Applied Mathematical Sciences, Vol. 2, 2008, no. 47, 2309 - 2322

(h, k)-Dichotomy and Periodic Solutions of a Kind of Integro-Differential Equations[1]

Dexian Sun

College of Mathematics and Computer Science
Fuzhou University, Fuzhou, 350002, China
sdexian@163.com

Abstract

In the present paper, we investigate some important properties of (h, k)-dichotomy. Furthermore, we study the higher dimensional periodic system

$$x'(t) = A(t)x(t) + \int_{-\infty}^{t} C(t,s)P(x(s))\mathrm{d}s + \sum_{i=1}^{l} g_i(t, x(t-\tau_i(t))) + f(t).$$

Based on the properties of (h, k)-dichotomy obtained in this paper, some new criteria for the existence and uniqueness of periodic solution of the above system are obtained. The approaches are based on the fixed point theory and functional analysis method. Due to the generality of (h, k)-dichotomy, our results generalize some previously known results.

Mathematics Subject Classification: 34C25, 34K15, 34A12

Keywords: Periodic solution, (h, k)-dichotomy, integro-differential equations

1 Introduction

The properties of the exponential dichotomy and its applications have been investigated extensively in theory and in practice (see e.g.[1-9,23-32] and the references cited therein). One of the important applications is to use the exponential dichotomy theory to study the existence and uniqueness of (almost) periodic solutions to the integro-differential equations.

[1]This work was supported by the Natural Science Foundation of Fujian Province of China under Grant (No. S0750008) and the Educational Breau Foundation of Fujian Province of China under Grant (No. JB06029).).

The existence of periodic solution of nonlinear Volterra equation with infinitely delay has been well discussed by Burton under the boundedness condition (see [10]), after introducing the space of ($\mathrm{BC}_{-\infty}$,ρ) by combining Lyapunov function (functional) and fixed point theory, Huang (see [11]) obtained the sufficient conditions which guarantee the existence of periodic solutions of following infinite delay system

$$x'(t) = f(t, x_t). \tag{1.1}$$

By introducing the C_h space, Wang and Huang [12] generalized the Yoshizawa theorem [7, Theorem 37.1] to the Eq.(1.1). Also, paper [11,12] studied the following equations successively,

$$x'(t) = A(t)x(t) + \int_{-\infty}^{t} C(t,s)x(s)\mathrm{d}s + f(t) \tag{1.2}$$

some frondose sufficient conditions which guarantee the existence of periodic solution of system (1.2) are obtained. Wang [14] and Chen [33] considered a kind of differential equation which is more generalized than (1.2), that is

$$x'(t) = A(t)x(t) + \int_{-\infty}^{t} C(t,s)x(s)\mathrm{d}s + g(t,x(t)) + f(t). \tag{1.3}$$

By using exponential dichotomy and fixed point theorem, they discussed the existence, uniqueness and stability of periodic solution of (1.3), respectively.

Indeed, the exponential dichotomy theory plays great role in the ordinary differential equations and functional equations. However, the requirements of the exponential dichotomy are relatively stronger. More specifically, consider the following linear system

$$\dot{x}(t) = A(t)x(t). \tag{1.4}$$

The linear system (1.4) is said to possess an exponential dichotomy, if there is a projection matrix P and positive constants $K_-, K_+ \in R$, α such that the following inequalities hold:

$$\begin{cases} \|U(t)PU^{-1}(s)\| \le K_- e^{-\alpha(t-s)}, & t \ge s, \\ \|U(t)(I-P)U^{-1}(s)\| \le K_+ e^{\alpha(t-s)}, & t \le s, \end{cases} \tag{1.5}$$

where $U(t)$ is the fundamental matrix of system (1.4). A solution-matrix $U(t)$ is said to be a fundamental-matrix, if $U(0) = I$.

But in bad situations, system (1.4) can not satisfy (1.5). For example,

$$\begin{cases} \|U(t)PU^{-1}(s)\| \le K_- e^{-\alpha(t-s)}, & t \ge s, \\ \|U(t)(I-P)U^{-1}(s)\| \le K_+, & t \le s, \end{cases} \tag{1.6}$$

which is so-called *expo-ordinary dichotomy.* Even for the autonomous linear systems, the exponential dichotomy is not to cover all the possibilities. This occurs, for instance, for system (1.4) with $A(t) = A$ an $n \times n$ constant matrix having eigenvalues λ so that $Re\lambda \neq 0$ or $Re\lambda = 0$ for λ simple. The situation with multiple eigenvalues λ such that $Re\lambda = 0$ give rise to other possibilities. For this reason, Fenner, Pinto, Naulín etc. generalize the concept of exponential dichotomy to (h, k)-dichotomy (see e.g. [15-21]). Though the definition has many forms, they are the same in essence and we use the definition proposed by Fenner [15].

Definition 1.1 *Let two positive continuous functions $h(t)$, $k(t) : R \to R_+$ be given. The linear system (1.4) is said to possess an (h, k) dichotomy, if there is a projection matrix P and positive constants $K_-, K_+ \in R$, α such that the following inequalities hold:*

$$\begin{cases} \|U(t)PU^{-1}(s)\| \leq K_- h(t)h(s)^{-1}e^{-\alpha(t-s)}, & t \geq s, \\ \|U(t)(I-P)U^{-1}(s)\| \leq K_+ k(t)k(s)^{-1}e^{\alpha(t-s)}, & t \leq s, \end{cases} \tag{1.7}$$

where $U(t)$ is the fundamental matrix of system (1.4) with $U(0) = I$.

Remark 1.1 Obviously, the exponential dichotomy is a special case of the (h, k)-dichotomy. In fact take $h = k = 1$, (h, k)-dichotomy yields an exponential dichotomy.

