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Strong Convergence Theorem for Solving Equilibrium Problems by Hybrid Method of Asymptotically k-Strictly Pseudo-Contractive Mappings in Hilbert Spaces

Chanan Sudsukh[1]

Department of Mathematics, Faculty of Liberal Arts and Science
Kasetsart University, Kamphaeng Saen Campus
Nakhonpathom 73140, Thailand
faaschs@ku.ac.th

Issara Inchan

Department of Mathematics and Computer
Uttaradit Rajabhat University
Uttaradit 53000, Thailand
peissara@uru.ac.th

Abstract

In this paper, we introduce the iterative schemes by using hybrid methods for finding a common element of the set of an equilibrium problem and the set of fixed points of asymptotically k-strictly pseudo-contractive mapping in a Hilbert space. We show that the iterative sequence converges strongly to a common element of two sets. Our result combine the ideas of Inchan [Strong convergence theorems of modified Mann iteration methods for asymptotically nonexpansive mappings in Hilbert spaces, Int. Journal of Math. Analysis, Vol 2, 2008, no. 23, 1135-1145], Kim and Xu [T. H. Kim, H. K. Xu, Convergence of the modified Mann's iteration method for asymptotically strict pseudo-contractions, Nonlinear Analysis 68 (2008) 2828-2836] and connected with Ceng et al's result [L.C. Ceng, S. Al-Homidan, Q.H. Ansari and J.C. Yao, An iterative scheme for equilibrium problems and xed point problems of strict pseudo-contraction mappings, J. comput. Appl. Math. (2008), doi:10.1016/j.cam.2008.03.032.] and other authors.

[1]Corresponding author: faaschs@ku.ac.th

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1 Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let F be a bifunction of $C \times C$ into $\mathbf{R}$, where $\mathbf{R}$ is the real numbers. The equilibrium problem for $F : C \times C \to \mathbf{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \text{for all} \quad y \in C. \tag{1}$$

The set of solutions of (1) is denoted by $EP(F)$. Given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then, $z \in EP(F)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$. Numerous problems in physics, optimization, and economics reduce to find a solution of (1). In 1997, Combettes and Hirstoaga [3] introduced an iterative scheme of finding the best approximation to the innitial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

We know that a Hilbert space H satisfies Opial's condition [13], that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Recall that a mapping $T : C \to C$ is said to be a strict pseudo-contractive mapping [1] if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \tag{2}$$

for all $x, y \in C$. (If (2) holds, we also say that T is a k-strict pseudo-contraction.)

It is know that if T is 0-strict pseudo-contractive mapping, T is nonexpansive mapping.

In this paper we will consider an iteration method of modified Mann for asymptotically k-strict pseudo-contractive mapping. We say that $T : C \to C$ is an asymptotically k-strict pseudo-contractive mapping if there exists a constant $0 \leq k < 1$ satisfying

$$\|T^n x - T^n y\|^2 \leq (1 + \gamma_n)\|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2, \tag{3}$$

for all $x, y \in C$ and for all $n \in \mathbf{N}$ where $\gamma_n \geq 0$ for all n such that $\lim_{n\to\infty} \gamma_n = 0$. We see that if $k = 0$, then T is asymptotically nonexpansive mapping. By

Goebel and Kirk [6], T is asymptotically nonexpansive mapping if there exists a sequence $\{\gamma_n\}$ of nonnegative numbers with $\lim_{n\to\infty}\gamma_n = 0$ and such that

$$\|T^n x - T^n y\|^2 \leq (1+\gamma_n)\|x-y\|^2, \tag{4}$$

for all $x, y \in C$ and all integers $n \geq 1$.

In 1967, Browder and Petryshyn [1] established the first convergence result for -strict pseudo-contraction in real Hilbert spaces. They proved weak and strong convergence theorem by using iteration (1.4) with a constant control sequence $\{\alpha_n\} = \alpha$ for all n. Many authors have appeared in the literature on the existence of solution equilibrium, see also, for example [4, 17, 18] and references therein. To find an element of $EP(F) \cap F(T)$, Tada and Takahashi [16] introduced the iterative scheme for nonexpansive mappings by the hybrid method in a Hilbert space.

