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The Most Powerful Tests for Fuzzy Hypotheses Testing with Vague Data

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Abstract

In hypotheses testing, such as other statistical problems, we may confront imprecise concepts. One case is a situation in which both hypotheses and observations are imprecise.

In this paper, we redefine some concepts about fuzzy hypotheses testing, and then we give the Neyman-Pearson lemma for fuzzy hypotheses testing with fuzzy observations. Finally, we give some applied examples.

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1 Introduction

Fuzzy set theory is a powerful and known tool for formulation and analysis of imprecise and subjective situations where exact analysis is either difficult or impossible.

Decision making in classical statistical inference is based on crispness of data, random variables, exact hypotheses, decision rules and so on. As there are many different situations in which the above assumptions are rather irrealistic, there have been some attempts to analyze these situations with fuzzy set theory proposed by Zadeh [38].

Some methods in descriptive statistics with vague data and some aspects of statistical inference is proposed in [17]. Fuzzy random variables were introduced by Kwakernaak [18], Puri and Ralescu [24] as a generalization of compact random sets, Kruse and Mayer[17] and were developed by others such as Juning and wang [12], Ralescu [25], López-Díaz and Gil [22], [23], and Liu [21].

In this paper, because of our main purpose (statistical inference about a parametric population with fuzzy data), we only consider and discuss Fuzzy-valued random variables which associate with an ordinary random variable.

One of the primary purpose of statistical inference is to test hypotheses. In the traditional approach to hypotheses testing all the concepts are precise and well defined(see e.g. [19], [8], and [30]). However, if we introduce vagueness into hypotheses, we face quite new and interesting problems. Arnold [2] considered statistical tests under fuzzy constraints on the type I and II errors. Testing fuzzy hypotheses was discussed by Arnold [1] and [3], Delgado et al [10], Saade and Schwarzlader [29], Saade [28], Watanabe and Imaizumi [37], Taheri and Behboodian [33], [34], and [35], and Grzegorzewski [15] and [16]. Kruse and Meyer [17], Taheri and Behboodian [35] considered problem of testing vague hypotheses in the presence of vague hypothesis. Up to the present testing hypotheses with fuzzy data was considered by Casals et al [7], and Son et al [31], see for more references about fuzzy testing problem, Taheri [36].Also, for more details about ordinary sequential probability ratio test, see e.g. [14], [26].

This paper is organized in the following way:
In Section 2, we provide some definitions and preliminaries. Fuzzy hypotheses testing is defined and reviewed in Section 3. The Neyman-Pearson lemma for fuzzy hypotheses testing with vague data is introduced in Section 4 and finally, some applied examples are given in Section 5.

2 preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A random variable (henceforth RV) X is a measurable function from $(\Omega, \mathcal{F}, P)$ to $(\mathbb{R}, \mathcal{B}, P_X)$, where P_X is a probability measure induced by X and is called the distribution of the RV X, i.e.,

$$P_X(A) = P(X \in A) = \int_{X \in A} dP, \quad A \in \mathcal{B}.$$

Using “the change of variable rule”, (see for example [5], pp. 215 and 216 or [30], P. 13), we have

$$P_X(A) = \int_A dPoX^{-1}(x) = \int_A dP_X(x), \quad A \in \mathcal{B}.$$

If P_X is dominated by a σ-finite measure ν, i.e., $P_X << \nu$, then using the Radon-Nikodym theorem, (see for example [5] pp. 422 and 423 or [30], P. 14), we have

$$P_X(A) = \int_A f(x) d\nu(x),$$

where $f(x)$ is the Radon-Nikodym derivative of P_X with respect to ν and is called the probability density function (henceforth PDF) of X with respect to ν.

In the statistical texts, the measure ν usually is "counting measure" or "Lebesgue measure"; hence $P_X(A)$ is calculated by $\sum_{x \in A} f(x)$ or $\int_A f(x)dx$, respectively.

Let $\mathcal{X} = \{x \in \mathbb{R} | f(x) > 0\}$. The set $\mathcal{X}$ is usually called "support" or "sample space" of X. A random vector $\mathbf{X} = (X_1, ..., X_n)$ is said a random sample of size n from a population with PDF $f(x)$, if X_i's are independent and whose PDF's are $f(x)$, i.e., X_i's are independent and identically distributed (IID). In this case, we have

$$f_{\mathbf{X}}(\mathbf{x}) = f(x_1)...f(x_n), \quad \forall x_i \in \mathbb{R},$$

where $\mathbf{x} = (x_1, ..., x_n)$ is an observed value of $\mathbf{X}$.

In the following, we present two definitions from introduction of Casals et al. [7], but in a slightly different way.

Definition 2.1. A fuzzy sample space $\tilde{\mathcal{X}}$ is defined as a fuzzy partition (Ruspini partition) of $\mathcal{X}$, i.e., a set of fuzzy subsets of $\mathcal{X}$ whose membership functions are Borel measurable and satisfy the orthogonality constraint: $\sum_{\tilde{x} \in \tilde{\mathcal{X}}} \mu_{\tilde{x}}(x) = 1$, for each $x \in \mathcal{X}$.