A very important property on the roughness of the (h, k)-dichotomy has been well studied in [18]. Moreover, the (h, k)-dichotomy has been applied to study the asymptotic behavior of linear or nonlinear systems (see [19,21,22]).

However, there are few paper considering the existence of periodic solution for nonlinear system by using (h, k)-dichotomy. There is still no published paper applying the (h, k)-dichotomy to study the integral-differential equation. So, in the present paper, we consider following more generalized system

$$x'(t) = A(t)x(t) + \int_{-\infty}^{t} C(t, s)P(x(s))\mathrm{d}s + \sum_{i=1}^{l} g_i(t, x(t - \tau_i(t))) + f(t). \tag{1.8}$$

Due to the generality of (h, k) dichotomy, the criteria established for the existence, uniqueness and stability of the periodic solution are new and more general.

2 Properties of (h, k)-dichotomy

Let $\mathbb{R}^n$ denote the set of read n-vector, and $|x|$ any convenient norm for $x \in \mathbb{R}^n$, also let $\mathbb{R} = \mathbb{R}^1$.

Thanks to the works of Pinto, Naulín and Fenner etc. (see [15-21]), the notion of (h, k)-dichotomy spectrum for linear nonautonomous ordinary differential equations has been well defined. Here, we summarize those definitions as follows.

Consider the linear nonautonomous system

$$\dot{x} = A(t)x, \tag{2.1}$$

Definition 2.1 *h and k are said to fulfill a compensation law* (**CL**) *if there exists a positive constant $C_{h,k}$ such that*

$$h(t)h(s)^{-1} \leq C_{h,k}k(t)k(s)^{-1}, \quad \forall\ t \geq s. \tag{2.2}$$

Definition 2.2 *The system is said to be an h-system, if it has an (h, h)-dichotomy.*

Remark 2.1 Clearly, a system having an (h, k)-dichotomy with compensation law (**CL**) belongs to the class of h-systems.

Definition 2.3 *System (2.1) is said to satisfy an integral condition (***I condition***), if there exists a projection P and $\mu > 0$ such that*

$$\int_{-\infty}^{t} \|U(t)PU^{-1}(s)\|\mathrm{d}s + \int_{t}^{+\infty} \|U(t)(I-P)U^{-1}(s)\|\mathrm{d}s \leq \mu \tag{2.3}$$

uniformly in $t \in \mathbb{R}$.

Definition 2.4 *We say that (h, k) are integral if there exists $\mu > 0$ such that*

$$K_{-}\int_{-\infty}^{t} h(t)h(s)^{-1}e^{-\alpha(t-s)}\mathrm{d}s + K_{+}\int_{t}^{+\infty} k(t)k(s)^{-1}e^{\alpha(t-s)}\mathrm{d}s \leq \mu \tag{2.4}$$

uniformly in $t \in \mathbb{R}$.

Remark 2.2 Clearly, if the system has an (h, k)-dichotomy with integral functions, then necessarily the **I condition** is satisfied.

Remark 2.3 Obviously, the case $h = k = 1$, i.e., exponential dichotomy, the **I condition** is always satisfied. In fact, one can take $\mu = \frac{1}{\alpha}[K_{-} + K_{+}]$.

Theorem 2.1 *Under the* **I condition***, if system (2.1) has the (h, k)-dichotomy of the form (1.7), then $x(t) = 0$ is the unique bounded solutions of system (2.1).*

Proof Define $B_0 \subset R^n$ to be the set of initial conditions $\xi \in R^n$ pertaining to bounded solutions of Eq. (2.1). Take any n-vector $\xi \in R^n$ and assume first that $(I-P)\xi \neq 0$. Define $\phi(t) = 1/\|U(t)(I-P)\xi\|$, by using $(I-P)^2 = I-P$ we may write

$$\int_{t}^{+\infty} \phi(s)U(t)(I-P)\xi\mathrm{d}s = \int_{t}^{\infty} U(t)(I-P)U^{-1}(s)U(s)(I-P)\xi\phi(s)\mathrm{d}s \tag{2.5}$$

so that upon taking norms and using the (h, k) dichotomy as well as the **I condition**, we have

$$\int_{t}^{\infty} \phi(s)\mathrm{d}s \leq \mu\phi(t), \quad \text{uniformly in } t. \tag{2.6}$$

Calculating the derivative both sides and by the comparison theorem, one has $-\phi(t) \le \mu\frac{\mathrm{d}\phi(t)}{\mathrm{d}t}$, which implies $\phi(t) \ge \phi(0)e^{-\frac{1}{\mu}t}$. For this, it must be that $\lim \inf_{s\in[t,\infty)} \phi(s) = 0$, which means that $\|U(t)(I-P)\xi\|$ must be unbounded.

If we assume now that $P\xi \neq 0$, then defining $\phi(t) = 1/\|U(t)P\xi\|$, then defining $\phi(t) = 1/\|U(t)P\xi\|$, we perform the same procedure, with the integral over the interval $(-\infty, t]$; we conclude that $\lim \inf_{s\in(-\infty,t]} = 0$, which means $\|U(t)P\xi\|$ must be unbounded. Thus the only possibility for boundedness of the solutions of system (2.1) is that $B_0 = \{0\}$ i.e. $x(t) = 0$.