In 2007, Kim and Xu [8], introduced the sequence generated by CQ algorithm for k-strict pseudo-contractions T is constructed as follows:

$$\begin{cases} x_0 \in C \text{ chosen arbitrary}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (1-\alpha_n)(k-\alpha_n)\|x_n - Tx_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ u_{n+1} = P_{C_n \cap Q_n} x_0, \ n \in \mathbf{N}, \end{cases} \tag{5}$$

where $\{\alpha_n\}$ with $\alpha_n < 1$, then $\{x_n\}$ is generated in (6) converge strongly to $P_{F(T)}x_0$.

In 2008, I. Inchan [7], introduce the modified Mann iteration processes for an asymptotically nonexpansive mapping. Let C be a closed bounded convex subset of a Hilbert space H, T be an asymptotically nonexpansive mapping of C into itself and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ n \in \mathbf{N}, \end{cases} \tag{6}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(diamC)^2 \to 0$ as $n \to \infty$ and $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbf{N}$. Then him prove that $\{x_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Recently, Ceng et al. [5] established an iterative scheme for finding a common element of the set of solution of an equilibrium problem (1.1) and the set of fixed point of a k-strict pseudo-contraction mapping in the setting of real Hilbert space. They also studied some weak and strong convergence theorem for k-strict pseudo-contraction mapping of the sequence generated by their algorithm.

Our iteration method for finding a fixed point of an asymptotically k-strict pseudo-contractive mapping T is the modified Mann's iteration method studied in [9, 8, 14, 15, 19] which generates a sequence $\{x_n\}$ via

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T^n x_n, \quad n \geq 0, \tag{7}$$

where the initial guess $x_0 \in C$ is arbitrary and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ line in the interval $(0, 1)$.

Motivated and inspired by the above results, in this paper, we consider the mixed iterative scheme (5), (6) and [5] from nonexpansive mappings to more general asymptotically k-strict pseudo-contraction mappings for finding a common element of the set of solutions of an equilibrium problem and the set of solutions of fixed points of a asymptotically k-strict pseudo-contraction mappings in Hilbert spaces. Moreover, we show that $\{x_n\}$ and $\{u_n\}$ converge strongly to $F(S) \cap EP(F)$ by the hybrid methods under mild assumption on parameters.

2 Preliminary Notes

Let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\| \text{ for all } x, y \in C.$$

P_C is called the metric projection of H onto C. It well known that P_C is a nonexpansive. Next, we collect some lemmas which will be used in the proof for the main result in next section.

Lemma 2.1 *[12] There holds the identity in a Hilbert space H:*
(i) $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle, \forall x, y \in H$.
(ii) $\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$ for all $x, y \in H$ and $\lambda \in [0, 1]$.

Lemma 2.2 *[12] Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geq 0, \quad \forall z \in C.$$

Lemma 2.3 *[10] Let C be a closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $z = P_C u$. If $\omega_w(x_n) \subset C$ and satisfies the condition $\|x_n - u\| \leq \|u - z\|$ for all n. Then $x_n \to p$.*

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbf{R}$, let us assume that F satisfies the following condition:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y \in C$,

$$\lim_{t \to 0} F(tz + (1-t)x, y) \leq F(x,y);$$

(A4) for each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous.
The following lemma appears implicitly in [2].

Lemma 2.4 *[2] Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbf{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad \textit{for all } y \in C.$$

The following lemma was also given in [3].

Lemma 2.5 *[3] Assume that $F : C \times C \to \mathbf{R}$ satisfying (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping*

$$T_r(x) = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\}$$

for all $z \in H$. Then, the following hold:

1. T_r is single-valued;

2. T_r is firmly nonexpansive, i. e., $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$ for all $x, y \in H$;

3. $F(T_r) = EP(F)$;

4. $EP(F)$ is closed and convex.