Definition 2.3. A Fuzzy-valued random sample (henceforth FVRS) of size n, say $\tilde{\mathbf{X}} = (\tilde{X}_1, ..., \tilde{X}_n)$, associated with PDF $f(x)$ is a measurable function from Ω to $\tilde{\mathcal{X}}^n$ (the support of $\tilde{\mathbf{X}}$), such that its PDF is given by

$$\tilde{f}(\tilde{x}_1, ..., \tilde{x}_n) = \tilde{P}(\tilde{\mathbf{X}} = \tilde{\mathbf{x}}) = \int_{\mathcal{X}^n} \prod_{i=1}^{n} \mu_{\tilde{x}_i}(x_i) f(x_i) d\nu(x_i),$$

Where $\mathcal{X}^n$ is the support of random sample $\mathbf{X}$.

The above definition is according to Zadeh [39]. Note that using Fubini's theorem (see [5] pp. 233-234), we obtain independency of $\tilde{X}_i$'s, i.e.,

$$\tilde{f}(\tilde{x}_1, ..., \tilde{x}_n) = \tilde{f}(\tilde{x}_1)...\tilde{f}(\tilde{x}_n), \quad \forall \tilde{x}_i \in \tilde{\mathcal{X}}$$

where

$$\tilde{f}(\tilde{x}_i) = \int_{\mathcal{X}} \mu_{\tilde{x}_i}(x_i) f(x_i) d\nu(x_i),$$

and $\tilde{f}(\tilde{x}_i)$ is the PDF of Fuzzy-valued random variable (henceforth FVRV) $\tilde{X}_i$, for each $i = 1, \cdots, n$. The $\tilde{f}(\tilde{x}_i)$ really is a PDF on $\tilde{\mathcal{X}}$, because by the orthogonality property of $\mu_{\tilde{x}_i}$'s, we have

$$\begin{aligned}\sum_{\tilde{x}_i\in\tilde{\mathcal{X}}}\tilde{f}(\tilde{x}_i) &= \sum_{\tilde{x}_i\in\tilde{\mathcal{X}}}\int_{\mathcal{X}}\mu_{\tilde{x}_i}(x_i)f(x_i)d\nu(x_i)\\ &= \int_{\mathcal{X}}f(x_i)(\sum_{\tilde{x}_i\in\tilde{\mathcal{X}}}\mu_{\tilde{x}_i}(x_i))d\nu(x_i)\\ &= \int_{\mathcal{X}}f(x_i)d\nu(x_i)=1.\end{aligned}$$

Theorem 2.1 . *If g is a measurable function from $\tilde{\mathcal{X}}^n$ to $\mathbb{R}$, then $Y=g(\tilde{\mathbf{X}})$ is an ordinary random variable.*

Proof. $\tilde{\mathbf{X}}$ is a measurable function from Ω to $\tilde{\mathcal{X}}^n$ and g is a measurable function from $\tilde{\mathcal{X}}^n$ to $\mathbb{R}$. Hence $g(\tilde{\mathbf{X}}(\omega))=go\tilde{\mathbf{X}}(\omega)$ is a composition of two measurable functions, therefore is measurable from Ω to $\mathbb{R}$. (see [5], P. 182.) □

Note that using Theorem 2.1, we can define and use all related concepts of the ordinary random variables, such as expectation, variance, etc.

Theorem 2.2 . *Let $\tilde{\mathbf{X}}$ be a Fuzzy-valued random sample with fuzzy sample space $\tilde{\mathcal{X}}^n$, and g be a measurable function from $\tilde{\mathcal{X}}^n$ to $\mathbb{R}$. The expectation of $g(\tilde{\mathbf{X}})$ is calculated by*

$$E\left[g(\tilde{\mathbf{X}})\right]=\sum_{\tilde{\mathbf{x}}\in\tilde{\mathcal{X}}^n}g(\tilde{\mathbf{x}})\tilde{f}(\tilde{\mathbf{x}}).$$

Proof. Using the change of variable rule and the Radon-Nikodym theorem, we have

$$\begin{aligned}E\left[g(\tilde{\mathbf{X}})\right] &= \int_{\Omega}g(\tilde{\mathbf{X}}(\omega))dP(\omega)\\ &= \int_{\tilde{\mathcal{X}}^n}g(\tilde{\mathbf{x}})dPo\tilde{\mathbf{X}}^{-1}(\tilde{\mathbf{x}})\\ &= \sum_{\tilde{\mathbf{x}}\in\tilde{\mathcal{X}}^n}g(\tilde{\mathbf{x}})\tilde{f}(\tilde{\mathbf{x}}).\end{aligned}$$

□

For more details about the properties of the ordinary RV's and their moments, see for e.g. [4], [5], [9], [11], and [27].