Theorem 2.2 *Under the* **I condition***, if the linear system (2.1) possesses an (h,k)-dichotomy of the form (1.7), then the projector P is unique, i.e., P was decided uniquely by (h,k).*

Proof Assume that there exists another projector $\widetilde{P}$, and positive constants $\widetilde{K}_\pm$, $\widetilde{\alpha}$ satisfying the formula (1.7), i.e.,

$$\begin{aligned} &\|U(t)\widetilde{P}U^{-1}(s)\| \le \widetilde{K}_- h(t)h(s)^{-1}e^{-\widetilde{\alpha}(t-s)}, \quad t \ge s, \\ &\|U(t)(I-\widetilde{P})U^{-1}(s)\| \le \widetilde{K}_+ k(t)k(s)^{-1}e^{\widetilde{\alpha}(t-s)}, \quad t \le s, \end{aligned} \tag{2.7}$$

and **I condition**

$$\widetilde{K}_- \int_{-\infty}^{t} h(t)h(s)^{-1}e^{-\widetilde{\alpha}(t-s)}\mathrm{d}s + \widetilde{K}_+ \int_{t}^{+\infty} k(t)k(s)^{-1}e^{\widetilde{\alpha}(t-s)}\mathrm{d}s \le \mu. \tag{2.8}$$

It follows from (2.8) that $K_- h(t)e^{-\alpha t}\int_{-\infty}^{t} h(s)^{-1}e^{\alpha s}\mathrm{d}s \le \mu$. Let $\varphi(t) = h(t)e^{-\alpha t}$, we have $\int_{-\infty}^{t}\frac{1}{\varphi(s)}\mathrm{d}s \le K_-^{-1}\mu\frac{1}{\varphi(t)}$ calculating the derivative with respect to t, one has $\varphi'(t) \le -K_-\mu^{-1}\varphi(t)$ which implies $\varphi(t) \le \varphi(0)\exp\{-K_-\mu^{-1}t\}$. Thus, there exists $M_1 > 0$ such that

$$\varphi(t) = h(t)e^{-\alpha t} \le M_1, \quad \text{for all} \ \ t \ge 0. \tag{2.9}$$

On the other hand, it also follows from (2.8) that $K_+\int_{t}^{+\infty} k(t)k(s)^{-1}e^{\alpha(t-s)}\mathrm{d}s \le \mu$. Let $\psi(t) = k(t)e^{\alpha t}$, we have $\psi'(t) \le K_+\mu^{-1}\psi(t)$, which implies $\psi(t) \le \psi(0)\exp\{K_+\mu^{-1}t\}$. Therefore, there exists $M_2 > 0$ such that

$$\psi(t) = k(t)e^{\alpha t} \le M_2, \quad \text{for all} \ \ t \le 0. \tag{2.10}$$

Similar to the discussion of (2.9), (2.10), there exists $\widetilde{M_1}$, $\widetilde{M_2} > 0$ such that

$$h(t)e^{-\widetilde{\alpha}t} \le \widetilde{M_1}, \quad \text{for all } t \ge 0, \qquad k(t)e^{\widetilde{\alpha}t} \le \widetilde{M_2}, \quad \text{for all} \ \ t \le 0. \tag{2.11}$$

Take any $\xi \in \mathbb{R}^n$, for $t \ge 0$, it follows from (1.7), (2.7) and (2.9) that

$$\|U(t)P(I-\widetilde{P})\xi\| \le K_- h(t)h(0)^{-1}e^{-\alpha t}\|(I-\widetilde{P})\xi\| \le K_- h(0)^{-1}M_1\|(I-\widetilde{P})\xi\|, \tag{2.12}$$

$(t \ge 0)$.

On the other hand, for $t \le 0$, it follows from (1.7), (2.7) and (2.11) that

$$\begin{aligned}\|U(t)P(I-\widetilde{P})\xi\| &\le \|U(t)PU^{-1}(t)\| \cdot \|U(t)(I-\widetilde{P})U^{-1}(0)\| \cdot \|U(0)(I-\widetilde{P})\xi\| \\ &\le K_-h(t)h(t)^{-1} \cdot \widetilde{K}_+k(t)k(0)^{-1}e^{\widetilde{\alpha}t}\|(I-\widetilde{P})\xi\| \\ &\le K_-\widetilde{K}_+k(0)^{-1}\widetilde{M}_2\|(I-\widetilde{P})\xi\|, \ (t \le 0). \end{aligned} \tag{2.13}$$

It follows from (2.12) and (2.13) that for any $\xi \in \mathbb{R}^n$, $x(t) = U(t)P(I-\widetilde{P})$ is the bounded solution of system (2.1). By Theorem 2.1, we have $P(I-\widetilde{P})\xi = 0$, which implies $P = P\widetilde{P}$. Similar to the above discussion, we also have $(I-P)\widetilde{P} = 0$, i.e., $\widetilde{P} = P\widetilde{P}$. Therefore, $P = P\widetilde{P} = \widetilde{P}$. This shows that the projection P is unique. □

Theorem 2.3 *Under the* **I condition***, let the linear system (2.1) possesses an (h,k)-dichotomy of the form (1.7) with the projection P, if further assume that $A(t+T) = A(t)$, $U(t)$ is the fundamental matrix. Then $U(t)PU^{-1}(t)$ is also a T-periodic function.*

Proof By the periodicity, we note that $U(t+T)$ is also a solution matrix of (2.1). Obviously, we have $U(t+T) = U(t)C$. By using $U(0) = I$, we have $C = U(T)$. Therefore, $U(t+T) = U(t)U(T)$. Since (2.1) possesses an (h,k)-dichotomy of the form (1.7), we have

$$\begin{cases} \|U(t)U(T)PU^{-1}(T)U^{-1}(s)\| \le K_-h(t)h(s)^{-1}e^{-\alpha(t-s)}, & t \ge s, \\ \|U(t)U(T)(I-P)U^{-1}(T)U^{-1}(s)\| \le K_+k(t)k(s)^{-1}e^{\alpha(t-s)}, & t \le s, \end{cases}$$

we note that $U(T)PU^{-1}(T)$ is also a projection. By projection 2.2, the projection P is unique. Thus, $U(T)PU^{-1}(T) = P$. Therefore, $U(t+T)PU^{-1}(t+T) = U(t)U(T)PU^{-1}(T)$
$U^{-1}(t) = U(t)PU^{-1}(t)$, i.e., $U(t)PU^{-1}(t)$ is T-periodic function. □

Consider the following system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + f(t) \tag{2.14}$$

where $A(t)$ is continuous matrix function, $f(t)$ is continuous bounded function.

