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On a Direct Study of an Operator Riccati Equation Appearing in Boundary Value Problems Factorization

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Abstract

In this paper we propose a direct study of the Riccati equation satisfied by the Dirichlet-Neumann operator defined on a moving section for an elliptic boundary value problem set in a cylindrical domain [3]. Similar equations are studied in control theory. The additional difficulty of this problem is due to the unboundedness of the right-hand side and of the solution of the equation. The Yosida regularization is used to overcome it.

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1 Introduction

Operator Riccati differential equations are used in the theory of linear-quadratic optimal control of infinite dimensional systems. The operator is the adjoint state to state operator and it allows to compute the optimal feedback control. It is derived by using the invariant embeding technique of R. Bellman. These equations were studied by J. L. Lions [7] using a Galerkin method and also by A. Bensoussan [1] in the context of Kalman filtering. In [9] a direct study of Riccati equations was made in a Hilbert-Schmidt operator framework. Also Tartar [8], at the same time, studied these equations by a fixed point argument. The same argument is used in the book [2]. In [6] a nonlinear semigroup approach is used.

In this paper we present a study of a similar Riccati equation arising in the theory of elliptic boundary value problems factorization. Here the operator satisfying the Riccati equation represents a Dirichlet to Neumann operator on a section of the domain. This kind of Riccati operator equation is not usually found in the litterature of control of infinite dimensional systems as here the right hand side, as well as the solution of the equation, are unbounded operators. In [3] its well-posedness

was proved by adapting the Galerkin method used by J.L. Lions in [7]. In [4] a direct study of the operator equation was made in a Hilbert-Schmidt operator framework inspired from [9]. Due to the unboundedness of the operator, the fixed point argument used in [2] and [8] does not work any more. In [2](p. 405) the case of the unbounded observation is studied, but the assumptions relates this unboundedness to the one of the generator of the evolution semi-group (which is here 0) and are not satisfied in our case. Here we present a direct study of the particular operator Riccati equation arising from the factorization of the Poisson equation in a cylindrical domain using a Yosida regularization.

In section 2 we recall the formal derivation of the factorization of a boundary value problem. In section 3 we introduce the Yosida regularization and in section 4 we derive the main result by passing to the limit in the regularization.

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2 Method of factorization by space invariant embeding

In this section we recall from [3] the factorization of boundary value problems for elliptic equation by space invariant embeding (in this paper the Neumann-Dirichlet operator was studied, but the approach is quite similar). The method is most easily explained in the case of a cylinder. Let Ω be the cylinder $\Omega =]0, a[\times \mathcal{O}$, $x' = (x, y) \in \mathbb{R}^n$, where x is the coordinate along the axis of the cylinder and $\mathcal{O}$ bounded open set in $\mathbb{R}^{n-1}$ is the section of the cylinder. Let $\Sigma =]0, a[\times \partial\mathcal{O}$ be the lateral boundary and $\Gamma_0 = \{0\} \times \mathcal{O}$, $\Gamma_a = \{a\} \times \mathcal{O}$ be the faces of the cylinder.

We consider the following Poisson equation with mixed boundary conditions

$$(\mathcal{P}_0) \begin{cases} -\Delta u = -\frac{\partial^2 u}{\partial x^2} - \Delta_y u = f \quad \text{in } \Omega, \\ u|_\Sigma = 0, \\ -\dfrac{\partial u}{\partial x}|_{\Gamma_0} = -u_0, \quad u|_{\Gamma_a} = u_1. \end{cases}$$

We embed this problem in the family of similar problems defined over subcylinders limited by the “moving boundary” Γ_s

$$(\mathcal{P}_{s,h}) \begin{cases} -\Delta u = f \quad \text{in } \Omega_s =]0, s[\times \mathcal{O} \\ u|_\Sigma = 0, \\ -\dfrac{\partial u}{\partial x}|_{\Gamma_0} = -u_0, \quad u|_{\Gamma_s} = h. \end{cases}$$