In this paper, we suppose that the PDF of the population is known but it has an unknown parameter $\theta\in\Theta$. In this case, we index $\tilde{f}$ by θ and we write $\tilde{f}(\tilde{\mathbf{x}};\theta)$.

Example 2.1. Let X be distributed according to the Bernoulli distribution with parameter θ, i.e.,

$$f(x;\theta)=\theta^x(1-\theta)^{1-x},\quad x=0,1,\quad 0<\theta<1.$$

We have $\mathcal{X}=\{0,1\}$. Let $\tilde{x}_1$ and $\tilde{x}_2$ be two fuzzy subsets of $\mathcal{X}$ with membership functions $\mu_{\tilde{x}_1}(x)$ and $\mu_{\tilde{x}_2}(x)$ as follows

$$\mu_{\tilde{x}_1}(x)=\begin{cases}0.9, & x=0\\ 0.1, & x=1\end{cases}$$

and

$$\mu_{\tilde{x}_2}(x) = \begin{cases} 0.1, & x = 0 \\ 0.9, & x = 1 \end{cases},$$

respectively. Note that $\tilde{x}_1$ and $\tilde{x}_2$ are stated "approximately zero" and "approximately one" values, respectively. In this example, the support of $\tilde{X}$ is $\tilde{\mathcal{X}} = \{\tilde{x}_1, \tilde{x}_2\}$. Therefore the PDF of $\tilde{X}$ is

$$\begin{aligned} \tilde{f}(\tilde{x};\theta) &= \textstyle\sum_{\mathcal{X}} \mu_{\tilde{x}_1}(x) f(x;\theta) \\ &= \begin{cases} 0.9(1-\theta) + 0.1\theta, & \tilde{x} = \tilde{x}_1 \\ 0.1(1\theta) + 0.9\theta, & \tilde{x} = \tilde{x}_2 \end{cases} \\ &= \begin{cases} 0.9 - 0.8\theta, & \tilde{x} = \tilde{x}_1 \\ 0.1 + 0.8\theta, & \tilde{x} = \tilde{x}_2 \end{cases}. \end{aligned}$$

Let $Y = \begin{cases} 0.1, & \tilde{x} = \tilde{x}_1 \\ 0.9, & \tilde{x} = \tilde{x}_2 \end{cases}$. Note that Y is a measurable function from $\tilde{\mathcal{X}}$ to $\mathbb{R}$ and therefore is a classical random variable. In the following, we calculate the mean and the variance of Y. The PDF of Y is

$$f_Y(y) = \begin{cases} 0.9 - 0.8\theta, & y = 0.1 \\ 0.1 + 0.8\theta, & y = 0.9 \end{cases}.$$

Therefore, the expectation and the variance of Y are calculated as the following

$$E[Y] = 0.1(0.9 - 0.8\theta) + 0.9(0.1 + 0.8\theta) = 0.18 + 0.64\theta$$

and

$$\begin{aligned} E[Y^2] = &\ 0.01(0.9 - 0.8\theta) + 0.81(0.1 + 0.8\theta) \\ &= 0.09 + 0.64\theta \Rightarrow \\ Var[Y] &= 0.09 + 0.64\theta - (0.18 + 0.64\theta)^2 \\ &= 0.0576 + 0.4096\theta - 0.4096\theta^2. \end{aligned}$$

3 Fuzzy hypotheses testing

In this section, we introduce some concepts about fuzzy hypotheses testing (henceforth FHT).

Definition 3.1. Any hypothesis of the form "$H : \theta$ is $H(\theta)$" is called a fuzzy hypothesis, where "$H : \theta$ is $H(\theta)$" implies that θ is in a fuzzy set of Θ (the parameter space) with membership function $H(\theta)$ i.e. a function from Θ to $[0, 1]$.

Note that the ordinary hypothesis $H : \theta \in \Theta$ is a fuzzy hypothesis with membership function $H(\theta) = 1$ at $\theta \in \Theta$, and zero otherwise, i.e., the indicator function of the crisp set Θ.

Example 3.1. Let θ be the parameter of a Bernoulli distribution. Consider the following function:

$$H(\theta)=\begin{cases} 2\theta, & 0<\theta<1/2 \\ 2-2\theta, & 1/2\leq\theta<1 \end{cases}$$

The hypothesis "$H:\theta$ is $H(\theta)$" is a fuzzy hypothesis and it means that "θ is approximately 1/2".

In FHT with fuzzy data, the main problem is testing

$$\begin{cases} H_0: \ \theta \ is \ H_0(\theta) \\ H_1: \ \theta \ is \ H_1(\theta) \end{cases}, \tag{3.1}$$

according to a FVRS $\tilde{\mathbf{X}}=(\tilde{X}_1,\cdots,\tilde{X}_n)$ from a parametric fuzzy population with the PDF $\tilde{f}(\tilde{x};\theta)$.

In the following, we give some definitions in FHT theory with fuzzy data.

