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Inequalities for Convolution Ratios under Local Approximation

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Abstract

This paper studies preservation of the convolution property under locally approximating functions for sums of independent random variables. Upper and lower bounds for limits of convolution ratios are given.

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1 Introduction

This paper studies preservation of the convolution property under locally approximating functions for sums of independent random variables. In particular, suppose X is a random variable satisfying $E(X) = \mu$ and $Var(X) = \sigma^2 < \infty$ and that $X_1, X_2, \ldots$ is a sequence of independent and identically distributed random variables each with the same distribution as X. In addition, suppose that X has probability function p, where p is either a density or a probability mass function, and denote the support set of X by $\mathcal{S}_X = \{x : p(x) > 0\}$. Let $p^{(i)}$ be the i-fold convolution of p (i.e. the probability function for the partial sum $S_i = X_1 + X_2 + \cdots + X_i$).

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Let $\phi_{n\mu,n\sigma^2}$ be the density for a normal random variable, N, with mean $n\mu$ and variance $n\sigma^2$, i.e.

$$\phi_{n\mu,n\sigma^2}(x) = \frac{1}{\sqrt{2\pi}\sigma\sqrt{n}} e^{-\frac{1}{2}\left(\frac{x-n\mu}{\sqrt{n}\sigma}\right)^2}, \qquad -\infty < x < \infty. \tag{1}$$

For any random variable Y, let m_Y denote the moment generating function (m.g.f.) of Y defined via

$$m_Y(t) = E(e^{tY}) \tag{2}$$

for $-\infty < t < \infty$. As well, for random variables Y_1 and Y_2, define R_{Y_1,Y_2} via

$$R_{Y_1,Y_2}(t) = \frac{m_{Y_1}(t)}{m_{Y_2}(t)}. \tag{3}$$

Note that for a normal random variable, N, with density as in (1) for $n = 1$,

$$m_N(t) = e^{t\mu + \frac{t^2\sigma^2}{2}}. \tag{4}$$

For the remainder of the paper N will always denote a random variable with moment generating function as in (4) where the values of μ and σ should be clear from the context.

There are many well-known local limit theorems for random variables where the associated attracting distribution is the normal (see Gnedenko and Kolomogrov [6] and Petrov [8], and many others). Two such results are the following two theorems.

Theorem 1 *Suppose X takes values of the form $b + Mh$, $M = 0, \pm1, \pm2, ...$, with probability 1, where b is a constant and $h > 0$ is the maximal span of the distribution. Then*

$$\sup_M \sigma\sqrt{n} \left| p^{(n)}(nb + Mh) - h\phi_{n\mu,n\sigma^2}(nb + Mh) \right| \to 0. \tag{5}$$

□

Theorem 2 *Suppose p is a density, and that there exists a K such that $p_K(x)$ is bounded. Then*

$$\sup_x \sigma\sqrt{n} \left| p^{(n)}(x) - \phi_{n\mu,n\sigma^2}(x) \right| \to 0. \tag{6}$$

□

For related results see for instance [4–6, 8–12].

Motivated by the following simple one-step conditioning equality implied by independence

$$p^{(n)}(x) = E(p^{(n-1)}(x - X)), \tag{7}$$

we have the following definition.

Definition 1 *For any two functions f and g, define the "convolution ratio", $H^X_{f,g}$ of f and g with respect to X via*

$$H^X_{f,g}(x) = \frac{E(f(x - X))}{g(x)} \tag{8}$$

for all $x \in \mathbb{R}$ such that $g(x) \neq 0$. □

Note that if Y is a random variable, independent of X, f is the probability function for Y and g is the probability function for $X + Y$, then $H^X_{f,g} \equiv 1$. In particular, we have that for $n > 1$,

$$H^X_{p^{(n-1)},p^{(n)}}(x) = 1, \qquad x \in \mathcal{S}_{S_n}, \tag{9}$$

where $\mathcal{S}_{S_n}$ is the support set of S_n.

Along the lines suggested by (7) and Theorems 1 and 2, we have the following definition.

Definition 2 *For $n > 1$, set*

$$C_{X,\tilde{\boldsymbol{p}}}(x, n) = H^X_{\tilde{p}_{n-1},\tilde{p}_n}(x) = \frac{E(\tilde{p}_{n-1}(x - X))}{\tilde{p}_n(x)} \tag{10}$$

where $\tilde{\boldsymbol{p}} = (\tilde{p}_i)_{i\geq 1}$ is a sequence of real valued functions. □

For our current consideration, $\tilde{p}_i$ in (10) will be a candidate approximating function for $p^{(i)}$.

The following result, which will be useful for our computations, was recently proven (see [1]).