3 Main Results

In this section, we prove the iterative sequence convergence strongly to element of the set of an equilibrium problem and the set of fixed points of asymptotically k-strictly pseudo-contractive mapping in a Hilbert space.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbf{R}$ satisfying $(A1) - (A4)$ and let S be an asymptoticall k-strictly pseudo-contractive mapping for some*

$0 \leq k < 1$ such that $F(S) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequence generated by $C_1 = C \subset H, x_1 = x \in C$ and let

$$\begin{cases} u_n \in C \text{ such that } F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ y_n = \alpha_n u_n + (1-\alpha_n)S^n u_n, \\ c_{n+1} = \{z \in C : \|y_n - z\|^2 \leq \|z - x_n\|^2 + [(k-\alpha_n)(1-\alpha_n)]\|u_n - S^n u_n\|^2 + \theta_n\}, \\ x_{n+1} = P_{Cn+1}x_0, \end{cases} \tag{8}$$

for all $n \in \mathbf{N}$, $\theta_n = (1-\alpha_n)\gamma_n(diamC)^2 \to 0$ as $n \to \infty$, where $\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in [k, 1)$, and $\{r_n\} \subset (0, \infty)$ satisfying $\liminf_{n\to\infty} r_n > 0$ and $\Sigma_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$. Then $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap EP(F)}x_0$.

Proof. We first show that $F(S) \cap EP(F) \subset C_n$ for all $n \in \mathbf{N}$, by induction. For any $z \in F(S) \cap EP(F)$ we have $z \in C = C_1$ hence $F(S) \cap EP(F) \subset C_1$. Let $F(S) \cap EP(F) \subset C_m$ for some $m \in \mathbf{N}$. Then we have, for $p \in F(S) \cap EP(F) \subset C_m$ and from $u_n = T_{r_n}x_n$, we note that

$$\|u_n - p\| = \|T_{r_n}x_n - T_{r_n}p\| \leq \|x_n - p\|, \tag{9}$$

for all $n \in \mathbf{N}$. Thus we have

$$\begin{aligned} \|y_m - p\|^2 &= \|\alpha_m u_m + (1-\alpha_m)S^m u_m - p\|^2 \\ &= \|\alpha_m(u_m - p) + (1-\alpha_m)(S^m u_m - p)\|^2 \\ &= \alpha_m\|u_m - p\|^2 + (1-\alpha_m)\|S^m u_m - p\|^2 - \alpha_m(1-\alpha_m)\|u_m - S^m u_m\|^2 \\ &\leq \alpha_m\|x_m - p\|^2 + (1-\alpha_m)[(1+\gamma_m)\|u_m - p\|^2 + k\|u_m - S^m u_m\|^2] \\ &\quad -\alpha_m(1-\alpha_m)\|u_m - S^m u_m\|^2 \\ &\leq (1 + (1-\alpha_m)\gamma_m)\|x_m - p\|^2 + [(k-\alpha_m)(1-\alpha_m)]\|u_m - S^m u_m\|^2 \\ &\leq \|x_m - p\|^2 + [(k-\alpha_m)(1-\alpha_m)]\|u_m - S^m u_m\|^2 + (1-\alpha_m)\gamma_m\|x_m - p\|^2 \\ &\leq \|x_m - p\|^2 + [(k-\alpha_m)(1-\alpha_m)]\|u_m - S^m u_m\|^2 + \theta_m. \end{aligned}$$

It follows that $p \in C_{m+1}$ and $F(S) \cap EP(F) \subset C_{m+1}$, hence $F(S) \cap EP(F) \subset C_n$ for all $n \in \mathbf{N}$. Next, we show that C_n is closed and convex for all $n \in \mathbf{N}$. It follows obvious that $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for each $m \in \mathbf{N}$. Let $z_j \in C_{m+1} \subset C_m$ with $z_j \to z$. Since C_m is closed, $z \in C_m$ and $\|y_m - z_j\|^2 \leq \|z_j - x_m\|^2 + [(k-\alpha_m)(1-\alpha_m)]\|u_m - S^m u_m\|^2 + \theta_m$. Then

$$\begin{aligned} \|y_m - z\|^2 &= \|y_m - z_j + z_j - z\|^2 \\ &= \|y_m - z_j\|^2 + \|z_j - z\|^2 + 2\langle y_m - z_j, z_j - z\rangle \\ &\leq \|z_j - x_m\|^2 + [(k-\alpha_m)(1-\alpha_m)]\|u_m - S^m u_m\|^2 + \theta_m + \|z_j - z\|^2 \\ &\quad +2\|y_m - z_j\|\|z_j - z\|. \end{aligned}$$