Definition 3.2. The normalized membership function of $H_j(\theta)$ is defined by

$$H_j^*(\theta)=H_j(\theta)/\int_\Theta H_j(\theta)d\theta, \quad j=0,1.$$

providing to $\int_\Theta H_j(\theta)d\theta<\infty$. Substitute $\int$ by $\sum$ in discrete cases.

Note that the normalized membership function is not necessarily a membership function, i.e., it may be greater than 1 for some values of θ.

In FTH with fuzzy data, like the traditional hypotheses testing, we must give a test function $\tilde{\Phi}(\tilde{\mathbf{X}})$. In the following, we define fuzzy test function.

Definition 3.3. Let $\tilde{\mathbf{X}}$ be a FVRS with the PDF $\tilde{f}(\tilde{\mathbf{x}};\theta)$. $\tilde{\Phi}(\tilde{\mathbf{X}})$ is called a fuzzy test function if it is the probability of rejecting H_0 providing to $\tilde{\mathbf{X}}=\tilde{\mathbf{x}}$ is observed.

Definition 3.4. Let FVRV $\tilde{X}$ have the PDF $\tilde{f}(\tilde{x};\theta)$. Under the hypothesis $H_j(\theta)$, $j=0,1$, the weighted probability density function (henceforth WPDF) of $\tilde{X}$ is defined by

$$\tilde{f}_j(x)=\int_\Theta H_j^*(\theta)\tilde{f}(\tilde{x};\theta)d\theta,$$

i.e., the expected value of $\tilde{f}(\tilde{x};\theta)$ over $H_j^*(\theta)$, $j=0,1$. If $\tilde{\mathbf{X}}$ is a FVRS from the PDF $\tilde{f}(.;\theta)$, then the joint WPDF of $\tilde{\mathbf{X}}$ is defined by

$$\tilde{f}_j(\tilde{\mathbf{x}})=\prod_{i=1}^n\tilde{f}_j(\tilde{x}_i).$$

Remark 3.1. $\tilde{f}_j(\tilde{x})$ is a PDF, since $\tilde{f}_j(\tilde{x})$ is nonnegative and

$$\begin{array}{ll} \sum_{\tilde{\mathcal{X}}} \tilde{f}_j(\tilde{x}) & = \sum_{\tilde{\mathcal{X}}} \int_\Theta H_j^*(\theta)\tilde{f}(\tilde{x};\theta)d\theta \\ & = \int_\Theta H_j^*(\theta)[\sum_{\tilde{\mathcal{X}}} \tilde{f}(\tilde{x};\theta)]d\theta \\ & = \int_\Theta H_j^*(\theta)d\theta \\ & = 1 \end{array}.$$

Hence $\tilde{f}_j(\tilde{x}_1, \cdots, \tilde{x}_n)$ is also a joint PDF.

Remark 3.2. If H_j is the crisp hypothesis $H_j : \theta = \theta_j$, then $\tilde{f}_j(x) = \tilde{f}(\tilde{x}; \theta_j)$ and then $\tilde{f}_j(\tilde{x}_1, \cdots, \tilde{x}_n) = \tilde{f}(\tilde{x}_1, \cdots, \tilde{x}_n; \theta_j)$, $j = 0, 1$.

Definition 3.5. Let $\tilde{\Phi}(\tilde{\mathbf{X}})$ be a fuzzy test function. The probability of type I and II errors of $\tilde{\Phi}(\tilde{\mathbf{X}})$ for the fuzzy testing problem (3.1) is defined by $\alpha_{\tilde{\Phi}} = E_0[\tilde{\Phi}(\tilde{\mathbf{X}})]$, and $\beta_{\tilde{\Phi}} = 1 - E_1[\tilde{\Phi}(\tilde{\mathbf{X}})]$, respectively, in which $E_j[\tilde{\Phi}(\tilde{\mathbf{X}})]$ means the expected value of $\tilde{\Phi}(\tilde{\mathbf{X}})$ over the joint WPDF $\tilde{f}_j(\tilde{\mathbf{x}})$, $j = 0, 1$.

Note that in the case of simple crisp hypothesis against simple crisp alternative, i.e.,

$$\begin{cases} H_0 : & \theta = \theta_0 \\ H_1 : & \theta = \theta_1 \end{cases},$$

and crisp observations, the above definition of $\alpha_{\tilde{\Phi}}$ and $\beta_{\tilde{\Phi}}$ gives the classical probability of errors.

Regarding to definitions of error sizes, it is concluded that fuzzy hypotheses testing (3.1) is equivalent to the following ordinary hypotheses testing

$$\begin{cases} H_0' : & \tilde{\mathbf{X}} \sim \tilde{f}_0 \\ H_1' : & \tilde{\mathbf{X}} \sim \tilde{f}_1 \end{cases} \qquad (2.2)$$

Definition 3.6. A fuzzy testing problem with a fuzzy test function $\tilde{\Phi}$ is said to be a test of (significance) level α if $\alpha_{\tilde{\Phi}} \leq \alpha$, where $\alpha \in [0, 1]$. We call $\alpha_{\tilde{\Phi}}$ as the size of $\tilde{\Phi}$.