Theorem 2.4 *Under the* **I condition***, if the homogeneous linear system $\dot{x}(t) = A(t)x$ possess (h,k)-dichotomy, then system (2.14) has exactly one bounded solution which can be represented as follows*

$$x(t) = \int_{-\infty}^{t} U(t)PU^{-1}(s)f(s)\mathrm{d}s - \int_{t}^{+\infty} U(t)(I-P)U^{-1}(s)f(s)\mathrm{d}s. \tag{2.15}$$

Proof Let $x(t) = \int_{-\infty}^{t} U(t)PU^{-1}(s)f(s)\mathrm{d}s - \int_{t}^{+\infty} U(t)(I-P)U^{-1}(s)f(s)\mathrm{d}s$. It is not difficult to check that $x(t)$ is a solution of (2.15). By **I condition**, simple computation shows $\|x(t)\| \leq \mu\|f\|$, which implies $x(t)$ is a bounded solution of (2.15).

If there exists another bounded solution $y(t)$, obviously, $x(t) - y(t)$ is a bounded solution of the homogeneous linear system (2.1). By Theorem 2.1, $x(t) - y(t) \equiv 0$, i.e., $x(t) \equiv y(t)$. The uniqueness of the bounded solution of (2.15) is proved. □.

Theorem 2.5 *Let the all the conditions in Theorem 2.4 hold, if further assume that* $A(t+T) = A(t)$ *and* $f(t+T) = f(t)$*. Then system (2.14) has exactly one* T*-periodic solution, which can be represented as (2.15).*

Proof By Theorem 2.3, it is not difficult to check that $x(t)$ is a T-periodic solution, we note that periodic solution is a bounded solution. Then, by Theorem 2.4, Theorem 2.5 is proved immediately.

3 Existence of T-periodic solutions

Let $BC(-\infty, t_0] = \{\varphi(t) |\ \varphi : (-\infty, t_0] \to \mathbb{R}^n$ is a bounded continuous function $\}$, for any $\varphi \in BC(-\infty, t_0]$, let the norm of φ be $\|\varphi\| = \sup\{\|\varphi(t)\| :\ t \in (-\infty, t_0]\}$. Let $x(t, t_0, \varphi)$ (or $x(t, \varphi), x(t)$ for convenience) denote the solution of system (1.8) with bounded continuous initial function $\varphi \in BC(-\infty, t_0]$.

Considering the following nonlinear integro-differential equations with both continuous delay and discrete delay of the form

$$x'(t) = A(t)x(t) + \int_{-\infty}^{t} C(t,s)P(x(s))\mathrm{d}s + \sum_{i=1}^{l} g_i(t, x(t - \tau_i(t))) + b(t) \quad (3.1)$$

and its linear system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x(t) \quad (3.2)$$

where $x \in \mathbb{R}$, $A(t) = (a_{ij}(t))_{n\times n}$, $C(t,s) = (c_{ij}(t,s))_{n\times n}$ are $n \times n$ matrix functions; $A(t)$ is continuous on $\mathbb{R}$, $C(t,s)$ is continuous on $R \times R$; $b : R \to \mathbb{R}^n$ is continuous; $g_i : R \times R^n \to R^n$ is continuous, T-periodic function in t ($i = 1, \cdots, l$). Now we introduce the following conditions:

•(H_1) The linear system (3.2) possesses a (h,k)-dichotomy under the **I condition**.

•(H_2) $A(t+T) = A(t)$, $b(t+T) = b(t)$, $\tau_i(t+T) = \tau_i(t)$, $i = 1, \cdots, l$.

•(H_3) There exists nonnegative T-periodic functions $b_i(t)$ such that

$$\|g_i(t,x) - g_i(t,y)\| \leq b_i(t)\|x - y\|, \quad i = 1, 2, \cdots, l,\ x, y \in R^n.$$

•(H_4) $\int_{-\infty}^{t} \|C(t,s)\|\mathrm{d}t$ is bounded and $C(t+T, s+T) = C(t,s)$, for any $t, s \in \mathbb{R}$.

•(H_5) For arbitrary $\varepsilon > 0$, there exists $L_0 = L_0(\varepsilon) > 0$, such that

$$\int_{-\infty}^{t_1} \|C(t,s)\|\mathrm{d}s < \varepsilon, \quad \text{for any} \quad t_1 > -\infty, \quad t - t_1 \geq L_0,$$

•(H_6) There exists nonnegative T-periodic functions $d_i(t)$ such that

$$\lim_{n\to+\infty} \frac{1}{n} \sup_{\|x\|\leq n} \|g_i(t,x)\| = d_i(t), \quad i = 1, 2, \cdots, l, \quad \text{uniformly for all } t \in \mathbb{R}.$$

•(H_7) $P(0) = 0$, $\|P(x) - P(y)\| \leq L\|x - y\|$, $x, y \in R^n$, L is a positive constant.

•(H_8) There exists a positive constant $\rho_1 < \frac{1}{\mu}$ such that $L\int_{-\infty}^{t} C(t,s) + \sum_{i=1}^{l} b_i(t) \leq \rho_1$.