Taking $j \to \infty$,

$$\|y_m - z\|^2 \leq \|z - x_m\|^2 + [(k-\alpha_m)(1-\alpha_m)]\|u_m - S^m u_m\|^2 + \theta_m.$$

Hence $z \in C_{m+1}$. Let $x, y \in C_{m+1} \subset C_m$ with $z = \alpha x + (1-\alpha)y$ where $\alpha \in [0, 1]$. Since C_m is convex, $z \in C_m$ and $\|y_m - x\|^2 \leq \|x - x_m\|^2 + [(k-\alpha_m)(1-\alpha_m)]\|u_m -$

$S^m u_m\|^2+\theta_m$, $\|y_m-y\|^2 \le \|y-x_m\|^2+[(k-\alpha_m)(1-\alpha_m)]\|u_m-S^m u_m\|^2+\theta_m$, we have

$$\begin{aligned}
\|y_m-z\|^2 &= \|y_m-(\alpha x+(1-\alpha)y)\|^2\\
&= \|\alpha(y_m-x)+(1-\alpha)(y_m-y)\|^2\\
&= \alpha\|y_m-x\|^2+(1-\alpha)\|y_m-y\|^2-\alpha(1-\alpha)\|(y_m-x)-(y_m-y)\|^2\\
&\le \alpha(\|x-x_m\|^2+[(k-\alpha_m)(1-\alpha_m)]\|u_m-S^m u_m\|^2+\theta_m)\\
&\quad +(1-\alpha)(\|y-x_m\|^2+[(k-\alpha_m)(1-\alpha_m)]\|u_m-S^m u_m\|^2+\theta_m)\\
&\quad -\alpha(1-\alpha)\|y-x\|^2\\
&= \alpha\|x-x_m\|^2+(1-\alpha)\|y-x_m\|^2-\alpha(1-\alpha)\|(x_m-x)-(x_m-y)\|^2\\
&\quad +[(k-\alpha_m)(1-\alpha_m)]\|u_m-S^m u_m\|^2+\theta_m\\
&= \|\alpha(x_m-x)+(1-\alpha)(x_m-y)\|^2+[(k-\alpha_m)(1-\alpha_m)]\|u_m-S^m u_m\|^2+\theta_m\\
&= \|x_m-z\|^2+[(k-\alpha_m)(1-\alpha_m)]\|u_m-S^m u_m\|^2+\theta_m.
\end{aligned}$$

Then $z \in C_{m+1}$, it follows that C_{m+1} is closed and convex. Hence C_n is closed and convex for all $n \in \mathbf{N}$. This implies that $\{x_n\}$ is well-defined. From $x_n = P_{C_n}x_0$, we have

$$\langle x_0-x_n, x_n-y\rangle \ge 0, \text{ for all } y \in C_n.$$

Since $F(S)\cap EP(F) \subset C_n$, we have

$$\langle x_0-x_n, x_n-u\rangle \ge 0 \text{ for all } u \in F(S)\cap EP(F) \text{ and } n \in \mathbf{N}. \tag{10}$$

So, for $u \in F(S)\cap EP(F)$, we have

$$\begin{aligned}
0 \le \langle x_0-x_n, x_n-u\rangle &= \langle x_0-x_n, x_n-x_0+x_0-u\rangle\\
&= -\langle x_n-x_0, x_n-x_0\rangle+\langle x_0-x_n, x_0-u\rangle\\
&\le -\|x_n-x_0\|^2+\|x_0-x_n\|\|x_0-u\|.
\end{aligned}$$

This implies that

$$\|x_0-x_n\|^2 \le \|x_0-x_n\|\|x_0-u\|,$$

hence

$$\|x_0-x_n\| \le \|x_0-u\| \text{ for all } u \in F(S)\cap EP(F) \text{ and } n \in \mathbf{N}. \tag{11}$$