Definition 3.7. A fuzzy test $\tilde{\Phi}$ of level α is said to be the most powerful test of level α if $\beta_{\tilde{\Phi}} \leq \beta^*_{\tilde{\Phi}}$, for all test $\tilde{\Phi}^*$ of level α.

4 Neyman-Pearson lemma for FHT with vague data

In this section, we state and prove the Neyman-Pearson lemma for FHT with vague data.

Theorem 4.1 . *Let $\tilde{\mathbf{X}} = (\tilde{X}_1, ..., \tilde{X}_n)$ be a FVRS with the observed value $\tilde{\mathbf{x}} = (\tilde{x}_1, ..., \tilde{x}_n)$, and the PDF $f(\tilde{\mathbf{x}};\theta)$, where $\theta \in \Theta$ is a parameter. For testing*

$$\begin{cases} H_0: \ \theta \ is \ H_0(\theta) \\ H_1: \ \theta \ is \ H_1(\theta) \end{cases},$$

(a). *any test with fuzzy test function*

$$\tilde{\Phi}(\tilde{\mathbf{x}}) = \begin{cases} 1, & if \ \tilde{f}_0(\tilde{\mathbf{x}})/\tilde{f}_1(\tilde{\mathbf{x}}) < k \\ \delta(\tilde{\mathbf{x}}), & if \ \tilde{f}_0(\tilde{\mathbf{x}})/\tilde{f}_1(\tilde{\mathbf{x}}) = k \\ 0, & if \ \tilde{f}_0(\tilde{\mathbf{x}})/\tilde{f}_1(\tilde{\mathbf{x}}) > k \end{cases}, \qquad (4.1)$$

for some $k \geq 0$ and $0 \leq \delta(\tilde{x}) \leq 1$, is the MP test of level α, where $\alpha = \alpha_\Phi$ and $\tilde{f}_j(\tilde{\mathbf{x}})$ is the joint WPDF of $\tilde{f}(\tilde{x};\theta)$ over $H_j^*(\theta)$, i.e., $\tilde{f}_j(\tilde{\mathbf{x}}) = \prod_{i=1}^n \tilde{f}_j(\tilde{x}_i) = \prod_{i=1}^n \int_\Theta H_j^*(\theta) f(\tilde{x}_i;\theta) d\theta$, $j = 0, 1$.

If $k = 0$, then the test

$$\tilde{\Phi}(\tilde{\mathbf{x}}) = \begin{cases} 1, & if \ \tilde{f}_0(\tilde{\mathbf{x}}) = 0 \\ 0, & if \ \tilde{f}_0(\tilde{\mathbf{x}}) > 0 \end{cases}, \qquad (4.2)$$

is the MP test of size zero.

(b). *for $0 \leq \alpha \leq 1$, there is a test of the form (4.1) or (4.2) with $\delta(\tilde{x}) = \delta$ (a constant), for which $\alpha_{\tilde{\Phi}} = \alpha$.*

Proof. (a). We have $\alpha_{\tilde{\Phi}} = \alpha$, hence $E_0[\tilde{\Phi}(\tilde{\mathbf{X}})] = \sum_{\tilde{\mathbf{x}} \in \tilde{\mathcal{X}}^n} f_0(\tilde{\mathbf{x}}) = \alpha$. Let $\tilde{\Phi}^*(\tilde{\mathbf{x}})$ be another fuzzy test of level α, i.e., $E_0[\tilde{\Phi}^*(\tilde{\mathbf{X}})] \leq \alpha$; we must prove that $\beta_{\tilde{\Phi}^*} \geq \beta_{\tilde{\Phi}}$.

Let $\tilde{A} = \{\tilde{\mathbf{x}} | \tilde{f}_0(\tilde{\mathbf{x}})/\tilde{f}_1(\tilde{\mathbf{x}}) < k\}$. If $\tilde{\mathbf{x}} \in \tilde{A}$ or $\tilde{\mathbf{x}} \in \tilde{A}$, then

$$[\tilde{\Phi}(\tilde{\mathbf{x}}) - \tilde{\Phi}^*(\tilde{\mathbf{x}})][\tilde{f}_0(\tilde{\mathbf{x}}) - k\tilde{f}_1(\tilde{\mathbf{x}})] \leq 0,$$

Hence

$$\sum_{\tilde{\mathcal{X}}^n} \Big[[\tilde{\Phi}(\tilde{\mathbf{x}}) - \tilde{\Phi}^*(\tilde{\mathbf{x}})][\tilde{f}_0(\tilde{\mathbf{x}}) - k\tilde{f}_1(\tilde{\mathbf{x}})] \Big] \leq 0.$$

Thus

$$\sum_{\tilde{\mathcal{X}}^n} [\tilde{\Phi}(\tilde{\mathbf{x}}) - \tilde{\Phi}^*(\tilde{\mathbf{x}})]\tilde{f}_0(\tilde{\mathbf{x}}) \leq k \sum_{\tilde{\mathcal{X}}^n} [\tilde{\Phi}(\tilde{\mathbf{x}}) - \tilde{\Phi}^*(\tilde{\mathbf{x}})]\tilde{f}_1(\tilde{\mathbf{x}}),$$

Therefore using Theorem 2.2, we have

$$\alpha_{\tilde{\Phi}} - \alpha_{\tilde{\Phi}^*} \leq k[(1 - \beta_{\tilde{\Phi}}) - (1 - \beta_{\tilde{\Phi}^*})].$$

But both the left side of the above inequality and k are non-negative, hence $\beta_{\tilde{\Phi}^*} \geq \beta_{\tilde{\Phi}}$.