The Dirichlet-Neumann map on $\Gamma_s : h \mapsto \frac{\partial u}{\partial x}|_{\Gamma_s}$, is affine, so there exist an operator $P(s)$ and a residual $w(s)$ such that

$$\frac{\partial u}{\partial x}|_{\Gamma_s} = P(s)h + w(s). \tag{1}$$

Let us denote by $u(s) = u|_{\Gamma_s}$ and $u_{s,h}$ the solution of $(\mathcal{P}_{s,h})$. Then considering the problem restricted to Ω_x for $0 < x \leq s \leq a$ it is clear that $u_{s,h}(x)$ is $u_{x,u_{s,h}(x)}(x)$. Then one has the relation

$$\frac{\partial u_{s,h}}{\partial x}(x) = P(x)u_{s,h}(x) + w(x). \tag{2}$$

Now to derive the equation satisfied by P and w one has to take the derivative with respect to x. This computation is formal as one does not know if these functions are derivable. Skipping the indices for the sake of clarity, one has

$$\frac{\partial^2 u}{\partial x^2} = -\Delta_y u - f = \frac{dP}{dx}u + P\frac{\partial u}{\partial x} + \frac{\partial w}{\partial x},$$

substituting $\dfrac{\partial u}{\partial x}$ from (2)

$$0 = \left(\frac{dP}{dx} + P^2 + \Delta_y\right) u + \frac{\partial w}{\partial x} + Pw + f.$$

Now setting $x = s$ and h being arbitrary, we can identify to zero the term depending linearly on h and the term independent. One gets the decoupled system

$$\frac{dP}{dx} + P^2 + \Delta_y = 0; \qquad P(0) = 0, \tag{3}$$

$$\frac{dw}{dx} + Pw = -f; \qquad w(0) = u_0, \tag{4}$$

$$-\frac{du}{dx} + Pu = -w; \qquad u(a) = u_1, \tag{5}$$

where P and w are to be integrated from 0 to a then u backwards from a to 0. We stress that P is an operator on functions defined on $\mathcal{O}$ satisfying a Riccati equation. The initial conditions for P and w are obtained from (1) written at $x = 0$. The notation Δ_y in equation (3) has to be understood as the abstract isomorphism from $H_0^1(\mathcal{O})$ to $H^{-1}(\mathcal{O})$ corresponding to the laplacian with homogeneous Dirichlet boundary condition.

We obtain the *factorization* of the elliptic boundary value problem $(\mathcal{P}_0)$ in the product of two Cauchy problems of parabolic type in opposite directions

$$\text{“} - \Delta\text{”} = -\left(\frac{d}{dx} + P\right)\left(\frac{d}{dx} - P\right)$$

We insist that this factorization should be understood not only in terms of differential operator but also in terms of boundary conditions.

Remark 1 Once the Riccati equation for P is integrated, for a new set of data u_0, u_1, f it suffices to integrate two uncoupled first order equations for w and u. This recalls a similar property for the Gauss LU factorization of matrices. In fact it can be shown that this factoriation can be viewed as the infinite dimensional limit of a block LU factorization of a discretization of the problem $(\mathcal{P}_0)$.

3 Yosida regularization

Let A be the unbounded operator $-\Delta_y$ in the Hilbert space $H = L^2(\mathcal{O})$, with domain $D(A) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$, and let A_n be its Yosida regularized $A_n = nI - n^2 R(n, -A) = nI - n^2(nI + A)^{-1}$. We know that

$$\lim_{n \to +\infty} A_n h = Ah, \quad \forall h \in D(A).$$

Each A_n is a linear, bounded, self adjoint and positive operator in H, and so we can define $A_n^{\frac{1}{2}}$, the positive square root of A_n, which is a linear, bounded, self adjoint and positive operator, which also verifies:

$$\left\| A_n^{\frac{1}{2}} \right\|_{\mathcal{L}(H)} = \| A_n \|_{\mathcal{L}(H)}^{\frac{1}{2}}, \forall n \in \mathbb{N}. \tag{6}$$

Let P_n be the solution of the corresponding Riccati equation:

$$\frac{dP_n}{dx} + P_n^2 = A_n, \quad P_n(0) = 0. \tag{7}$$

From [2], the equation (7) admits a solution given by:

$$P_n(x) = A_n^{\frac{1}{2}}(\exp(2xA_n^{\frac{1}{2}}) - I)(\exp(2xA_n^{\frac{1}{2}}) + I)^{-1}. \tag{8}$$