Theorem 3 *Suppose $\tilde{p}_i = \phi_{n\mu,n\sigma^2}$, if X is continuous (or $\tilde{p}_i = h\phi_{n\mu,n\sigma^2}$, if X is defined on a lattice with maximal span h), and let N be a normally distributed random variable with mean μ and variance σ^2. Then*

$$\lim_{n\to\infty} C_{X,\tilde{\boldsymbol{p}}}(tn, n) = \frac{m_X\left(\frac{t-\mu}{\sigma^2}\right)}{m_N\left(\frac{t-\mu}{\sigma^2}\right)} = R_{X,N}\left(\frac{t-\mu}{\sigma^2}\right), \tag{11}$$

provided that the numerator in the middle quotient exists, and otherwise $C_{X,\tilde{\boldsymbol{p}}}(tn, n)$ is unbounded. We will refer to the quantity in (11) as $L_X(t)$.

Note that when $\tilde{p}_i(x)$ is taken as in Theorem 3, then $\tilde{p}_i(x) > 0$ for all (i, x).

If $\tilde{p}_i = p^{(i)}$ for all i (i.e. the approximation is exact for all (i, x)), then, by (7), $C_{X,\tilde{\boldsymbol{p}}}(x, n) = 1$ for $x \in \mathcal{S}_{S_n}$ (the support set of S_n). In this sense $C_{X,\tilde{\boldsymbol{p}}}(x, n)$ (for $x \in \mathcal{S}_{S_n}$) may be viewed as a measure of the quality of the n^{th} order approximation, locally at x, with regard to preservation of the inherent convolution property.

Remark. The value $\lim_{n\to\infty} C_{X,\tilde{\boldsymbol{p}}}(tn, n)$ in (11) addresses preservation of the convolution property, for values t in the support set of $\bar{X}_n = S_n/n$, for large n. Note that if $t \in \mathcal{S}_{S_m/m}$ for some $m > 1$ then $t \in \mathcal{S}_{S_n/n}$ for infinitely many n. Note as well that the set $\mathcal{U}_X = \bigcup_{n>1} \mathcal{S}_{S_n/n}$ is dense in the interval $\{x : \inf\{\mathcal{S}_X\} \leq x \leq \sup\{\mathcal{S}_X\}\}$ (where the infimum or supremum may be infinite). In what follows, we will be interested in developing bounds for $L_X(t)$ for $t \in \mathcal{U}_X$ for some distributions which are heavily used in practice.

Remark. Note that ratios $R_{X,N}(t)$ occur when considering Cornish-Fisher expansions (see for instance [13]).

The remainder of the paper proceeds as follows. In Section 2, we consider bounds for $L_X(t)$ where X is Bernoulli with parameter p, while Section 3 includes discussion of uniformly distributed (continuous and discrete) random variables X. Section 4 addresses bounds for some X with unbounded support. Some bounds for a variety of distributions, discussed in the paper, are summarized in the following table.

Table 1: Optimal global bounds for $L_X(t)$ for $t \in \mathcal{U}_X$

X	Lower bound	Upper bound
Bernoulli(1/2)	$(e^{1/2} + e^{-3/2})/2$	1
Bernoulli(p)	$(e^{1/2} + e^{-3/2})/2$	Unbounded
Uniform on $\{0, 1, \ldots, T-1\}$	$e^{3/2}(1 - e^{-6})/6$	1
Uniform on $[a, b]$	$e^{3/2}(1 - e^{-6})/6$	1
Exponential(λ)	$e^{1/2}/2$	Unbounded
Poisson(λ)	$e^{-\lambda/2+\lambda/e}$	Unbounded
Geometric(p)	$e^{-1/2}/2$	Unbounded

2 Bernoulli Random Variables

Suppose X is a Bernoulli random variable with parameter p, i.e. $P(X = 1) = p$ and $P(X = 0) = 1 - p$ with $0 < p < 1$. Then $\mu = p$, $\sigma^2 = p(1 - p)$, $m_X(t) = pe^t + (1 - p)$ and $\mathcal{U}_X \subset [0, 1]$.

In light of Theorem 3, we have

$$\begin{aligned} L_X(t) &= \frac{m_X\left(\frac{t-\mu}{\sigma^2}\right)}{m_N\left(\frac{t-\mu}{\sigma^2}\right)} = R_{X,N}\left(\frac{t-p}{p(1-p)}\right) \\ &= (1-p)e^{\frac{(t^2-p^2)}{2p(p-1)}} + pe^{\frac{(t-p)(t+p-2)}{2p(p-1)}}. \end{aligned} \quad (12)$$

For simplicity, we will write $L(t,p)$ to represent $L_X(t)$.

We will prove the following theorem.

