Applied Mathematical Sciences, Vol. 3, 2009, no. 33, 1609 - 1618

Sequential Probability Ratio Tests for Fuzzy Hypotheses Testing

H. Torabi and S. M. Mirhosseini

Department of Statistics, Yazd University, Yazd, Iran
htorabi@yazduni.ac.ir

Abstract

In hypotheses testing, such as other statistical problems, we may confront imprecise concepts. One case is a situation in which hypotheses are imprecise.

In this paper, we redefine some concepts about fuzzy hypotheses testing, and then we introduce the sequential probability ratio test for fuzzy hypotheses testing. Finally, we give some examples.

Mathematics Subject Classification: 03E72, 62F03, 62G10

Keywords: Fuzzy hypothesis, critical region, probability of type I and II errors, sequential probability ratio test

1 Introduction

Decision making in classical statistical inference is based on crispness of data, random variables, exact hypotheses and decision rules. As there are many different situations in which the above assumptions are rather irrealistic, there have been some attempts to analyze these situations with fuzzy set theory proposed by Zadeh [25].

One of the primary purposes of statistical inference is to test hypotheses. In the traditional approach to hypotheses testing all the concepts are precise and well defined; see e.g. [13], [7], and [16]. However, if we introduce vagueness into hypotheses, we face quite new and interesting problems. Arnold [2] considered statistical tests under fuzzy constraints on the type I and II errors. Testing fuzzy hypotheses was discussed by Arnold [1] and [3], Delgado et al. [8], Watanabe and Imaizumi [24], Taheri and Behboodian [19], [20], and [21], and Grzegorzewski [10] and [11]. Kruse and Meyer [12], Taheri and Behboodian [21] considered problem of testing vague hypotheses in the presence of vague hypothesis. Up to the present testing hypotheses with fuzzy data was considered by Casals et al. [6], and Son et al. [17]; see for more references about fuzzy testing problem, Taheri [18].

In Section 2, we provide some definitions and preliminaries about fuzzy hypotheses testing. The sequential probability ratio test for fuzzy hypotheses testing is given in Section 3, and finally, some examples are presented in Section 4.

2 Fuzzy hypotheses testing

In this section, we consider the fuzzy hypotheses and testing such hypotheses. We may face a fuzzy hypothesis in practice. Let us begin with an example.

Example 2.1. Suppose we are interested in evaluating the diameters of produced washers of a factory, and we know that the distribution of such diameters is $N(\mu, \sigma^2)$, where σ^2 is known. In the ordinary case, the test $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ is used. Obviously in this example, if the diameter of a washer is slightly different from μ_0, H_0 is still acceptable, but a considerable difference from μ_0 makes H_0 unacceptable. Thus, it is reasonable to accept the products, if $\mu \approx \mu_0$ and to reject them if $\mu \not\approx \mu_0$. So the realistic hypotheses are

$$\begin{cases} H_0 : & \mu \text{ is close to } \mu_0 \\ H_1 : & \mu \text{ is away from } \mu_0. \end{cases}$$

In the following, we introduce some concepts about fuzzy hypotheses testing (henceforth FHT).

Definition 2.1. Any hypothesis of the form "$H : \theta$ is $H(\theta)$" is called a fuzzy hypothesis, where "$H : \theta$ is $H(\theta)$" implies that θ is in a fuzzy set of Θ (the parameter space) with membership function $H(\theta)$ i.e. a function from Θ to $[0, 1]$.

Note that the ordinary hypothesis $H : \theta \in \Theta$ is a fuzzy hypothesis with membership function $H(\theta) = 1$ at $\theta \in \Theta$, and zero otherwise, i.e., the indicator function of the crisp set Θ.