First of all, we notice that each operator $A_n^{\frac{1}{2}}$ is a linear bounded operator in H, and so it is the infinitesimal generator of an uniformly continuous semigroup of bounded linear operators in H:

$$T_n(x) = \exp(xA_n^{\frac{1}{2}}),\ x \geq 0,\ n \in \mathbb{N}, \tag{9}$$

and taking into account that $A_n^{\frac{1}{2}}$ is self adjoint $\forall n \in \mathbb{N}$, then we may conclude that $T_n(x)$ is self adjoint $\forall n \in \mathbb{N}$.

Theorem 3.1 *For each $x \geq 0$, $P_n(x)$ is well defined and $P_n(x) \in \mathcal{L}(H)$ is a positive and self adjoint operator in H. Moreover, we have that:*

$$P_n \in C^1([0, a]; \mathcal{L}(H)). \tag{10}$$

Proof. First of all, we notice that for each $x \geq 0$, the linear operator $\exp(2xA_n^{\frac{1}{2}}) + I$ is invertible in H. In fact, we know that $T_n(x) = \exp(xA_n^{\frac{1}{2}})$ is self adjoint in H, $\forall x \geq 0$, and so we have that:

$$((\exp(2xA_n^{\frac{1}{2}}) + I)h, h) = (\exp(xA_n^{\frac{1}{2}}) \exp(xA_n^{\frac{1}{2}})h, h) + (h, h) = \tag{11}$$

$$= (\exp(xA_n^{\frac{1}{2}})h, \exp(xA_n^{\frac{1}{2}})h) + \|h\|^2 \geq \tag{12}$$

$$\geq \|h\|^2, \forall h \in H \tag{13}$$

and, taking into account that $\exp(2xA_n^{\frac{1}{2}}) + I$ is continuous in H, then, by Lax-Milgram theorem, we may conclude that it is invertible in H, and we have also that:

$$\left\|(\exp(2xA_n^{\frac{1}{2}}) + I)^{-1}\right\|_{\mathcal{L}(H)} \leq 1, \forall x \geq 0. \tag{14}$$

Next, we remark that, for each $x \geq 0$, $P_n(x)$ is the product of self adjoint, positive and bounded operators in H, that commute with each other, and, consequently, we may conclude that $P_n(x)$ is a self adjoint, positive and bounded operator in H, $\forall x \geq 0$. Taking into account that $\exp(2xA_n^{\frac{1}{2}})$ is an uniformly continuous semigroup of bounded linear operators in H, it can be shown, in a straightforward way, that $P_n \in C([0,a];\mathcal{L}(H))$. From (8) we obtain:

$$\frac{dP_n}{dx} = 4A_n(\exp(2xA_n^{\frac{1}{2}}))(\exp(2xA_n^{\frac{1}{2}}) + I)^{-2}. \tag{15}$$

As before, we may conclude that $\frac{dP_n}{dx} \in C([0,a];\mathcal{L}(H))$.

Theorem 3.2 *For each $h \in D(A)$ there exists a constant $c(h) \geq 0$, such that*

$$\|P_n(x)h\| \leq c(h), \forall x \in [0,a], \forall n \in \mathbb{N}.$$

Proof. By (15) and, being the product of positive operators that commute with each other, we may conclude that:

$$\frac{dP_n}{dx} \geq 0, \quad \forall n \in \mathbb{N}, \quad \forall x \geq 0.$$

From the Riccati equation (7) we have:

$$(\frac{dP_n}{dx}h,h) + (P_n^2(x)h,h) = (A_nh,h),$$

P_n being self adjoint,

$$(\frac{dP_n}{dx}h,h) + \|P_n(x)h\|^2 = (A_nh,h), \quad \forall h \in H, \forall x \geq 0,$$

and so we may conclude that:

$$\|P_n(x)h\|^2 \leq (A_nh,h) \longrightarrow (Ah,h), \forall h \in D(A), \forall x \in [0,a],$$

and, consequently, for each $h \in D(A)$ there exists a constant $c(h) \geq 0$, such that:

$$\|P_n(x)h\| \leq c(h), \forall x \in [0,a], \forall n \in \mathbb{N}.$$

Theorem 3.3 *For each $h \in D(A)$, the following limit: $\lim\limits_{n\to+\infty} P_n(x)h$ exists strongly in H, uniformly in $x \in [0,a]$.*

