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Fixed Point Theorems for the Generalized C-Contractions

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Abstract

In this paper, by using the concept of p-convergency we obtain two fixed point theorems for (q_1, q_2)-contraction of (ϵ, λ)-type and generalized C-contraction.

Keywords: Probabilistic metric space, Fuzzy metric space, p-convergent, generalized C-contraction, (q_1, q_2)-contraction of (ϵ, λ)-type, fixed point

1 Introduction

Menger introduced a probabilistic metric space (briefly, PM-space) as a generalization of metric space [8]. In fact, he replaced the metric function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ with a distribution function $F_{pq} : \mathbb{R} \to [0, 1]$ and then for any number x the value $F_{pq}(x)$ was interpreted the probability that the distance between p and q is less than x.
The first fixed point theorem in probabilistic metric space was proved by Sehgal and Bharucha-Reid for t-norm T_M [14]. A very interesting approach to the fixed point theory in probabilistic metric space is given in Tradiff′s paper [15]. Radu introduced a stronger growth condition.
The class of probabilistic C-contractions which intensively studied in the fixed point theory in probabilistic metric spaces has been introduced by Hicks [6]. There are too many fixed point theorems which are in some sense generalizations of B-contraction and C-contraction classes. In [10], Mihet introduced

the notion of a q-contraction of (ϵ, λ)-type. In [11] generalization, a notion of C-contraction is given. In [9], Mihet proved the existence of fixed point for fuzzy contractive by using p-convergency. In [1], we proved alike theorems for B-contractions and C-contractions. Now, we extend these theorems to q-contractions of (ϵ, λ)-type and generalized C-contractions under weaker condition of p-convergency.

The structure of this paper is as follows: Section 2 recalls some notions and known results in fuzzy metric spaces and probabilistic contractions. In section 3, we will show two theorems for (q, q_1)-contraction of (ϵ, λ)-type and generalized C-contraction.

2 Preliminary Notes

We recall some concepts from probabilistic metric space, fuzzy metric space and contraction. For more details, we refer the reader to [5,13].

The class of generalized distribution functions (denoted by Δ_+) is the class of functions $F : [0, \infty] \to [0, 1]$ with the following properties:

1) $F(0) = 0$,
2) F is nondecreasing ,
3) F is left continuous on $(0, \infty)$.

If X is a nonempty set , a mapping F from $X \times X \to \Delta_+$ is called a probabilistic distance on X, and $F(x, y)$ is denoted by F_{xy}.

A triangular norm (shorter t-norm) is a binary operation on the unit interval $[0, 1]$ which the following conditions are satisfied:

1) $T(a, 1) = a$ for every $a \in [0, 1]$,
2) $T(a, b) = T(b, a)$ for every $a, b \in [0, 1]$,
3) $a \geq b, \ c \geq d \ \Rightarrow \ T(a, c) \geq T(b, d) \ \ a, b, c \in [0, 1]$.
4) $T(T(a, b), c) = T(a, T(b, c))$,

Basic examples are t-norms T_L (Lukasiewicz t-norm), T_P and T_M, defined by $T_L(a, b) = \max\{a + b - 1, 0\}$, $T_P(a, b) = ab$ and $T_M(a, b) = \min\{a, b\}$.

The triple (X, F, T), where X is a nonempty set, F is a probabilistic distance on X, and T is a t-norm, is called a generalized Menger space if it satisfies:

1) $F_{xx}(t) = 0$ for all $t > 0$,
2) $F_{xy}(t) = 1$ for all $t > 0$,
2) $F_{xy} = F_{yx} \quad \forall \ x, y \in X$,
3) $F_{xz}(s + t) \geq T(F_{xy}(s), F_{yz}(t)) \qquad \forall \ x, y, z \in X, \ \forall \ s, t \geq 0$.