Example 2.2. Let θ be the parameter of a Bernoulli distribution. Consider the following function:

$$H(\theta) = \begin{cases} 2\theta, & 0 < \theta < 1/2 \\ 2 - 2\theta, & 1/2 \leq \theta < 1 \end{cases}$$

The hypothesis "$H : \theta$ is $H(\theta)$" is a fuzzy hypothesis and it means that "θ is approximately 1/2".

In FHT with crisp data, the main problem is testing

$$\begin{cases} H_0: \ \theta \ is \ H_0(\theta) \\ H_1: \ \theta \ is \ H_1(\theta) \end{cases}, \tag{2.1}$$

according to a random sample $\mathbf{X} = (X_1, \cdots, X_n)$ from a parametric population with the PDF $f(x;\theta)$. It is clear that the PDF of the random sample $\mathbf{X}$ is

$$f(\mathbf{x};\theta) = \prod_{i=1}^{n} f(x_i;\theta).$$

In the following, we give some definitions in FHT theory.

Definition 2.2. The pseudo-membership function of $H_j(\theta)$ is defined by

$$H_j^*(\theta) = H_j(\theta) / \int_{\Theta} H_j(\theta) d\theta, \quad j = 0, 1.$$

Substitute $\int$ by $\sum$ in discrete cases; i.e., in the case that Θ has just countable values.

Note that the pseudo-membership function is not necessarily a membership function, i.e., it may be greater than 1 for some values of θ; see Example 4.3, $H_0^*(1/4) = 4$.

In FHT like the traditional hypotheses testing, we must give a test function $\phi(\mathbf{X})$. In the following, we define the test function.

Definition 2.3. Let $\mathbf{X}$ be a random sample with the PDF $f(\mathbf{x};\theta)$. $\phi(\mathbf{X})$ is called a test function if it is the probability of rejecting H_0 provided that $\mathbf{X} = \mathbf{x}$ is observed.

Definition 2.4. Let r.v. X have the PDF $f(x;\theta)$. Under the hypothesis $H_j(\theta)$, $j = 0, 1$, the weighted probability density function (henceforth WPDF) of X is defined by

$$f_j(x) = \int_{\Theta} H_j^*(\theta) f(x;\theta) d\theta,$$

i.e., the expected value of $f(x;\theta)$ over $H_j^*(\theta)$, $j = 0, 1$. If $\mathbf{X}$ is a random sample from PDF $f(.;\theta)$, then the joint WPDF of $\mathbf{X}$ is defined by

$$f_j(\mathbf{x}) = \prod_{i=1}^{n} f_j(x_i).$$

Remark 2.1. $f_j(x)$ is a PDF, since $f_j(x)$ is nonnegative and

$$\begin{aligned} \int_{\mathbb{R}} f_j(x) dx &= \int_R \int_{\Theta} H_j^*(\theta) f(x;\theta) d\theta dx \\ &= \int_{\Theta} H_j^*(\theta) [\int_{\mathbb{R}} f(x;\theta) dx] d\theta \\ &= \int_{\Theta} H_j^*(\theta) d\theta \\ &= 1 \end{aligned},$$

Substitute $\int$ by $\sum$ in the discrete cases. Hence $f_j(x_1, \cdots, x_n)$ is also a joint PDF. Note that using the Fubini's theorem (see for example Billingsley [4], pp. 233-234) the interchange of the double integrals is satisfied because the integrands are non-negative and integrable.

Remark 2.2. If H_j is the crisp hypothesis $H_j : \theta = \theta_j$, then $f_j(x) = f(x; \theta_j)$ and then $f_j(x_1, \cdots, x_n) = f(x_1, \cdots, x_n; \theta_j)$, $j = 0, 1$.

Definition 2.5. Let $\phi(\mathbf{X})$ be a test function. The probability of type I and II errors of $\phi(\mathbf{X})$ for the fuzzy testing problem (2.1) is defined by $\alpha_\phi = E_0[\phi(\mathbf{X})]$, and $\beta_\phi = 1 - E_1[\phi(\mathbf{X})]$, respectively, in which $E_j[\phi(\mathbf{x})]$ means the expected value of $\phi(\mathbf{X})$ over the joint WPDF $f_j(\mathbf{x})$,$j = 0, 1$.