Now consider the case $k = 0$. In this case, using Theorem 2.2, we have

$$\begin{aligned} \alpha_{\tilde{\Phi}} &= E_0[\tilde{\Phi}(\tilde{\mathbf{X}})] \\ &= \textstyle\sum_{\{\tilde{\mathbf{x}}|\tilde{f}_0(\tilde{x})=0\}} \tilde{f}_0(\tilde{x}) \\ &= 0. \end{aligned}$$

Now consider a test Φ^* with size zero. we define $\tilde{A} = \{\tilde{\mathbf{x}}|\tilde{f}_0(\tilde{\mathbf{x}}) = 0\}$. If $\tilde{\mathbf{x}} \in \tilde{A}$ or $\tilde{\mathbf{x}} \notin \tilde{A}$, It can been easily proved that $\tilde{\Phi}(\tilde{\mathbf{x}}) - \tilde{\Phi}^*(\tilde{\mathbf{x}}) \geq 0$; therefore

$$\beta_{\Phi^*} - \beta_{\Phi} = \sum_{\tilde{\mathcal{X}}^n} [\tilde{\Phi}(\tilde{\mathbf{x}}) - \tilde{\Phi}^*(\tilde{\mathbf{x}})]\tilde{f}_1(\theta) \geq 0.$$

(b). We discuss only the case $0 < \alpha \leq 1$, since the MP test of size zero is given by (4.2); ($\delta = 1$). The size of the test of form (4.1) must be α; hence we must have $E_0[\tilde{\Phi}(\tilde{\mathbf{X}})] = \alpha$. Let $\delta(\tilde{\mathbf{x}}) = \delta$ (be a constat). We prove that there exists δ, such that $\alpha_{\tilde{\Phi}} = \alpha$. Using Theorem 2.2, we have

$$\begin{aligned} \alpha_{\Phi} &= E_0[\tilde{\Phi}(\tilde{\mathbf{X}})] \\ &= P_0(\tilde{f}_0(\tilde{\mathbf{X}})/\tilde{f}_1(\tilde{\mathbf{X}}) < k) + \delta P_0(\tilde{f}_0(\tilde{\mathbf{X}})/\tilde{f}_1(\tilde{\mathbf{X}}) = k) \\ &= \alpha \end{aligned} \tag{4.3}$$

If there exist a k_0 such that

$$P_\theta(\tilde{f}_0(\tilde{\mathbf{X}})/\tilde{f}_0(\tilde{\mathbf{X}}) < k_0) = \alpha,$$

then we take $\delta = 0$ and $k = k_0$; otherwise, we define a RV R by $R = \tilde{f}_0(\tilde{\mathbf{X}})/\tilde{f}_0(\tilde{\mathbf{X}})$. Note that R is a real-valued RV defined from $\tilde{\mathcal{X}}^n$ to $\mathbb{R}$. We know that $P_0(R < k)$ is a non-decreasing and right continuous function of k. Therefore there exists a k_0, such that

$$P_0(R < k_0) < \alpha < P_0(R \leq k_0).$$

Therefore, in this case we take

$$\delta = \frac{\alpha - P_0(R < k_0)}{P_0(R = k_0)},$$

which satisfies (4.3). □

Note that using Theorem 4.1, the MP critical region is

$$C = \left\{ \tilde{\mathbf{X}} | \sum_{i=1}^{n} Y_i < k \right\},$$

where $Y_i = Ln(\tilde{f}_o(\tilde{X}_i)/\tilde{f}_1(\tilde{X}_i))$. It is clear that the Y_i's are IID, since $\tilde{X}_i$'s are IID. Therefore for finding k and then $\alpha_{\tilde{\Phi}}$ we can obtain the exact distribution of the RV $\sum_{i=1}^{n} Y_i$ for small n, see Example 5.1, or use the central limit theorem for large n, $n \geq 30$, under $H_0' : \tilde{X}_i \sim \tilde{f}_0(\tilde{x}_i)$. For $\beta_{\tilde{\Phi}}$, we can obtain the exact or asymptotically distribution of the RV $\sum_{i=1}^{n}$ under $H_1 : \tilde{X}_i \sim \tilde{f}_1(\tilde{x}_i)$.

5 Some Examples

In this section, we present some important examples to clarify the theoretical discussions so far.