•(H_9) There exists a positive constant $\rho_2 < \frac{1}{\mu}$ such that $L\int_{-\infty}^{t} C(t,s) + \sum_{i=1}^{l} d_i(t) \leq \rho_2$.

•(H_{10}) There exist positive continuously differentiable T-periodic functions $p_i(t)$, $i = 1, 2, ...n$ and continuous T-periodic function $\alpha_1(t)$ such that

$$\dot{p}_i(t)+p_i(t)a_{ii}(t)+\sum_{j=1,j\neq i}^{n} |a_{ji}(t)| \leq \alpha_1(t)p_i(t), \text{ and } k_1 = \exp\Big(\int_0^T \alpha_1(\tau)d\tau\Big) < 1.$$

•(H_{11}) There exist positive continuously differentiable T-periodic functions $p_i(t)$, $i = 1, 2, ...n$ and continuous T-periodic function $\alpha_2(t)$ such that

$$\dot{p}_i(t)+p_i(t)a_{ii}(t)+\sum_{j=1,j\neq i}^{n} |a_{ji}(t)| \geq \alpha_2(t)p_i(t), \text{ and } k_2 = \exp\Big(\int_0^T (-\alpha_2(\tau))d\tau\Big) < 1.$$

•(H_{12}) $\bar{p}(t) = \max_i\{p_i(t)\} \leq p_1$ and $p(t) = \min_i\{p_i(t)\} \geq p_2 > 0$.

In order to process further, we also need the following Lemmas.

Lemma 3.1[13,Lemma 3.2],[32,Lemma 3.4] *Let $C(t,s)$ be a continuous $n \times n$ matrix function, if the following assumptions are satisfied.*

(i) $\int_{-\infty}^{t} \|C(t,s)\|\mathrm{d}s$ is bounded and $C(t+T, s+T) = C(t,s)$ for any $t, s \in \mathbb{R}$.

(ii) For arbitrary $\varepsilon > 0$, there exists $L_0 = L_0(\varepsilon) > 0$ such that $\int_{-\infty}^{t_1} \|C(t,s)\|\mathrm{d}s < \varepsilon$, for any $t_1 > -\infty$, $t - t_1 \geq L_0$.

(iii) $f \in C(R, R^n)$, $f_1(t+T) = f_1(t)$.

Then $g(t) = \int_{-\infty}^{t} C(t,s)f_1(s)\mathrm{d}s$ is also a continuous T-periodic function.

Lemma 3.2[14,Lemma 2.1],[32,Lemma 3.1] *Let $U(t)$ be any fundamental matrix of system (3.2), if $A(t)$ satisfies H_{10} and H_{12}, then*
$\|U(t)U^{-1}(s)\| \leq \frac{p_1}{p_2} \exp\Big(\int_s^t \alpha_1(\tau)d\tau\Big)$, $t \geq s$. *If $A(t)$ satisfies H_{10} and H_{12},*

then
$\|U(t)U^{-1}(s)\| \le \frac{p_1}{p_2}\exp\Big(\int_t^s(-\alpha_2(\tau))d\tau\Big),\ t \le s.$

Lemma 3.3[13,Lemma 2.3],[32,Lemma 3.6] *If the T-periodic function $\alpha_1(t)$ satisfies $k_1 < 1$, then $\exp\Big(\int_s^t \alpha_1(\tau)d\tau\Big) \le \beta\exp(-\alpha(t-s)),\ t \ge s$, where $\alpha = -\frac{1}{T}\ln k_1 > 0, \beta = \exp(\alpha T)\sup\{\exp(\int_s^t \alpha_1(\tau)d\tau), 0 \le s \le t \le T\}$.*

Lemma 3.4[32,Lemma 3.6] *If the T-periodic function $\alpha_2(t)$ satisfies $k_2 < 1$, then $\exp\Big(-\int_t^s \alpha_1(\tau)d\tau\Big) \le \beta\exp(\alpha(t-s)),\ t \le s$, where $\alpha = -\frac{1}{T}\ln k_1 > 0, \beta = \exp(\alpha T)\sup\{\exp(-\int_t^s \alpha_1(\tau)d\tau), 0 \le t \le s \le T\}$.*

Now we are ready to state our results on the existence of T-periodic solution for the integro-differential equations.

Theorem 3.1 *If the assumptions (H_1)-(H_5), (H_7) and (H_8) hold, system (3.1) has exactly one T-periodic solution.*

Theorem 3.2 *If the assumptions (H_1)-(H_2), (H_4)-(H_7) and (H_9) hold, system (3.1) has at least one T-periodic solution.*

By Lemma 3.2 and Lemma 3.3, taking $P = I$ and $K_- = \beta\frac{p_1}{p_2}, \mu = \frac{K_-}{\alpha}$, system (3.2) possesses an exponential dichotomy with the projector $P = I$. Then we have following corollaries.

Corollary 3.1 *If the assumptions (H_2)-(H_5), (H_7), (H_8), (H_{10}) and (H_{12}) hold, system (3.1) has exactly one T-periodic solution.*

Corollary 3.2 *If the assumptions (H_2), (H_4)-(H_7), (H_9), (H_{10}) and (H_{12}) hold, system (3.1) has at least one T-periodic solution.*

By Lemma 3.2 and Lemma 3.4, taking $P = 0$ and $K_+ = \beta\frac{p_1}{p_2}, \mu = \frac{K_+}{\alpha}$, system (3.2) possesses an exponential dichotomy with the projector $P = 0$. Then we have following corollaries.