From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we also have

$$\langle x_0-x_n, x_n-x_{n+1}\rangle \ge 0 \text{ for all } n \in \mathbf{N}. \tag{12}$$

So, for $x_{n+1} \in C_n$, we have, for $n \in \mathbf{N}$

$$\begin{aligned}
0 \le \langle x_0-x_n, x_n-x_{n+1}\rangle &= \langle x_0-x_n, x_n-x_0+x_0-x_{n+1}\rangle\\
&= -\langle x_n-x_0, x_n-x_0\rangle+\langle x_0-x_n, x_0-x_{n+1}\rangle\\
&\le -\|x_n-x_0\|^2+\|x_0-x_n\|\|x_0-x_{n+1}\|.
\end{aligned}$$

This implies that

$$\|x_0-x_n\|^2 \le \|x_0-x_n\|\|x_0-x_{n+1}\|,$$

hence

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\| \text{ for all } n \in \mathbf{N}. \tag{13}$$

From (11) we have $\{x_n\}$ is bounded, $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Next, we show that $\|x_n - x_{n+1}\| \to 0$. In fact, from (12) we have

$$\begin{aligned}\|x_n - x_{n+1}\|^2 &= \|(x_n - x_0) + (x_0 - x_{n+1})\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_0 - x_n\rangle - 2\langle x_0 - x_n, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &\leq \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2.\end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - x_0\|$ exists, we have that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{14}$$

On the other hand, we note that

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in H, \tag{15}$$

and

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0, \ \forall y \in H. \tag{16}$$

Putting $y = u_{n+1}$ in (15) and $y = u_n$ in (17), we have

$F(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0$ and

$F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0$

and hence

$\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \geq 0.$

Since $\liminf_{n\to\infty} r_n > 0$, we assume that there exists a real number b such that $r_n > b > 0$ for all $n \in \mathbf{N}$. Thus, we have

$$\begin{aligned}\|u_{n+1} - u_n\|^2 &\leq \langle u_{n+1} - u_n, x_{n+1} - x_n + (1 - \tfrac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1})\rangle \\ &\leq \|u_{n+1} - u_n\|\{\|x_{n+1} - x_n\| + |1 - \tfrac{r_n}{r_{n+1}}|\|u_{n+1} - x_{n+1}\|\}\end{aligned}$$

and hence

$$\begin{aligned}\|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + |1 - \tfrac{r_n}{r_{n+1}}|\|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{b}|r_{n+1} - r_n|L \end{aligned} \tag{17}$$

where $L = \sup\{\|u_{n+1} - x_{n+1}\| : n \in \mathbf{N}\}$. From (14), we obtain

$$\lim_{n\to\infty} \|u_n - u_{n+1}\| = 0. \tag{18}$$

Since $x_{n+1} \in C_{n+1} \subset C_n$ implies that

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + [(k - \alpha_n)(1 - \alpha_n)]\|u_n - S^n u_n\|^2 + \theta_n. \tag{19}$$

By the definition of y_n and $k < \alpha \le \alpha_n \le \beta < 1$, we have

$$\begin{aligned}\|y_n - x_n\|^2 &= \|(y_n - x_{n+1}) + (x_{n+1} - x_n)\|^2 \\ &\le \|y_n - x_{n+1}\|^2 + 2\|y_n - x_{n+1}\|\|x_{n+1} - x_n\| + \|x_{n+1} - x_n\|^2 \\ &\le \|x_n - x_{n+1}\|^2 + [(k - \alpha_n)(1 - \alpha_n)]\|u_n - S^n u_n\|^2 + \theta_n \\ &\quad +2\|y_n - x_{n+1}\|\|x_{n+1} - x_n\| + \|x_{n+1} - x_n\|^2 \\ &= 2\|x_{n+1} - x_n\|(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|) \\ &\quad -[(\alpha_n - k)(1 - \alpha_n)]\|u_n - S^n u_n\|^2 + \theta_n \\ &\le 2\|x_{n+1} - x_n\|(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|) + \theta_n.\end{aligned}$$