Example 5.1. Let $X_1, X_2, \cdots, X_n$ be a random sample from $Ber(\theta)$, Bernoulli distribution, $0 < \theta < 1$.

We want to test

$$\begin{cases} H_0: \ \theta \approx \theta_0 \\ H_1: \ \theta \approx \theta_1 \end{cases},$$

where

$$H_0(\theta) = \theta^{\theta_0/(1-\theta_0)}(1-\theta), \ \theta \in (0,1),$$

and

$$H_1(\theta) = \theta(1-\theta)^{(1-\theta_1)/\theta_1}, \ \theta \in (0,1),$$

according to two fuzzy data (fuzzy subsets of $\mathcal{X} = \{0,1\}$) $\tilde{x}_I$ and $\tilde{x}_{II}$ where their membership functions are defined by

$$\mu_{\tilde{x}_I}(x) = \begin{cases} 0.9 & , \quad x = 0 \\ 0.1 & , \quad x = 1 \end{cases} \text{ and } \mu_{\tilde{x}_{II}}(x) = \begin{cases} 0.1 & , \quad x = 0 \\ 0.9 & , \quad x = 1 \end{cases}.$$

The normalized membership function of $H_0(\theta)$ and $H_1(\theta)$ is

$$H_0^*(\theta) = [(2-\theta_0)/(1-\theta_0)^2]\theta^{\theta_0/(1-\theta_0)}(1-\theta),$$

and

$$H_1^*(\theta) = [(\theta_1+1)/\theta_1^2]\theta(1-\theta)^{(1-\theta_1)/\theta_1},$$

respectively.

If we denote this FVRV, its fuzzy observation and its PDF by $\tilde{X}$, $\tilde{x}$, and $f(\tilde{x};\theta)$, respectively, then using Example 2.1, we have

$$f(\tilde{x};\theta) = \begin{cases} 0.9 - 0.8\theta & , \quad \tilde{x} = \tilde{x}_I \\ 0.1 + 0.8\theta & , \quad \tilde{x} = \tilde{x}_{II} \end{cases}.$$

It is easy to show that

$$\tilde{f}_0(\tilde{x}) = \begin{cases} 0.26 & , \quad \tilde{x} = \tilde{x}_I \\ 0.74 & , \quad \tilde{x} = \tilde{x}_{II} \end{cases} \quad \text{and } \tilde{f}_1(\tilde{x}) = \begin{cases} 0.74 & , \quad \tilde{x} = \tilde{x}_I \\ 0.26 & , \quad \tilde{x} = \tilde{x}_{II} \end{cases}.$$

Hence for $i = 1, 2, \cdots, n$, we obtain

$$y_i = Ln(\tilde{f}_0(\tilde{x}_i)/\tilde{f}_1(\tilde{x}_i)) = \begin{cases} -1.04597 & , \quad \tilde{x}_i = \tilde{x}_I \\ \ \ 1.04597 & , \quad \tilde{x}_i = \tilde{x}_{II} \end{cases}.$$

But using Theorem 4.1, the MP critical region is

$$C = \left\{ \tilde{\mathbf{x}} \middle| \sum_{i=1}^{n} y_i < k \right\}.$$

The PDF of Y_i under H_0 and H_1 is

$$f_0(y_i) = \begin{cases} 0.26 & , \quad y_i = -1.04597 \\ 0.74 & , \quad y_i = 1.04597 \end{cases} \quad \text{and } f_1(y_i) = \begin{cases} 0.74 & , \quad y_i = -1.04597 \\ 0.26 & , \quad y_i = 1.04597 \end{cases} ,$$

respectively.

Let $z_i = (1/2.09194)(y_i + 1.04597)$. Hence the MP critical region is $C = \{\tilde{\mathbf{x}} | \sum_{i=1}^{n} z_i < c\}$. Under H_0, $Z_i \sim Ber(0.74)$ and under H_1, $Z_i \sim Ber(0.26)$. Therefore we can easily find c for a test of known level or size α. Assume that $n = 10$, $\alpha = 0.05$. We obtain $P_0(\sum_{i=1}^{10} Z_i \leq 4) = 0.0239$ and $P_0(\sum_{i=1}^{10} Z_i \leq 5) = 0.09035$. Thus the critical region $\sum Z_i \leq 4$ is of level 0.05. The test function of the MP test of size α, using Theorem 4.1(b), is

$$\tilde{\Phi}(\tilde{\mathbf{X}}) = \begin{cases} 1 & , \quad \sum Z_i \leq 4 \\ 0.393 & , \quad \sum Z_i = 5 \\ 0 & , \quad \sum Z_i \geq 6 \end{cases}$$

The probability of type II error for this test is

$$\beta_{\tilde{\Phi}} = 1 - E_1[\tilde{\Phi}(\tilde{\mathbf{X}})] = 1 - 0.90965 - 0.393(0.06644) = 0.0642.$$

For $n \geq 30$, we can use the central limit theorem. we obtain $c = 0.74n - \sqrt{(0.74 \times 0.26)n}\, z_\alpha$, where z_α is the α-right side quantile of the standard normal distribution. For instance, let $n = 40$ and $\alpha = 0.025$. We obtain $c = 24.163$ and $\beta_{\tilde{\Phi}} = 1 - P_1(Z < 4.961) \approx 0$, where $Z \sim N(0, 1)$.