Corollary 3.3 *If the assumptions (H_2)-(H_5), (H_7), (H_8), (H_{11}) and (H_{12}) hold, system (3.1) has exactly one T-periodic solution.*

Corollary 3.4 *If the assumptions (H_2), (H_4)-(H_7), (H_9), (H_{11}) and (H_{12}) hold, system (3.1) has at least one T-periodic solution.*

Remark 3.1 Take $h = k \equiv 1$ in condition (H_1), then (h,k)-dichotomy reduces to the exponential dichotomy. Therefore, the previous known results which the exponential dichotomy theory is employed to analysis the integro-differential equations are special cases of Theorems 3.1 and 3.2. In condition (H_{11}), taking $p_i(t) \equiv 1$, conditions (H_{11}) reduces to $\alpha(t) = \max_{1\le j\le n}\{a_{jj}(t) + \sum_{j=1,j\ne i}^{n} |a_{ij}(t)|\}$ which is used in [13]. It is in some sense that we generalize those results [13].

Remark 3.2 Considering system (3.1) without delays, that is

$$x' = Ax + g(t,x) \tag{3.3}$$

Krasnoselski (see [6, P12]) proved that under the condition $\lim_{\|x\|\to+\infty}\frac{\|g(t,x)\|}{\|x\|} = 0$, system (3.3) has at least one T-periodic solution. In fact, applying Theorem 3.2 to system (3.3), we just require that $\overline{\lim\limits_{\|x\|\to+\infty}} \frac{\|g(t,x)\|}{\|x\|} = d(t)$, $d(t)$ is a nonnegative T-periodic function.

Remark 3.3 As a special case, when $C(t,s) = C(t-s)$, (H$_4$) can be reduced to $\int_{-\infty}^{t} \|C(t,s)\|\mathrm{d}s = \int_{-\infty}^{t} \|C(t-s)\|\mathrm{d}s = \int_{0}^{+\infty} \|C(s)\|\mathrm{d}s < +\infty$.

Proof of Theorem 3.1

$$\text{Let } B = \{u(t)|\ u : R \to \mathbb{R}^n \text{ is continuous } T\text{-periodic function }\}.$$

Obviously, B is Banach space with the norm $\|u\| = \sup\{|u(t)| :\ 0 \le t \le T\}$. For any $u \in B$, consider the following integro-differential equation

$$x'(t) = A(t)x(t) + \int_{-\infty}^{t} C(t,s)P(u(s))\mathrm{d}s + \sum_{i=1}^{l} g_i(t, u(t-\tau_i(t))) + b(t) \quad (3.4)$$

By the conditions (H$_1$)-(H$_7$) and Theorem 2.5, system (3.4) has exactly one T-periodic solution which can be written as

$$\begin{aligned} x_u(t) \ =& \int_{-\infty}^{t} U(t)PU^{-1}(r)\Big[\int_{-\infty}^{r} C(r,s)P(u(s))\mathrm{d}s + \sum_{i=1}^{l} g_i(r, u(r-\tau_i(r))) + b(r)\Big]\mathrm{d}r \\ &- \int_{t}^{+\infty} U(t)(I-P)U^{-1}(r)\Big[\int_{-\infty}^{r} C(r,s)P(u(s))\mathrm{d}s + \sum_{i=1}^{l} g_i(r, u(r-\tau_i(r))) + b(r)\Big]\mathrm{d}r. \quad (3.5) \end{aligned}$$

Define the operator $\mathcal{T} : B \to B$ as follows:

$$\mathcal{T}u(t) = x_u(t), \quad \forall\ u \in B.$$

By (H$_8$), we shall prove that $\mathcal{T}$ is a contraction mapping in B. In fact, for any $u_1, u_2 \in B$, it follows from (3.5) and the conditions in Theorem 3.1 that

$$\begin{aligned} \|\mathcal{T}u_1(t) - \mathcal{T}u_2(t)\| \ \le& \ \Big[\int_{-\infty}^{t} K_- h(t)h(s)^{-1}e^{-\alpha(t-r)}\big[L\int_{-\infty}^{r} \|C(r,s)\mathrm{d}s + \sum_{i=1}^{l} b_i(r)\big]\mathrm{d}r \\ &+ \int_{t}^{+\infty} K_+ k(t)k(s)^{-1}e^{\alpha(t-r)}\big[L\int_{-\infty}^{r} \|C(r,s)\mathrm{d}s + \sum_{i=1}^{l} b_i(r)\big]\mathrm{d}r\Big]\|u_1 - u_2\| \\ \le& \ \mu\rho_1\|u_1 - u_2\|. \end{aligned}$$

That is $\|\mathcal{T}u_1(t) - \mathcal{T}u_2(t)\| \le \mu\rho_1\|u_1 - u_2\|$. It follows from $\mu\rho_1 < 1$, that $\mathcal{T}$ is a contraction mapping. Therefore $\mathcal{T}$ has exactly one fixed point u^* in B. It is easy to check u^* is the unique T-periodic solution of (3.1).

Proof of Theorem 3.2

Take the Banach space B and the operator $\mathcal{T}$ which have been defined in the proof of Theorem 3.1. Now by using Schauder's fixed point theorem, we shall prove that $\mathcal{T}$ has at least a fixed point under the assumption of Theorem 3.2. In order to prove this, we set

$$D_n = \{u|\ u \in B \text{ and } \|u\| \le n,\ n \in \mathbb{N}\}.$$

Step 1 First, we claim that there exists $N \in \mathbb{N}$ such that $\mathcal{T}: D_N \to D_N$. Or else, for any $n \in \mathbb{N}$, there exists $u_n \in D_n$ such that $\|\mathcal{T}u_n\| \geq n$. Since $b(t)$ is a continuous T-periodic function, there exists $M > 0$ such that $\|b(t)\| \leq M$, $t \in \mathbb{R}$. For any sufficiently small ε, it follows from (H$_6$) that there exists sufficient large $N_i \in \mathbb{N}$ such that if $n > N_i$ $(i = 1, \cdots, l)$ then