From (14), we have

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{20}$$

From Lemma 2.1 (ii), we have

$$\begin{aligned}\|y_n - p\|^2 &\le \alpha_n\|u_n - p\|^2 + (1 - \alpha_n)\|S^n u_n - p\| - \alpha_n(1 - \alpha_n)\|S^n u_n - u_n\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)[(1 + \gamma_n)\|u_n - p\|^2 + k\|S^n u_n - u_n\|^2] \\ &\quad -\alpha_n(1 - \alpha_n)\|S^n u_n - u_n\|^2 \\ &= (1 + (1 - \alpha_n)\gamma_n)\|x_n - p\|^2 + [(k - \alpha_n)(1 - \alpha_n))]\|S^n u_n - u_n\|^2\end{aligned}$$

$$= \|x_n - p\|^2 - [(\alpha_n - k)(1 - \alpha_n))]\|S^n u_n - u_n\|^2 + (1 - \alpha_n)\gamma_n\|x_n - p\|^2, \tag{21}$$

it follows that

$$(\alpha_n - k)(1 - \alpha_n)\|S^n u_n - u_n\|^2 \le \|x_n - p\|^2 - \|y_n - p\|^2 + \theta_n. \tag{22}$$

From (22) and $k < \alpha \le \alpha_n \le \beta < 1$, we also have

$$\begin{aligned}(\alpha - k)(1 - \beta)\|S^n u_n - u_n\|^2 &\le (\alpha_n - k)(1 - \alpha_n)\|S^n u_n - u_n\|^2 \\ &\le \|x_n - p\|^2 - \|y_n - p\|^2 + \theta_n \\ &\le \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|) + \theta_n.\end{aligned}$$

By (20), we obtain that

$$\lim_{n\to\infty} \|S^n u_n - u_n\| = 0. \tag{23}$$

Next, we show that $\lim_{n\to\infty} \|Su_n - u_n\| = 0$. From (18) and (23), we have

$$\begin{aligned}\|Su_n - u_n\| &\le \|Su_n - S^{n+1}u_n\| + \|S^{n+1}u_n - S^{n+1}u_{n+1}\| \\ &\quad +\|S^{n+1}u_{n+1} - u_{n+1}\| + \|u_{n+1} - u_n\|\end{aligned}$$

$$\le L_1\|u_n - S^n u_n\| + \|S^{n+1}u_{n+1} - u_{n+1}\| + (1 + L_{n+1})\|u_n - u_{n+1}\| \to 0 \text{ as } n \to \infty. \tag{24}$$

Next, we show that $\lim_{n\to\infty} \|x_n - u_n\| = 0$. For $p \in F(S) \cap EP(F)$ and from $u_n = T_{r_n}x_n$, we have

$$\begin{aligned}\|u_n - p\|^2 &\le \langle T_{r_n}x_n - T_{r_n}p, x_n - p\rangle = \langle u_n - p, x_n - p\rangle \\ &= \tfrac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2)\end{aligned}$$

and hence

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{25}$$

From (9) and (25), we have

$$\begin{aligned}\|y_n - p\|^2 &\le \alpha_n\|u_n - p\|^2 + (1-\alpha_n)\|S^n u_n - p\| - \alpha_n(1-\alpha_n)\|S^n u_n - u_n\|^2\\ &\le \alpha_n\|u_n - p\|^2 + (1-\alpha_n)[(1+\gamma_n)\|u_n - p\|^2 + k\|S^n u_n - u_n\|^2] -\\ &\quad \alpha_n(1-\alpha_n)\|S^n u_n - u_n\|^2\\ &= \|u_n - p\|^2 - [(\alpha_n - k)(1-\alpha_n))]\|S^n u_n - u_n\|^2 + (1-\alpha_n)\gamma_n\|u_n - p\|^2\\ &\le (1 + (1-\alpha_n)\gamma_n)\|u_n - p\|^2\\ &= \|u_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2\\ &\le \|u_n - p\|^2 + (1-\alpha_n)\gamma_n\|x_n - p\|^2\\ &\le \|x_n - p\|^2 - \|x_n - u_n\|^2 + \theta_n.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_n - u_n\|^2 &\le \|x_n - p\|^2 - \|y_n - p\|^2 + \theta_n\\ &\le \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|) + \theta_n.\end{aligned}$$