Note that in the case of simple crisp hypothesis against simple crisp alternative, i.e.,

$$\begin{cases} H_0 : & \theta = \theta_0 \\ H_1 : & \theta = \theta_1 \end{cases},$$

the above definition of α_ϕ and β_ϕ gives the classical probability of errors.

Example 2.3. Let X be a random variable from the normal distribution with mean μ and variance 1. Suppose that crisp hypothesis $H_0 : \mu = 0$ against crisp hypothesis $H_1 : \mu = 1$ is rejected if $X > 2$. The test function of this test is

$$\phi(X) = \begin{cases} 1, & X > 2 \\ 0, & X \leq 2. \end{cases}$$

In this problem, we have $\alpha = E_0[\phi(X)] = P(X > 2|\mu = 0) \approx 0.02$ and $\beta = 1 - E_1[\phi(X)] = P(X \leq 2|\mu = 1) \approx 0.84$.

Regarding to the definitions of error sizes, it is concluded that fuzzy hypotheses testing problem (2.1) is equivalent to the following ordinary hypotheses testing

$$\begin{cases} H'_0 : & \mathbf{X} \sim f_0 \\ H'_1 : & \mathbf{X} \sim f_1 \end{cases}$$

Definition 2.6. A fuzzy testing problem with a test function ϕ is said to be a test of (significance) level α if $\alpha_\phi \leq \alpha$, where $\alpha \in [0, 1]$. We call α_ϕ as the size of ϕ.

3 Sequential probability ratio test for fuzzy hypotheses testing

In this section, first, we define the sequential probability ratio test(henceforth SPRT) for the ordinary simple hypotheses testing and then the sequential probability ratio test for ordinary hypotheses. Concerning Section 2, we extend this case to the FHT.

Consider testing a simple null hypothesis against a simple alternative hypothesis. In other words, suppose a sample can be drawn from one of two known distributions and it is desired to test that the sample came from one distribution against the possibility that it came from the other. If $X_1, X_2, \cdots$ denote the iid r.v.'s, we want to test $H_0 : X_i \sim f_0(.)$ versus $H_1 : X_i \sim f_1(.)$. For a sample of size m, the Neyman-Pearson criterion rejects H_0 if $R_m(\mathbf{x}) = L_0(\mathbf{x})/L_1(\mathbf{x}) < k$, for some constant $k > 0$, where $\mathbf{x} = (x_1, \cdots, x_m)$, $L_j(\mathbf{x}) = \prod_{i=1}^{m} f_j(x_i)$, $j = 0, 1$. Compute sequentially $R_1, R_2, \cdots$. For fixed k_0 and k_1 satisfying $0 < k_0 < k_1$, adopt the following procedure: Take observation x_1 and compute R_1; if $R_1 \le k_0$, reject H_0; if $R_1 \ge k_1$, accept H_0; and if $k_0 < R_1 < k_1$, take observation x_2, and compute R_2; if $R_2 \le k_0$, reject H_0; if $R_2 \ge k_1$, accept H_0; and if $k_0 < R_2 < k_1$, take observation x_3, etc. The idea is to continue sampling as long as $k_0 < R_j < k_1$ and stop as soon as $R_m \le k_0$ or $R_m \ge k_1$, rejecting H_0 if $R_m \le k_0$ and accepting H_0 if $R_m \ge k_1$. The critical region of the described sequential test can be defined as $C = \bigcup_{n=1}^{\infty} C_n$, where

$$C_n = \{(x_1, \cdots, x_n) | k_0 < R_j(x_1, \cdots, x_j) < k_1, \quad j = 1, \cdots, n-1, \\ R_n(x_1, \cdots, x_n) \le k_0\}.$$