Proof. For each $h \in H$ and each $x \in [0,a]$, we consider the sequence $\{P_n(x)h\}_{n\in\mathbb{N}}$. Now, it can be shown that for each $x \geq 0$, $P_n(x)$ and $P_m(x)$ commute with each other, $\forall m,n \in \mathbb{N}$, and so we have that:

$$\frac{d}{dx}(P_n(x) - P_m(x))h + (P_n(x) + P_m(x))(P_n(x) - P_m(x))h = (A_n - A_m)h. \tag{16}$$

Multiplying by $(P_n(x) - P_m(x))h$ it results that

$$\begin{aligned}
&(\tfrac{d}{dx}(P_n(x) - P_m(x))h, (P_n(x) - P_m(x))h)+ \\
&+((P_n(x) + P_m(x))(P_n(x) - P_m(x))h, (P_n(x) - P_m(x))h) = \\
&= ((A_n - A_m)h, (P_n(x) - P_m(x))h) \Rightarrow \\
&\Rightarrow \tfrac{1}{2}\tfrac{d}{dx}((P_n(x) - P_m(x))h, (P_n(x) - P_m(x))h) \leq \\
&\leq ((A_n - A_m)h, (P_n(x) - P_m(x))h) \leq \\
&\leq \|(A_n - A_m)h\| \, \|(P_n(x) - P_m(x))h\| \Leftrightarrow \\
&\Leftrightarrow \tfrac{1}{2}\tfrac{d}{dx} \|(P_n(x) - P_m(x))h\|^2 \\
&= \|(P_n(x) - P_m(x))h\| \tfrac{d}{dx} \|(P_n(x) - P_m(x))h\| \leq \\
&\leq \|(A_n - A_m)h\| \, \|(P_n(x) - P_m(x))h\| \\
&\Rightarrow \tfrac{d}{dx} \|(P_n(x) - P_m(x))h\| \leq \|(A_n - A_m)h\| \\
&\Rightarrow \|(P_n(x) - P_m(x))h\| \leq x \|(A_n - A_m)h\| \\
&\leq a \|(A_n - A_m)h\|, \forall x \in [0, a], \forall h \in H.
\end{aligned}$$

We obtain the estimate

$$\|(P_n(x) - P_m(x))h\| \leq a \|(A_n - A_m)h\|, \forall x \in [0, a], \forall h \in H. \tag{17}$$

Since $A_n h \to Ah, \forall h \in D(A)$, and remarking that $\|(A_n - A_m)h\|$ does not depend on x, we conclude that, for each $h \in D(A)$, the sequence $\{P_n(x)h\}_{n\in\mathbb{N}}$ is a Cauchy sequence, uniformly in $x \in [0, a]$, and, consequently, for each $h \in D(A)$, the sequence $\{P_n(x)h\}_{n\in\mathbb{N}}$ is strongly convergent in H, uniformly in $x \in [0, a]$.

4 Passing to the limit

We now define, for each $h \in D(A)$ and each $x \in [0, a]$: $P(x)h = \lim_{n\to+\infty} P_n(x)h$, which, by linearity, defines the operator $P(x)$ from $D(A)$ to H.

Theorem 4.1 *The operator P verifies the Riccati equation (3) in the following sense:*

$$\frac{d}{dx}(P(x)h, \overline{h}) + (P(x)h, P(x)\overline{h}) = (-\Delta_y h, \overline{h}), \quad \forall h, \overline{h} \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}), \tag{18}$$

and it verifies also $P(0) = 0$.