The (ϵ, λ)-topology on X is introduced by the family of neighborhoods

$\{U_x(\epsilon,\lambda)\}_{x\in X,\ \epsilon>0,\lambda\in(0,1)}$, where $U_x(\epsilon,\lambda)=\{y\in X\,|F_{x,y}(\epsilon)>1-\lambda\}$. If (X,F,T) is a generalized Menger space with $sup_{\ 0\leq\ t\ <1}T(t,t)=1$, then in the (ϵ,λ)-topology, X is a metrizable topological space.

A triple (X,M,T), where X is nonempty set, T is a continuous t-norm and M is a fuzzy set on $X^2\times[0,\infty)$, is called a fuzzy metric space in the sense of Kramosil and Michalek, when the following properties are satisfied [7]:

1) $M(x,y,0)=0\ \ \forall x,y\in X$,
2) $M(x,y,t)=1\ \ \forall t>0\ \ \Rightarrow\ x=y$,
3) $M(x,y,t)=M(y,x,t)$,
4) $M(x,y,.):\ [0,\infty]\to[0,1]$ is leftcontinuous, $\forall x,y\in X$,
5) $M(x,z,t+s)\geq T(M(x,y,t)\ ,M(y,z,s)),\ \ \forall\ x,y,z\ \in\ X$ and $\forall t,s>0$.

In [2] George and Veeramani defined, M is a fuzzy set on $X^2\times(0,\infty)$ and $(1),(4)$ are replaced, respectively, with $(1)^{'}$,$(4)^{'}$ below:
$1)^{'}\ \ M(x,y,t)>0$,
$4)^{'}\ \ M(x,y,.):[0,\infty]\to[0,1]$ is continuous, $\forall x,y\in X$,

A sequence $(x_n)_{n\in\ N}$ in a fuzzy metric space (X,M,T) is a M-cauchy sequence in the sense of George and Veeramani if for every $\epsilon>0$ and $\delta\in(0,1)$ there exists $n_0\in N$ such that $M(x_n,x_m,\epsilon)>1-\delta,\forall n\geq m\geq n_0$ and a G-cauchy sequence if $\lim_{n\to\infty}M(x_n,x_{n+m},t)=1$ for each $m\in\mathbb{N},\ t>0$ [2,3,16]

Definition 2.1 [14]: A B-contraction on a probabilistic space (X,F) is a self-mapping f of X for which

$$F_{f(p)f(q)}(kt)\geq F_{pq}(t)\quad\forall p,q\in X\ ,\forall t>0,$$

where k is a fixed constant in $(0,1)$.

Definition 2.2 [6]: Let X be a nonempty set and F be a probabilistic distance on X. A mapping $f:X\to X$ is called a C-contraction if there exists $k\in(0,1)$ such that for all $x,y\in X,\ t>0$

$$F_{xy}(t)>1-t\ \Rightarrow\ F_{f(x)f(y)}(kt)>1-kt.$$

There are some definitions for other types of contractions. For more details we refer the readers to [5].
Radu [12] showed that every C-contraction in a Menger space (X,F,T) with $T\geq T_L$ is actually a Banach contraction in the metric space (X,K), where $K(x,y)=\sup\{t\geq 0\mid t\leq 1-F_{xy}(t)\}$ and proved a more general result.
There are some other types of contraction which are beyond the scope of this

paper but now we bring some generalization of these contractions.
In [10], Mihet introduced the following definition.

Definition 2.3: Let (X, F) be a probabilistic metric space. A mapping $f : X \to X$ is said to be a (q, q_1)-contraction of (ϵ, λ)-type, where $q, q_1 \in (0, 1)$ if the following implication holds for every p_1, p_2 :

$$\forall \epsilon > 0,\ \forall \lambda \in (0,1) \ \ Fp_1, p_2(\epsilon) > 1 - \lambda \ \Rightarrow \ F_{f(p_1), f(p_2)}(q\epsilon) > 1 - q_1\lambda.$$

Now for presenting generalized C-contraction, let M be the family of all the mappings $m : \overline{R} \to \overline{R}$ such that the following conditions are satisfied:
1) $\forall\ t, s \geq\ 0 :\ m(t + s) \geq\ m(t) + m(s)$;
2) $m(t) = 0\ \Leftrightarrow\ t = 0$;
3) m is continuous.

