$p_k(x) = \prod_{i=1}^{m}\prod_{j=1}^{k}[a_{ij}\left(a_{ij} - x_{ij}z\right)^{x_{ij}-1} e^{-\left(a_{ij} - x_{ij}z\right)} / \left(x_{ij}\right)!] \qquad \text{..……...…(13)}$$

where $\sum_j jx_{ij} = x_i$ and each x_{ij} being a non-negative integer.

The p.f. (2.13) is known as the MGPD of order k and for k = 1, the p.f. (2.13) reduces to MGPD (Janardan, 1975).

V. Finally, we derive the quasi negative multinomial distribution of order k.

Let us consider the series function, $f(a) = (b-a)^{-n}$, where $a = (a_{11}, ..., a_{mk})$ and $a = a_{11} + ... + a_{mk}$. Then the multivariate Abel series expansion of $(b-a)^{-n}$ of order k is,

$$(b-a)^{-n} = \sum_{\sum_j jx_{ij}=x_i} \frac{\Gamma\left(n+\sum_i\sum_j x_{ij}\right)}{\left(\prod_i\prod_j x_{ij}!\right)\Gamma(n)} \prod_i\prod_j a_{ij}\left(a_{ij}-x_{ij}z\right)^{x_{ij}-1}\left(b-\sum_i\sum_j x_{ij}z\right)^{-n-\sum_i\sum_j x_{ij}}$$

where $\sum_{\sum_j jx_{ij}=x_i}$ are given in (2.1).

Then the associated family of the MASD of order *k* has the p.f.

$$p_k(x) = \frac{\Gamma\left(n+\sum_i\sum_j x_{ij}\right)}{\left(\prod_i\prod_j x_{ij}!\right)\Gamma(n)} \prod_i\prod_j a_{ij}\left(a_{ij}-x_{ij}z\right)^{x_{ij}-1}\left(b-\sum_i\sum_j x_{ij}z\right)^{-n-\sum_i\sum_j x_{ij}} (b-a)^n$$

……..…(14)

$; x_{ij} \geq 0; i = 1(1)m$ and $j = 1(1)k$.

Suppose, $p_{ij} = \frac{a_{ij}}{(b-a)}; i = 1(1)m \;\&\; j = 1(1)k;$

$$Q = \frac{b}{(b-a)} \text{ and } \phi = \frac{z}{(b-a)}$$ ………...(15)

Applying (2.15) to the p.f. (2.14), we get

$$p_k(x) = \frac{\Gamma\left(n+\sum_i\sum_j x_{ij}\right)}{\left(\prod_i\prod_j x_{ij}!\right)\Gamma(n)} \prod_i\prod_j P_{ij}\left(P_{ij}-x_{ij}\phi\right)^{x_{ij}-1}\left(Q-\sum_i\sum_j x_{ij}\phi\right)^{-n-\sum_i\sum_j x_{ij}}$$ …..(16)

where $P_{ij} - x_{ij}\phi \geq 0$ and $Q - \sum_i\sum_j P_{ij} = 1$

The probability functions (2.14) and (2.16) are called QNMD of order *k*.
If z = 0, then (2.14) and (2.16) reduces to negative multinomial distribution of order *k*.
If *k* = 1, then (2.14) and (2.16) reduces to QNMD and then for z = 0, it reduces to common negative multinomial distribution (Johnson and *K*otz, 1969, p. 292).

3. Properties of QNMD of order *k*

Limiting distributions

The QNMD of order *k*, (2.16) with P_{ij} (i = 1 (1) m, j = 1 (1) *k*), Q, ϕ and n tends to multiple generalized poisson distribution with parameters λ_{ij} (i = 1 (1) m, j = 1

(1) k) and φ, as $n \to \infty$, $P_{ij} \to 0$ and $\phi \to 0$, such that $nP_{ij} = \lambda_{ij}$ and $n\phi = \varphi$. The probability function of this limiting distribution is given in (2.13).

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