Similarly, the acceptance region can be defined as $A = \bigcup_{n=1}^{\infty} A_n$, where

$$A_n = \{(x_1, \cdots, x_n) | k_0 < R_j(x_1, \cdots, x_j) < k_1, \quad j = 1, \cdots, n-1, \\ R_n(x_1, \cdots, x_n) \ge k_1\}.$$

Definition 3.1. For fixed k_0 and k_1, a test as described above is defined to be a sequential probability ratio test (henceforth SPRT). Therefore for the SPRT, the probability of type I and II errors is calculated by $\alpha = \sum_{n=1}^{\infty} \int_{C_n} L_0(\mathbf{x}) d\mathbf{x}$, and $\beta = \sum_{n=1}^{\infty} \int_{A_n} L_1(\mathbf{x}) d\mathbf{x}$, respectively.

In the following, we briefly state some results from classical SPRT without proofs, see for more details Mood et al. [15] or Hogg and Craig [9].

Let k_0 and k_1 be defined so that the SPRT has the fixed probability of type I and II errors α and β. Then k_0 and k_1 can be approximated by $k'_0 = \alpha/(1-\beta)$ and $k'_1 = (1-\alpha)/\beta$, respectively. If α' and β' are the error sizes of the SPRT defined by k_0 and k_1, then $\alpha' + \beta' \le \alpha + \beta$.

If $Z_i = Ln(f_0(X_i)/f_1(X_i))$ with observed value $z_i = Ln(f_0(x_i)/f_1(x_i))$, then an equivalent test to the SPRT is given by the following: continue sampling as long as $Ln(k_0) < \sum_{i=1}^{m} z_i < Ln(k_1)$, and stop sampling as soon as $\sum_{i=1}^{m} z_i \le Ln(k_0)$ (and then reject H_0) or $\sum_{i=1}^{m} z_i \ge Ln(k_1)$ (and then accept H_0).

Let N be the r.v. denoting the sample size of the SPRT. The SPRT with error sizes α and β minimizes both $E[N|H_0 \text{ is true }]$ and $E[N|H_1 \text{ is true }]$ among all tests (sequential or not) which satisfy the following:

$$P(H_0 \text{ is rejected } |H_0 \text{ is true}) \leq \alpha, \text{ and } P(H_0 \text{ is accepted } |H_0 \text{ is false}) \leq \beta.$$

Using Wald's equation, it is obtained that $E[N] = E[Z_1+\cdots+Z_N]/E[Z_1]$. But $E[Z_1 + \cdots + Z_N] \approx \rho Ln(k_0) + (1-\rho)Ln(k_1)$, where $\rho = P($ rejection of $H_0)$. Hence

$$E[N|H_0 \text{ is true }] \approx \frac{\alpha Ln[\alpha/(1-\beta)] + (1-\alpha)Ln[(1-\alpha)/\beta]}{E[Z_1|H_0 \text{ is true }]},$$

and

$$E[N|H_1 \text{ is true }] \approx \frac{(1-\beta)Ln[\alpha/(1-\beta)] + \beta Ln[(1-\alpha)/\beta]}{E[Z_1|H_1 \text{ is true }]}.$$

Now, we are ready to state the SPRT for fuzzy hypotheses testing.

Definition 3.2. Let $X_1, X_2, \cdots$ be a sequence of r.v.'s from a population with PDF $f(.;\theta)$. We propose to consider testing

$$\begin{cases} H_0' : & \mathbf{X} \sim f_0 \\ H_1' : & \mathbf{X} \sim f_1 \end{cases},$$

as a SPRT for fuzzy hypotheses testing (2.1), in which $f_j(x)$ is the WPDF of $f(x;\theta)$ under the hypothesis $H_j(\theta)$, $j = 0, 1$ (see Definition 2.4), i.e. the critical region of the described SPRT for fuzzy hypotheses testing (2.1) is defined as $C = \bigcup_{n=1}^{\infty} C_n$, where

$$C_n = \{(x_1, \cdots, x_n)|k_0 < R_j(x_1, \cdots, x_j) < k_1, \quad j = 1, \cdots, n-1, \\ R_n(x_1, \cdots, x_n) \leq k_0\}.$$