Example 5.2. Let $X_1, X_2, \cdots, X_n$ be a random sample from a Normal population with mean μ and variance σ^2, i.e.,

$$f(x; \theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}, \quad \mu \in \mathbb{R}, \quad \sigma > 0.$$

We want to test

$$\begin{cases} H_0 : \mu \approx \mu_0 \\ H_1 : \mu \approx \mu_1 \end{cases}$$

with membership functions

$$H_j(\mu) = e^{-(\mu-\mu_j)^2/(2\sigma_0^2)}, \quad j = 0, 1, \quad \mu \in \mathbb{R}, \ \sigma_0 > 0,$$

in two cases $\mu_1 > \mu_0$ and $\mu_1 > \mu_0$, according to three fuzzy data (fuzzy subsets of $\mathcal{X} = (-\infty, +\infty)$) $\tilde{x}_I$, $\tilde{x}_{II}$, and $\tilde{x}_{III}$ whose membership functions are defined by

$$\mu_{\tilde{x}_I}(x) = \begin{cases} 1 - e^{-x^2/2} & , \quad x < 0 \\ 0 & , \quad x \geq 0 \end{cases} , \qquad \mu_{\tilde{x}_{II}}(x) = e^{-x^2/2}, \quad x \in \mathbb{R}, \quad \text{and}$$

$$\mu_{\tilde{x}_{III}}(x) = \begin{cases} 0 & , \quad x < 0 \\ 1 - e^{-x^2/2} & , \quad x \geq 0 \end{cases} .$$

The fuzzy subsets $\tilde{x}_I, \tilde{x}_{II}$, and $\tilde{x}_{III}$ can be interpreted as the values where are "very small", "near to zero", and "very large".

Note that μ's are measurable and satisfy the orthogonality constraint; see Definitions 2.2 and 2.3.

The normalized membership function of $H_j(\theta)$ is

$$H_j^*(\mu) = 1/(\sigma_0\sqrt{2\pi})e^{-(\mu-\mu_j)^2/(2\sigma_0^2)}, \quad j=0,1, \quad \mu\in\mathbb{R},\ \sigma_0>0,$$

Denote this FVRV, its fuzzy observation and its PDF by $\tilde{X}$, $\tilde{x}$, and $f(\tilde{x};\theta)$, respectively. Let $\mu_0=0$, $\mu_1=1$, $\sigma^2=4$, and $\sigma_0^2=0.5$. It is easy to show that

$$\tilde{f}_0(\tilde{x}) = \begin{cases} 0.3796 & , \quad \tilde{x}=\tilde{x}_I \\ 0.2408 & , \quad \tilde{x}=\tilde{x}_{II} \\ 0.3796 & , \quad \tilde{x}=\tilde{x}_{III} \end{cases},$$

and

$$\tilde{f}_1(\tilde{x}) = \begin{cases} 0.2907 & , \quad \tilde{x}=\tilde{x}_I \\ 0.2339 & , \quad \tilde{x}=\tilde{x}_{II} \\ 0.4754 & , \quad \tilde{x}=\tilde{x}_{III} \end{cases}.$$

Hence for $i=1,2,\cdots,n$, we obtain

$$y_i = Ln(\tilde{f}_0(\tilde{x}_i)/\tilde{f}_1(\tilde{x}_i)) = \begin{cases} 0.2668 & , \quad \tilde{x}_i=\tilde{x}_I \\ 0.0291 & , \quad \tilde{x}_i=\tilde{x}_{II} \\ -0.2250 & , \quad \tilde{x}_i=\tilde{x}_{III} \end{cases}.$$

Therefore the MP critical region is $C=\{\tilde{\mathbf{x}}|\sum y_i<k\}$. But the PDF of Y_i under H_0 and H_1 is

$$f_0(y_i) = \begin{cases} 0.3796 & , \quad y_i = \ \ 0.2668 \\ 0.2408 & , \quad y_i = \ \ 0.0291 \\ 0.3796 & , \quad y_i = -0.2250 \end{cases} \text{ and } f_1(y_i) = \begin{cases} 0.2907 & , \quad y_i = \ \ 0.2668 \\ 0.2339 & , \quad y_i = \ \ 0.0291 \\ 0.4754 & , \quad y_i = -0.2250 \end{cases},$$

respectively.

Suppose that $n=80$ and $\alpha=0.1$. It is easy to show that $E_0[Y_i] = 0.022867, Var_0[Y_i] = 0.04592$, $E_1[Y_i] = -0.022607$, and $Var_1[Y_i] = 0.04444$. Thus using the central limit theorem, we obtain $k=2.1768$ and $\beta=0.2643$.

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