$$\frac{1}{n}\sup_{\|x\|\leq n}\|g_i(t,x)\| \leq d_i(t) + \frac{\varepsilon}{2l}, \quad i = 1, 2, \cdots, l. \tag{3.6}$$

and that there exist sufficient large $N_{l+1} \in \mathbb{N}$ such that

$$\frac{M}{n} < \frac{\varepsilon}{2}. \tag{3.7}$$

Take $N_0 = \max\{N_1, N_2, \cdots, N_l, N_{l+1}\} \in \mathbb{N}$, it follows from (3.6) and (3.7) that if $n > N_0$, then

$$\frac{1}{n}\sup_{\|x\|\leq n}\|g_i(t,x)\| \leq c_i(t) + \frac{\varepsilon}{2l}, \quad \frac{M}{n} < \frac{\varepsilon}{2}, \quad i = 1, 2, \cdots, l. \tag{3.8}$$

Therefore, it follows from the assumptions in Theorem 3.2 and (3.8) that

$$\begin{aligned}\frac{\|\mathcal{T}u_n\|}{n} &\leq \frac{1}{n}\Big\{\int_{-\infty}^{t}\|U(t)PU^{-1}(r)\|\Big[\int_{-\infty}^{r}\|C(r,s)\|L\|u_n(s)\|\mathrm{d}s + \Big\|\sum_{i=1}^{l}g_i(r,u_n(r-\tau_i(r)))\Big\| + M\Big]\mathrm{d}r\\ &\leq \mu\rho_2 + \mu\varepsilon.\end{aligned} \tag{3.9}$$

By the condition (H$_9$) and take sufficient small ε, we have $\mu\rho_2 + \mu\varepsilon < 1$. Therefore, it follows from (3.9) that $\overline{\lim\limits_{n\to+\infty}} \frac{\|\mathcal{T}u_n\|}{n} < 1$ which implies that for sufficient large n, $\frac{\|\mathcal{T}u_n\|}{n} < 1$. This is a contradiction to $\|\mathcal{T}u_n\| \geq n$. Thus, there exists $N \in \mathbb{N}$ such that $\mathcal{T}: D_N \to D_N$.

Step 2 We claim that $\mathcal{T}D_N$ is a compact subset of B. In fact, since $\mathcal{T}D_N \subseteq D_N$, $\{\mathcal{T}u(t) |\ u \in D_N\}$ is uniformly bounded. We set

$$\begin{aligned} a_1 &= \sup\{\|A(t)\|\Big|\ t \in [0,T]\},\\ a_2 &= \sup\Big\{\int_{-\infty}^{t}\|C(t,s)\|\mathrm{d}s\Big|\ t \in [0,T]\Big\},\\ a_3 &= \sup\Big\{\sum_{i=1}^{l}\|g_i(t,x)\|\Big|\ (t,x) \in [0,T] \times R_N\Big\}\end{aligned}$$

where $R_N = \{x|\ x \in \mathbb{R}^n,\ \|x\| \leq N\}$. For any $u \in D_N$, we have

$$\frac{\mathrm{d}\mathcal{T}u(t)}{\mathrm{d}t} = \frac{\mathrm{d}x_u(t)}{\mathrm{d}t} = A(t)x_u(t) + \int_{-\infty}^{t}C(t,s)p(u(s))\mathrm{d}s + \sum_{i=1}^{l}g_i(t,u(t-\tau_i(t))) + b(t).$$

Simple computation shows $\left\|\frac{\mathrm{d}\mathcal{T}u(t)}{\mathrm{d}t}\right\| \le a_1N+a_2N+a_3+M$. Therefore, $\{\mathcal{T}u(t)|\ u \in D_N\}$ is equal-continuous. It follows from Ascoli-Arzela theorem that $\mathcal{T}D_N$ is a compact subset of B.

Step 3 We claim that $\mathcal{T}$ is continuous on D_N. In fact, since $g_i(t,x)$ is uniformly continuous on $[0,T] \times R_N$ and $g_i(t+T,x) = g_i(t,x)$, $g_i(t,x)$ is uniformly continuous on $R \times R_N$. Therefore, for any $\varepsilon > 0$, there exists $\delta_i = \delta_i(\varepsilon) > 0$ $(i = 1, 2, \cdots, l)$ such that if $\|x_1 - x_2\| < \delta_i$ $(x_1, x_2 \in R_N)$, then

$$\|g_i(t,x_1) - g_i(t,x_2)\| \le \frac{\varepsilon}{3\mu l} \quad (t \in R).$$

Take $\delta' = \min\limits_{1\le i\le l}\left\{\delta_i(\varepsilon), \frac{\varepsilon}{3\mu\rho_2}\right\}$, for any $u_1, u_2 \in D_N$, $t \in R$, if $\|u_1 - u_2\| \le \delta'$, then we have $\|\mathcal{T}u_1(t) - \mathcal{T}u_2(t)\| \le \mu\rho_2\delta' + \frac{\varepsilon}{3} < \varepsilon$. Therefore, if $\|u_1 - u_2\| \to 0$, we have $\|\mathcal{T}u_1 - \mathcal{T}u_2\| \to 0$, i.e., $\mathcal{T}$ is continuous on D_N.

From the above three Steps, in a word, $\mathcal{T} : D_N \to D_N$ is completely continuous. Therefore by Schauder's fixed point theorem, there exists at least fixed point in D_N. It follows from (3.1), (3.4) and (3.5) that the fixed point is just the T-periodic solution of system (3.1). The proof of Theorem 3.2 is complete. □

References

[1] W.A.Coppel, Stability and Asymptotic Behaveiour of Differential Equations, D.C.Heath, Boston, 1965.