Similarly, the acceptance region can be defined as $A = \bigcup_{n=1}^{\infty} A_n$, where

$$A_n = \{(x_1, \cdots, x_n)|k_0 < R_j(x_1, \cdots, x_j) < k_1, \quad j = 1, \cdots, n-1, \\ R_n(x_1, \cdots, x_n) \geq k_1\}.,$$

in which

$$\begin{aligned} R_m(x_1, \cdots, x_m) &= L_0(\mathbf{x})/L_1(\mathbf{x}) \\ &= \prod_{i=1}^{m}[f_0(x_i)/f_1(x_i)] \\ &= \prod_{i=1}^{m}[\int_\Theta H_0^*(\theta)f(x_i;\theta)d\theta / \int_\Theta H_1^*(\theta)f(x_i;\theta)d\theta]. \end{aligned}$$

Regarding to the definition of WPDF, α, β and other related concepts, all results of the ordinary SPRT are satisfied for this case, of course with some modifications. For instance, $Z_i = Ln[\int_\Theta H_0^*(\theta)f(X_i;\theta)d\theta / \int_\Theta H_1^*(\theta)f(X_i;\theta)d\theta]$, and

$$E[Z_i|Hj \text{ is true }] \\ = \int Ln[\int_\Theta H_0^*(\theta)f(x_i;\theta)d\theta / \int_\Theta H_1^*(\theta)f(x_i;\theta)d\theta]f_j(x_i)dx_i, \ j = 0, 1.$$

4 Numerical examples

In this section, we present two important examples to clarify the theoretical discussions so far.

Example 4.1. Let $X_1, X_2, \cdots$ be a sequence of iid r.v.'s from a Normal population with mean μ and variance σ^2, i.e.,

$$f(x;\theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}, \quad \mu \in \mathbb{R}, \quad \sigma > 0.$$

We want to test

$$\begin{cases} H_0 : \mu \simeq \mu_0 \\ H_1 : \mu \simeq \mu_1 \end{cases} \tag{4.1}$$

with membership functions

$$H_j(\mu) = e^{-(\mu-\mu_j)^2/(2\sigma_0^2)}, \quad j = 0, 1, \quad \mu \in \mathbb{R}, \ \sigma_0 > 0,$$

in two cases $\mu_1 > \mu_0$ and $\mu_1 < \mu_0$, using SPRT.

The pseudo-membership function of $H_j(\theta)$ is

$$H_j^*(\mu) = 1/(\sigma\sqrt{2\pi}) e^{-(\mu-\mu_j)^2/(2\sigma_0^2)}, \quad j = 0, 1, \quad \mu \in \mathbb{R}, \ \sigma_0 > 0,$$

It is easy to show that

$$\begin{aligned} f_j(x) &= \textstyle\int_{-\infty}^{\infty} H_0^*(\mu) f(x;\mu) d\mu \\ &= \sqrt{(\sigma^2 + \sigma_0^2)(2\pi)} \exp\{-(x-\mu_j)^2/(2(\sigma^2+\sigma_0^2))\}. \end{aligned}$$

Hence the FHT (4.1) is equivalent to

$$\begin{cases} H_0' : & X \sim N(\mu_0, \sigma^2 + \sigma_0^2) \\ H_1' : & X \sim N(\mu_1, \sigma^2 + \sigma_0^2) \end{cases}$$