Definition 2.4 [11]: Let (X, F) be a probabilistic metric space and $f : X \to X$. The mapping f is a generalized C-contraction if there exist a continuous, decreasing function $h :\ [0, 1] \to\ [0, \infty]$ such that $h(1) = 0$, $m_1, m_2 \in\ M$, and $k\ \in (0, 1)$ such that the following implication holds for every $p, q\ \in\ X$ and for every $t > 0$:

$$hoF_{p,q}(m_2(t)) < m_1(t) \ \Rightarrow \ hoF_{f(p),f(q)}(m_2(kt)) < m_1(kt).$$

If $m_1(s) = m_2(s) = s$, and $h(s) = 1 - s$ for every $s \in [0, 1]$, we obtain the Hicks definition.

3 Main Results

The notion of point convergence introduced by Gregori and Romaguera in [4]. We recall this in definition 3.1.

Definition 3.1 [4]: Let (X, M, T) be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be point convergent to $x \in X$ (shown as $x_n \xrightarrow{p} x$) if there exists $t > 0$ such that

$$\lim_{n\to\infty} M(x_n, x, t) = 1.$$

A George and Veeramani fuzzy metric space (X, M, T) with the point convergence is a space with convergence in sense of Frechet [1,9].
Mihet in [9] showed the existence of fixed point for fuzzy contractive mapping

in the case of p-convergency subsequence. Now, we will prove two alike theorems for these kinds of contractions.

Theorem 3.1: Let (X, M, T) be a George and Veeramani fuzzy metric space and $sup_{\ 0\ \leq\ a\ <1}\ T(a, a) = 1$ and $A : X \to X$ be a (q, q_1)-contraction (ϵ, λ)-type, where $q, q_1 \in (0, 1)$. If T is q_1-convergent ,i.e.,

$$\lim_{n\to\infty} T_{i=n}^{\infty}(1 - q_1^i) = 1, \qquad (1)$$

and for some $x \in X$ the sequence $A^n(x)$ has a p-convergent subsequence, then A has a fixed point.

Proof: A is (q, q_1)-contraction of (ϵ, λ)-type, where $q, q_1 \in (0, 1)$, so for every $\epsilon > 0$ and $\lambda \in (0, 1)$.

$$M(u, v, \epsilon) > 1 - \lambda \ \Rightarrow\ M(Au, Av, q\epsilon) > 1 - q_1\lambda, \quad \text{forall } u, v \in X. \quad (2)$$

Let $x_n = A^n(x) = A(x_{n-1})$for every $n \in \mathbb{N}$ and $(x = A^0(x))$. Also $x \in X$ and $\delta > 0$ be such that $M(x, A(x), \delta) > 0$. Since $M(x, A(x), .) \in D_+$ such a δ exists. Now let $\lambda_1 \in (0, 1)$ be such that $M(x, A(x), \delta) > 1 - \lambda_1$. From (2) we have $M(A(x), A^2(x), q\delta) > 1 - q_1\lambda_1$, and generally for every $n \in \mathbb{N}$

$$M(A^n(x), A^{n+1}(x), q^n\delta) > 1 - q_1^n\lambda_1. \qquad (3)$$

We prove that $\{A^n(x)\}$ is a cauchy sequence; i.e. that for every $\epsilon > 0$ and $\lambda \in (0, 1)$ there exist $n_0(\epsilon, \lambda) \in \mathbb{N}$ such that

$$M(A^n(x), A^{n+m}(x), \epsilon) > 1 - \lambda \quad \forall n \geq n_0(\epsilon, \lambda) \text{ and } \forall m \in \mathbb{N}.$$