Proof. We know that each $P_n$verifies:

$$(\frac{dP_n}{dx}h,\overline{h})+(P_n(x)h,P_n(x)\overline{h})=(A_nh,\overline{h}),\quad \forall h,\overline{h}\in H.$$

Let $\varphi\in\mathcal{D}(]0,a[)$. Then:

$$-\int_0^a(P_n(x)h,\overline{h})\,\varphi'(x)dx+\int_0^a(P_n(x)h,P_n(x)\overline{h})\,\varphi(x)dx=\int_0^a(A_nh,\overline{h})\,\varphi(x)dx. \tag{19}$$

For each h, $\overline{h}$ fixed in $D(A)=H^2(\mathcal{O})\cap H_0^1(\mathcal{O})$, we have:

$$\lim_{n\to+\infty}(P_n(x)h,\overline{h})\,\varphi'(x)=(P(x)h,\overline{h})\,\varphi'(x),\quad \forall x\in[0,a] \tag{20}$$

and

$$\left|(P_n(x)h,\overline{h})\,\varphi'(x)\right|\leq c(h).\left\|\overline{h}\right\|\left|\varphi'(x)\right|,\forall x\in[0,a]\,,\forall n\in\mathbb{N}. \tag{21}$$

We also have that

$$\lim_{n\to+\infty}(P_n(x)h,P_n(x)\overline{h})\,\varphi(x)=(P(x)h,P(x)\overline{h})\,\varphi(x),\forall x\in[0,a] \tag{22}$$

and

$$\left|(P_n(x)h,P_n(x)\overline{h})\,\varphi(x)\right|\leq c(h).c(\overline{h})\left|\varphi(x)\right|,\ \forall x\in[0,a]\,,\forall n\in\mathbb{N}. \tag{23}$$

Finally, we notice that:

$$\lim_{n\to+\infty}\int_0^a(A_nh,\overline{h})\,\varphi(x)dx=(Ah,\overline{h})\int_0^a\ \varphi(x)dx. \tag{24}$$

Consequently, we may use the Lebesgue dominated convergence theorem to pass to the limit for each term of (19). Then (18) is satisfied in the sense of distributions on $]0,a[$ for each $h,\overline{h}\in D(A)$. Furthermore, since $P_n(0)h=0$ for all $n\in\mathbb{N}$ and $h\in D(A)$ then, by Theorem 3.3, $P(0)=0$.

Theorem 4.2 *The operator $P(x)$ verifies:*

$$(\frac{dP}{dx}h,h)\geq 0,\forall h\in H^2(\mathcal{O})\cap H_0^1(\mathcal{O}). \tag{25}$$

Proof. From Theorem 4.1, we know that $P(x)$ verifies:

$$(\frac{dP}{dx}h,h)+\left\|P(x)h\right\|^2=(-\Delta_yh,h),\forall h\in H^2(\mathcal{O})\cap H_0^1(\mathcal{O}). \tag{26}$$

We also know that, for each $n \in \mathbb{N}$, $P_n(x)$ verifies:

$$(\frac{dP_n}{dx}h, h) + \|P_n(x)h\|^2 = (A_n h, h), \forall h \in L^2(\mathcal{O}).$$

Now, taking into account that, for each $h \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$, $P_n(x)h \underset{n\to+\infty}{\longrightarrow} P(x)h$, strongly in $L^2(\mathcal{O})$, uniformly in $x \in [0, a]$, and

$$(A_n h, h) \underset{n\to+\infty}{\longrightarrow} (-\Delta_y h, h), \forall h \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}), \tag{27}$$

we may conclude that:

$$(\frac{dP_n}{dx}h, h) \underset{n\to+\infty}{\longrightarrow} (\frac{dP}{dx}h, h), \forall h \in D(A).$$

On the other hand, we know that $\frac{dP_n}{dx} \geq 0$ in $L^2(\mathcal{O})$, and so we conclude that:

$$(\frac{dP}{dx}h, h) \geq 0, \forall h \in D(A).$$

Theorem 4.3 *The operator $P(x)$ verifies*

$$\|P(x)h\|_{L^2(\mathcal{O})} \leq \|h\|_{H_0^1(\mathcal{O})}, \forall h \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}), \forall x \in [0, a],$$

and, consequently, it also verifies

$$P \in L^\infty(0, a; \mathcal{L}(H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}), L^2(\mathcal{O}))). \tag{28}$$