But $Z_i = Ln[f_0(X_i)/f_1(X_i)] = X_i \frac{\mu_0-\mu_1}{\sigma^2+\sigma_0^2} - \frac{\mu_0^2-\mu_1^2}{2(\sigma^2+\sigma_0^2)}$, hence the critical region in the cases $\mu_1 > \mu_0$ and $\mu_1 < \mu_0$ are $\sum_{i=1}^n X_i > c$ and $\sum_{i=1}^n X_i < c$, respectively. Let $\alpha = 0.01$, $\beta = 0.05$, $\mu_0 = 0$, $\mu_1 = 1$, $\sigma^2 = 4$, and $\sigma_0^2 = 1$, we obtain $Ln(k_0') = -4.5539$, $Ln(k_1') = 2.98568$, $E[Z_i|H_0 \text{ is true }] = 0.1$, $E[Z_i|H_1 \text{ is true }] = -0.1$. Hence $E[N|H_0 \text{ is true }] = 29.103$ and $E[N|H_1 \text{ is true }] = 41.769$.

Example 4.2. Let $X_1, X_2, \cdots$ be a sequence of iid r.v.'s from $Ber(\theta)$ (Bernoulli distribution), $0 < \theta < 1$.

We want to test

$$\begin{cases} H_0: \ \theta \approx \theta_0 \\ H_1: \ \theta \approx \theta_1 \end{cases}, \qquad (4.2)$$

where

$$H_j(\theta) = \theta^{\alpha_j - 1}(1-\theta)^{\beta_j - 1}, \ \forall \theta \in (0,1), \ \theta_j = \alpha_j/(\alpha_j + \beta_j), \ j = 0, 1.$$

The pseudo-membership function of $H_j(\theta)$ is

$$H_j^*(\theta) = \frac{\Gamma(\alpha_j + \beta_j)}{\Gamma(\alpha_j)\Gamma(\beta_j)} \theta^{\alpha_j - 1}(1-\theta)^{\beta_j - 1}$$

It is easy to show that

$$\begin{aligned} f_j(x) &= \int_0^1 H_0^*(\theta) f(x;\theta) d\theta \\ &= \begin{cases} \beta_j/(\alpha_j + \beta_j), & x = 0 \\ \alpha_j/(\alpha_j + \beta_j), & x = 1 \end{cases} \end{aligned}$$

Hence the FHT (4.2) is equivalent to

$$\begin{cases} H_0': \ \ X \sim Ber(\alpha_0/(\alpha_0 + \beta_0)) \\ H_1': \ \ X \sim Ber(\alpha_1/(\alpha_1 + \beta_1)) \end{cases}$$

But we obtain $Z_i = X_i Ln(\alpha_0\beta_1/(\alpha_1\beta_0) + Ln(\beta_0(\alpha_1 + \beta_1)/[\beta_1(\alpha_0 + \beta_0)]$, hence the critical region in the cases $\alpha_0\beta_1/(\alpha_1\beta_0) > 1$ and $< 0\alpha_0\beta_1/(\alpha_1\beta_0) < 1$ are $\sum_{i=1}^n X_i > c$ and $\sum_{i=1}^n X_i < c$, respectively. Let $\alpha = 0.05$, $\beta = 0.1$, $\alpha_0 = 7$, $\alpha_1 = 1$, $\beta_0 = 1$, $\beta_1 = 7$, i.e., $\theta_0 = 7/8$ and $\theta_1 = 1/8$, we obtain $Ln(k_0') = -5.28827$, $Ln(k_1') = 4.60016$, $E[Z_i|H_0 \text{ is true }] = 1.45943$, $E[Z_i|H_1 \text{ is true }] = -1.45943$. Hence $E[N|H_0 \text{ is true }] = 3.118$ and $E[N|H_1$ is true $] = 3.556$.

References

[1] B.F. Arnold, Statistical tests optimally meeting certain fuzzy requirements on the power function and on the sample size, Fuzzy Sets and Systems, 75(2) (1995)365-372.

[2] B.F. Arnold, An approach to fuzzy hypotheses testing, Metrika, 44 (1996) 119-126.