Let $\epsilon > 0$ and $\lambda \in (0, 1)$ be given. Since the series $\sum_{n=1}^{\infty} q^n\delta$ converges, there exists $n_0(\epsilon)$ such that $\sum_{n=n_0}^{\infty} q^n\delta < \epsilon$. Then for every $n \geq n_0$:

$$M(A^n(x), A^{n+m}(x), \epsilon) \geq M(A^n(x), A^{n+m}(x), \sum_{n=n_0}^{\infty} q^n\delta)$$

$$\geq M(A^n(x), A^{n+m}(x), \sum_{i=n}^{n+m-1} q^n\delta)$$

$$\geq T(\ldots(T(M(A^n(x), A^{n+1}(x), q^n\delta), M(A^{n+1}(x), A^{n+2}(x), q^{n+1}\delta),$$

$$\ldots, M(A^{n+m-1}(x), A^{n+m}(x), q^{n+m-1}\delta)))$$

Let $n_1 = n_1(\lambda) \in \mathbb{N}$ be such that $T_{i=n_1}^{\infty}(1 - q_1^i) = 1 - \lambda$. Since (1) holds, such a number n_1 exists. By using (3), we obtain for every $n \geq \max(n_0, n_1)$ and every $m \in \mathbb{N}$:

$$\begin{aligned} M(A^n(x), A^{n+m}(x), \epsilon) &\geq T_{i=n}^{n+m-1}(1 - q_1^i \lambda_1) \\ &\geq T_{i=1}^{n+m-1}(1 - q_1^i) \\ &\geq T_{i=n}^{\infty}(1 - q_1^i \lambda_1) \\ &> 1 - \lambda. \end{aligned}$$

Suppose $\{x_n\}$ has a p-convergent subsequence $\{x_{n_j}\}$ which point converges to y_0. Then, there is a $t_0 > 0$ which $\lim_{n\to\infty} M(x_{n_j}, y_0, t_0) = 1$. Let $\lambda > 0$ be given since $sup_{\ 0\ \leq\ a\ <1}\ T(a,a) = 1$, there is a $\lambda_1 > 0$ such that $T((1-\lambda_1),(1-\lambda_1)) > 1 - \lambda$. Since $\{x_n\}$ and then $\{x_{n_j}\}$ are Cauchy sequences we can take n_0 large enough such that $M(x_{n_j}, x_{n_j+1}, t_0) > 1 - \lambda_1,\ \forall j \geq n_0$ and $M(x_{n_j}, y_0, t_0) > 1 - \lambda_1,\ \forall j \geq n_0$, then $M(x_{n_j+1}, y_0, 2t_0) \geq T((1-\lambda_1),(1-\lambda_1)) > 1 - \lambda$ which implies that:

$$x_{n_j+1} \xrightarrow{p} y_0,$$

and by (q, q_1)-contraction $(\epsilon,\ \lambda)$-type condition

$$M(x_{n_j}, y_0, 2t_0) > 1 - \lambda \ \Rightarrow\ M(A(x_{n_j}), A(y_0), 2qt_0) > 1 - 2q_1\lambda.$$

Then, $A(x_{n_j}) = x_{n_j+1} \xrightarrow{p} A(y_0)$, since the p-convergence is Frechet in a Goerge and Veeramani fuzzy metric space we get $A(y_0) = y_0$ which means y_0 is a fixed point.

Theorem 3.2: Let (X, M, T) be a George and Veermani fuzzy metric space and $A : X \to X$ be a generalized C-contraction and $sup_{\ 0\leq\ a<1} T(a,a) = 1$. Suppose that for some $x \in X$ the sequence $A^n(x)$ has a p-convergent subsequence. Then, A has a fixed point.