Proof. We know that $P(x)$ verifies

$$\begin{aligned}(\tfrac{dP}{dx}h, h) + \|P(x)h\|^2 &= (-\Delta_y h, h) = \|\nabla_y h\|^2 \\ &= \|h\|^2_{H_0^1(\mathcal{O})} \leq \|h\|^2_{H^2(\mathcal{O})}, \quad \forall h \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})\end{aligned}$$

and, taking into account that $(\frac{dP}{dx}h, h) \geq 0, \forall h \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$ the conclusion is now obvious.

Theorem 4.4 *There exists an operator*

$$P \in L^\infty(0, a; \mathcal{L}(H_0^1(\mathcal{O}), L^2(\mathcal{O}))) \tag{29}$$

solution of the Riccati equation

$$(\frac{dP}{dx}h, \overline{h}) + (P(x)h, P(x)\overline{h}) = (\nabla_y h, \nabla_y \overline{h}), \forall h, \overline{h} \in H_0^1(\mathcal{O}) \tag{30}$$

verifying $P(0) = 0$. This solution is strongly continuous in the sense that $P(x)h \in C([0, a]; L^2(\mathcal{O}))$, for all $h \in H_0^1(\mathcal{O})$.

Proof. The operator P, referred in the previous theorems, can be extended by density, taking Theorem 4.3 into account, to an operator $\overline{P}$ from $H_0^1(\mathcal{O})$ to $L^2(\mathcal{O})$. This extension is unique and we name $\overline{P}$ by P. From Theorem 4.3, we may conclude that:

$$\|P(x)h\|_{L^2(\mathcal{O})} \leq \|h\|_{H_0^1(\mathcal{O})}, \forall h \in H_0^1(\mathcal{O}), \forall x \in [0,a], \tag{31}$$

and, consequently, $P \in L^\infty(0,a;\mathcal{L}(H_0^1(\mathcal{O}), L^2(\mathcal{O})))$.

Following a similar way to the one we used in the proof of Theorem 4.1, it can be shown that $P(x)$ verifies the Riccati equation:

$$(\frac{dP}{dx}h, \overline{h}) + (P(x)h, P(x)\overline{h}) = (\nabla_y h, \nabla_y \overline{h}), \forall h, \overline{h} \in H_0^1(\mathcal{O}). \tag{32}$$

By Theorem 3.3, $P(x)h \in C([0,a]; L^2(\mathcal{O}))$, for all $h \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$. The remaining property follows from (31).

Remark 2 The uniqueness of the solution of (3) can be proved. The easiest way is to use the interpretation of P as a Dirichlet-Neumann operator and to use the uniqueness of the solution of the boundary value problem.

Remark 3 If instead $A = -\Delta_y$ we consider the operator

$$A = -\sum_{i,j=1}^{n-1} \frac{\partial}{\partial y_j}\left(a_{i,j}(y)\frac{\partial}{\partial y_i}\right)$$

with $a_{i,j}(y) = a_{j,i}(y)$, $\forall i,j$ and $\forall y \in \mathcal{O}$, $a_{i,j}$ being continuously differentiable in $\overline{\mathcal{O}}$, and A being $H_0^1(\mathcal{O})$-elliptic, we can do everything as before with almost no changes. In fact, A remains self adjoint, positive and an infinitesimal generator of a strongly continuous semigroup. The explicit formula at the begining remains. The norm of P in $L^\infty(0,a;\mathcal{L}(H_0^1(\mathcal{O}), L^2(\mathcal{O})))$ is now bounded by M for some M, instead of 1. *This is the unique little change.*

5 Conclusion

We have proved directly the existence of a solution of an operator Riccati equation having an unbounded solution that arises in the theory of factorization of elliptic linear boundary value problems. The problem studied is particular as it concerns the Laplacian in a cylindrical domain. Further work will study more general operators and more general geometries that give rise to more complex Riccati equations but with the same kind of unboundedness. For example the case of invariant embeding in a circular domain was presented in [5].

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