From (20), we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{26}$$

Since $\{x_n\}$ is bounded, there exists subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w$. From (26), we can assume that $u_{n_i} \rightharpoonup w$. We show that $w \in EP(F)$. It follows by (26) and $(A2)$ that

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge F(y, u_n)$$

and hence

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \ge F(y, u_{n_i}).$$

Since $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup w$, it follows by $(A4)$ that $0 \ge F(y, w)$ for all $y \in H$. For $t \in (0, 1]$ and $y \in H$ let $y_t = ty + (1-t)w$. Since $y, w \in H$, we have $y_t \in H$ and hence $F(y_t, w) \le 0$. So from $(A1)$ and $(A4)$ we have

$$0 = F(y_t, y_t) \le tF(y_t, y) + (1-t)F(y_t, w) \le tF(y_t, y)$$

and hence $0 \le F(y_t, y)$. From $(A3)$, we have $0 \le F(w, y)$ for all $y \in H$ and hence $w \in EP(F)$.

We shall show that $w \in F(S)$. Assume that $w \neq F(S)$. Since $u_{n_i} \rightharpoonup w$ and $w \neq Sw$, it follows by Opial's condition (see [13]) that

$$\begin{aligned}\liminf_{i\to\infty} \|u_{n_i} - w\| &< \liminf_{i\to\infty} \|u_{n_i} - Sw\|\\ &\le \liminf_{i\to\infty}(\|u_{n_i} - Su_{n_i}\| + \|Su_{n_i} - Sw\|)\\ &\le \liminf_{i\to\infty} \|u_{n_i} - w\|.\end{aligned}$$

A contradiction. So, we get $w \in F(S)$. Therefore $w \in F(T) \cap EP(F)$. If w is arbitrarily element of $\omega_w(x_n)$, by the same argument of the prove, we have $\omega_w(x_n) \subset F(S) \cap EP(F)$. From (11) and Lemma 2.3, we have $\{x_n\}$ strongly convergence to $z_0 = P_{F(S)\cap EP(F)}x_0$.This complete the proof. ⋄

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into* $\mathbf{R}$ *satisfying* $(A1) - (A4)$ *and let S be a k-strictly pseudo-contractive mapping for some* $0 \le k < 1$

such that $F(S) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequence generated by $C_1 = C \subset H, x_1 = x \in C$ and let

$$\begin{cases} u_n \in C \text{ such that } F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ y_n = \alpha_n u_n + (1-\alpha_n)Su_n, \\ c_{n+1} = \{z \in C : \|y_n - z\|^2 \leq \|z - x_n\|^2, \\ x_{n+1} = P_{Cn+1}x_0, \end{cases} \tag{27}$$

for all $n \in \mathbf{N}$, where $\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in [k, 1)$, and $\{r_n\} \subset (0, \infty)$ satisfying $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap EP(F)}x_0$.

Proof. Let $\gamma_n \equiv 0$ replace S from asymptotically k-strictly pseudo-contractive mapping into k-strictly pseudo-contractive mapping in Theorem 3.1, the conclusion follows. ⋄

Corollary 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction fron $C \times C$ into $\mathbf{R}$ satisfying $(A1) - (A4)$ and let S be an asymptotically nonexpansive mapping such that $F(S) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define $\{x_n\}$ as follows way:*

$$\begin{cases} u_n \in C \text{ such that } F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ y_n = \alpha_n x_n + (1-\alpha_n)S^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbf{N}, \end{cases} \tag{28}$$

where $\theta_n = (1-\alpha_n)\gamma_n^2(diamC)^2 \to 0$ as $n \to \infty$ and $\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in [k, 1)$, and $\{r_n\} \subset (0, \infty)$ satisfying $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap EP(F)}x_0$.

Proof. Putting $k = 0$ in Theorem 3.1, then the class of asymptotically k-strictly pseudo-contractive mapping reduced to asymptotically nonexpansive mapping, Then complete the proof. ⋄

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