[2] W.A.Coppel, Dichotomies in Stability Theory, Springer-Verlag, Berlin, 1978.

[3] J.K.Hale, S.M.Verduyn Lunel, Introduction to Functional Differential Equations, in Introduction to Functional Differential Equations, Springer-Verlag, New York, 1993.

[4] J.K.Hale, J.Kato, Phase space for retarded equations with infinite delay, *Funkcial, Ekvact.*, **21**(1978),11-41.

[5] Y.H.Xia, S.S.Cheng, Quasi-uniformly Asymptotic Stability and Existence of Almost Periodic Solutions of Difference Equations With Applications in Population Dynamic Systems, Journal of Difference Equations and Applications, 14:1 (2008), 59-81.

[6] Y. H. Xia, J. Cao, M. Han, A new analytical method for the linearization of dynamic equation on measure chains, *J. Differential Equations*, **235**:2 (2007), 527-543.

[7] Y. H. Xia, Z. Huang, M. Han, Existence of Almost Periodic Solutions for Forced Perturbed Systems with Piecewise Constant Argument, *J. Math. Anal. Appl.*, 333 (2007) 798-816.

[8] C. He, Almost periodic differential equations, Higher Education Press, 1992, (in Chinese).

[9] A.M.Fink, Almost Periodic Differential Equation, Springer-Verlag, Berlin, Heidleberg, New York, 1974.

[10] T.Burton, Stability and periodic solutions of ordinary and functional differential equations, Academic Press Inc., New York, 1985.

[11] Q.Huang, The existences of periodic solution for the functional equations with infinite delays, *Chinese Sciences, Ser. A*, **10**(1984),882-889(in Chinese).

[12] K.Wang, Q.Huang, C_h space and the boundness and periodic solution of functional differential equation with infinite delay, *Chineses Science*, Series A, **17**(1987),242-2525(in Chinese).

[13] Q.Wang, Existence, Uniqueness and stability of periodic solution of integrodifferenital equations with infinite delay, *Acta Mathematics Applications Sinica*, **21**(1998),312-318(in Chinese).

[14] Z.Zou, Periodic soltution for a class of higer dimesional retarded functinal differentail equations, *I. of Math. (PRC)*, **22**:4(2002),423-430(in Chinese).

[15] J.L.Fenner, M.Pinto, On a Hartman linearization theorem for a class of ODE with impulse effect, *Nonlinear Analysis*, **38**(1999), 307-325.

[16] J.L.Fenner, On (h, k)-trichotomies and asymptotices of non-autonomous difference systems, *Comp. Math. Appl.*, **33**:10(1997),105-124.

[17] J.L.Fenner, M.Pinto, On (h, k) manifolds with asymptotic phase, *J. Math. Anal. Appl.*, **216**(1997), 549-568.

[18] R.Naulíu, M.Pinto, Roughness of (h, k)-dichotomies, *J. Differential Equations*, **118**:1(1995),20-35.

[19] M.Pinto, Dichotomies and asymptotic formulas for the solutions of differential equations, *J. Math. Anal. Appl.*, **195**(1995),16-31.

[20] M.Pinto, Dichotomies for differential systems with impulsive effect, Proc. 1st World Congr. of Nonlinear Analysts, Walter De Gruyter, Berlin, 1966, pp.1181-1192.

[21] R.Naulíu, M.Pinto, Dichotomies and asymptotic solutions of nonlinear differentail systems,*Nonlinear Anal.*, **23**:7(1994),871-882.

[22] J.S.Muldowney, Dichotomies and asymptotic solutions of linear differentail systems,*Trans.Amer.Math.Soc.*, **283**:2(1984),465-484.

[23] R.J.Sacker, G.R.Sell, Lifting properties in skew-product flows with applications to differential equations, *Memoris American Mathematical Society*, **190**(1977).

[24] Y.Hino, Almost periodic solutions of a linear Volterra equation, *Differential and Integral Equations*, **3**(1990),495-501.

[25] Y.Hino, S.Murakami, Stability properties of linear Volterra equations, *Journal of Differential Equations*, **89**:1(1991),121-137.

[26] Y.H Xia, M. Lin, J. Cao, The Existence of Almost Periodic Solutions of Certain Perturbation Systems, *J. Math. Anal. Appl.*, 310:1(2005), 81-96.

[27] Y.H.Xia, J.Cao, Almost periodicity in an ecological model with M-predators and N-preys by "pure-delay type" system, *Nonlinear Dynamics*, 39:3(2005), 275-304.

[28] Y.H.Xia, J.Cao, Almost-periodic solutions for an ecological model with infinite delays, *Proc. Edinburgh Math. Soc.*,50:1(2007),229-249.

[29] Y.H.Xia, J. Cao, S. S. Cheng, Periodicity in a Lotka Volterra mutualism system with several delays, *Appl. Math. Modelling,* 31:9 (2007) 1960-1969.

[30] Y.H.Xia, J. Cao, S. S. Cheng, Multiple Periodic Solutions of a Delayed Stage-structured Predator-prey Model With Nonmonotone Functional Responses, *Appl. Math. Modelling,* 31:9 (2007) 1947-1959.

[31] Y.H. Xia, J.Cao, Global attractivity of a periodic ecological model with m-predators and n-preys by "pure-delay type" system, *Comput. Math. Appl.,* 52:6-7(2006), 829-852.

[32] Y.H.Xia, M.Han, Z. Huang, Global attractivity of an almost periodic N-species nonlinear ecological competitive model, *J. Math. Anal. Appl.*, 337 (2008) 144-168.

Received: January 14, 2008