Proof: A satisfies the following condition:

$$hoM(u, v, m_2(t)) < m_1(t) \ \Rightarrow\ hoM(A(u), A(v), m_2(kt)) < m_1(kt),$$

for all $u, v \in X$ and for all $t > 0$, where $k \in (0,1)$ and h, m_1, m_2 are given as in definition 2.4. Let $x_n = A^n(x) = A(x_{n-1})$ for every $n \in \mathbb{N}$ and $(A^0(x) = x)$. First we show that the sequence $A^n(x)$ is a Caushy sequence. We prove that for every $\epsilon > 0$ and $\lambda \in\ (0,1)$ there exists an integer $N = N(\epsilon, \lambda) \in N$ such that for every $x, y \in X$ and $n \geq N$, $M(A^n(x), A^n(y), \epsilon) > 1 - \lambda$.
Since $h(0) \in R$ and $\lim_{s\to\infty} m_1(s) = \infty$, from $M(x, y, m_2(s)) \geq 0$ it follows that there exists $s \in R$ such that $h(0) < m_1(s)$, also from $M(p, q, m_2(s)) \geq 0$

it follows that

$$hoM(x,y,m_2(s)) \leq h(0) < m_1(s),$$

which implies that $hoM(A(x),A(y),m_2(ks)) < m_1(ks)$, and then for every $n \in N$ by induction

$$hoM(A^n(x),A^n(y),m_2(k^n s)) < m_1(k^n s).$$

Suppose $N = N(\epsilon,\lambda)$ be such that $m_2(k^n s) < \epsilon$, $m_1(k^n s) < h(1-\lambda)$ for every $n > N$. Then $n > N$ implies that:

$$M(A^n(x),A^n(y),\epsilon) > M(A^n(x),A^n(y),m_2(k^n s)) > 1-\lambda.$$

Now let $y = A^m(x)$, then we obtain:

$$M(A^n(x),A^{n+m}(x),\epsilon) > 1-\lambda \ \ \forall\, n,m > N,$$

which means that $A^n(x)$ is a Cauchy sequence.
Suppose $\{x_n\}$ has a p-convergent subsequence $\{x_{n_j}\}$ which point converges to y_0. Then, there is a $t_0 > 0$, such that:

$$\lim_{j\to\infty} M(x_{n_j},y_0,t_0) = 1.$$

Let $\epsilon > 0$ be given since $sup_{\ 0\leq\ a<1} T(a,a) = 1$, there is a $\delta > 0$ such that $T((1-\delta),(1-\delta)) > 1-\epsilon$. Since $\{x_n\}$ and then $\{x_{n_j}\}$ are Cauchy sequences we can take n_0 large enough such that $M(x_{n_j},x_{n_j+1},t_0) > 1-\delta, \quad \forall j \geq n_0$ and $M(x_{n_j},y_0,t_0) > 1-\delta, \quad \forall j \geq n_0$, then $M(x_{n_j+1},y_0,2t_0) \geq T((1-\delta),(1-\delta)) > 1-\epsilon$ which implies that:

$$x_{n_j+1} \xrightarrow{p} y_0$$

.
Since $\{x_n\}$ and then $\{x_{n_j}\}$ are Cauchy sequences we can take n_0 large enough such that $M(x_{n_j},x_{n_j+1},t_0) > 1-\delta$, $\forall\, j \geq\ n_0$ and $M(x_{n_j},y_0,t_0) > 1-\delta$, $\forall\, j \geq n_0$, then $M(x_{n_j+1},y_0,2t_0) \geq\ T((1-\delta),(1-\delta)) > 1-\epsilon$ which implies that $x_{n_j+1} \xrightarrow{p} y_0$.
Now by generalized C-contraction condition

$$hoM(x_{n_j},y_0,m_2(2t_0)) < m_1(2t_0) \ \Rightarrow\ hoM(A(x_{n_j}),A(t_0),m_2(2kt_0)) < m_1(2kt_0),$$

then, $A(x_{n_j}) = x_{n_j+1} \xrightarrow{p} A(y_0)$ and since the p-convergence is Frechet in a Goerge and Veeramani fuzzy metric space, then $A(y_0) = y_0$ which means y_0 is a fixed point.

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