[3] B.F. Arnold, Testing fuzzy hypothsis with crisp data, Fuzzy Sets and Systems, 94(2) (1998) 323-333.

[4] P. Billingsley, Probability and measure, John Wiley & Sons, Inc, (1995).

[5] M.R. Casals, Bayesian testing of fuzzy parametric hypotheses from fuzzy information, RAIRO, Oper. Res., (1993) 189-199.

[6] M.R. Casals, M. R. Gil, and P. Gil, On the use of Zadeh's probabilistic definition for testing statistical hypotheses from fuzzy information, Fuzzy Sets and Systems, 20 (1986) 175-190.

[7] G. Casella, R.L. Berger, Statistical inference, 2nd Edition, Duxbury Press,(2002).

[8] M. Delgado, M. A. Verdegay and M. A. Vila, Testing fuzzy hypotheses: a Bayesian approach, in: M.M. Gupta et al.(Eds.), Approximate Reasoning in Expert Systems, North-Holland Publishing Co, Amsterdam, (1985), pp. 307-316.

[9] R.V. Hogg, A.T. Craig, Introduction to mathematical statistics, 5th Edition, Eaglewood Cliffs, NJ: Prentice-Hall, (1995).

[10] P. Grzegorzewski, Testing statistical hypotheses with vague data, Fuzzy Sets and Systems, 112 (2000) 501-510.

[11] P. Grzegorzewski, Testing fuzzy hypotheses with vague data, in: C. Bertoluzzi, editor, Statistical Modeling, Analysis and Management of Fuzzy Data, Physica-Verlag, Heidelberg, (2002) pp. 213-225.

[12] R. Kruse and K. D. Meyer, Statistics with vague data, Reidel Pub. Comp., Dordrecht, Netherlands, (1987).

[13] E.L. Lehmann, Testing statistical hypotheses, Chapman-Hall, New York, (1994).

[14] E.L. Lehmann, G. Casella, Theory of point estimation, Springer-Verlag, New York, (1998).

[15] A.M. Mood, F.A. Graybill and D.C. Boes, Introduction to the theory of statistics, 3rd Edition, McGRAW-HILL, (1974).

[16] J. Shao, Mathematical statistics, Springer-Verlag, New York, (1999).

[17] J. Ch. Son, I. Song , H. Y. Kim, A fuzzy decision problem based on the generalized Neyman-Pearson criteria, Fuzzy Sets and Systems, 47 (1992) 65-75.

[18] S.M. Taheri, Trends in fuzzy statistics, Austrian Journal of Statistics, 32 (2003) 239-257.

[19] S.M. Taheri, J. Behboodian, Neyman-Pearson lemma for fuzzy hypotheses testing, Metrika, 49 (1999) 3-17.

[20] S.M. Taheri, J. Behboodian, A Bayesian approach to fuzzy hypotheses testing, Fuzzy Sets and Systems, 123 (2001) 39-48.

[21] S.M. Taheri, J. Behboodian, Fuzzy hypotheses testing with fuzzy data: a Bayesian approach, in: N.R. Pal amd M. Sugeno (Eds.): AFSS 2002, Physica-Verlag, Heidelberg, (2002), 527-533.

[22] H. Torabi and J. Behboodian, Likelihood ratio test for fuzzy hypotheses testing, Statistical Papers, (2007), Vol. 48, No. 3, pp. 509-522.

[23] H. Torabi, J. Behboodian and S.M. Taheri, Neyman-Pearson lemma for fuzzy hypotheses testing with vague data, Metrika, (2006), Vol. 64, No. 3, pp. 289-304.

[24] N. Watanabe, T. Imaizumi, A fuzzy statistical test of fuzzy hypotheses, Fuzzy Sets and Systems, 53 (1993) 167-178.

[25] L.A. Zadeh, Fuzzy Sets, Information and control, 8 (1965) 338-353.

Received: December, 2008