Theorem 4 *Suppose X is a Bernoulli random variable with parameter p. Then, for all $0 < t < 1$,*

$$L_X(t) = L(t,p) \geq L(t,1/2) \geq L(0,1/2) = \frac{1}{2}(e^{\frac{1}{2}} + e^{-\frac{3}{2}}) \approx 0.9359. \quad (13)$$

Proof: For $0 < p < 1$, we have

$$L(t,p) = e^{\frac{(t^2-p^2)}{2p(p-1)}} - pe^{\frac{(t^2-p^2)}{2p(p-1)}} + pe^{\frac{(t-p)(t+p-2)}{2p(p-1)}}. \quad (14)$$

We will show that, as a function of p, $L(t,p)$ is increasing for $p \in (0,t)$, attains its maximum at $p = t$, and is decreasing for $p \in (t,1)$.

Differentiating with respect to p in (14) gives,

$$\begin{aligned} \frac{d}{dp}L(t,p) &= \frac{(t-p)(2p-1)}{2p(p-1)}\left(\frac{t+p}{p}e^{\frac{(t^2-p^2)}{2p(p-1)}} - \frac{t+p-2}{p-1}e^{\frac{(t-p)(t+p-2)}{2p(p-1)}}\right) \\ &= \frac{(t-p)(2p-1)(t+p-2)}{2p(p-1)^2}e^{\frac{(t-p)(t+p-2)}{2p(p-1)}}\left(\frac{(t+p)(p-1)}{p(p+t-2)}e^{\frac{(t-p)}{p(p-1)}} - 1\right) \\ &= \frac{(t-p)(2p-1)(t+p-2)}{2p(p-1)^2}e^{\frac{(t-p)(t+p-2)}{2p(p-1)}}g_p(t), \end{aligned} \quad (15)$$

say.

Now,

$$\begin{aligned} \frac{d}{dt}g_p(t) &= e^{\frac{t-p}{p(p-1)}}\frac{t^2-2t+2tp-p^2}{p^2(t+p-2)^2} \\ &= e^{\frac{t-p}{p(p-1)}}\frac{-2t(1-t)-(t-p)^2}{p^2(t+p-2)^2} < 0. \end{aligned} \quad (16)$$

Thus $g_p(t)$ is a decreasing function of t. Since $t = p$ is the only zero of $g_p(t)$, $g_p(t) > 0$ for $t \in (0,p)$, and $g_p(t) < 0$ for $t \in (p,1)$.

Employing (15), $\frac{d}{dp}L(t,p) < 0$ for $0 < p < \frac{1}{2}$, $\frac{d}{dp}L(t,p) > 0$ for $\frac{1}{2} < p < 1$, which implies that, for $0 < t < 1$,

$$\min_{0<p<1} L(t,p) = L(t,1/2), \quad (17)$$

hence we have the first inequality in (13).

Now, suppose $p = 1/2$, and let

$$\begin{aligned} h(t) &= L(t, 1/2) \\ &= \frac{1}{2}e^{-2(t^2-\frac{1}{4})} + \frac{1}{2}e^{-2(t-\frac{1}{2})(t-\frac{3}{2})}. \end{aligned} \tag{18}$$

Differentiating in (18) with respect to t, gives

$$\begin{aligned} h'(t) &= 2te^{-2(t-\frac{1}{2})(t-\frac{3}{2})}\left(\frac{1}{t} - 1 - e^{-(4t-2)}\right) \\ &= 2te^{-2(t-\frac{1}{2})(t-\frac{3}{2})}e^{2-4t}\left(\frac{t^{-1}-1}{e^{2-4t}} - 1\right) \\ &= 2te^{-2(t-\frac{1}{2})(t-\frac{3}{2})}e^{2-4t}\left(e^{q(t)} - 1\right), \end{aligned} \tag{19}$$

where

$$\begin{aligned} q(t) &= \log\left(\frac{t^{-1}-1}{e^{2-4t}}\right) = \log(t^{-1}-1) - (2-4t) \\ &= (2-4t)\sum_{i=1}^{\infty}\frac{1}{4^i}\frac{(2-4t)^{2i}}{2i+1}. \end{aligned} \tag{20}$$

Combining (19) and (20), gives that $L(t, 1/2)$ is increasing for $0 < t < \frac{1}{2}$, and decreasing for $\frac{1}{2} < t < 1$. Thus,

$$\max_{0<t<1} L(t, 1/2) = L(1/2, 1/2) = 1 \tag{21}$$

and

$$\min_{0<t<1} L(t, 1/2) = L(0, 1/2) = L(1, 1/2) = \frac{1}{2}(e^{\frac{1}{2}} + e^{-\frac{3}{2}}) \approx 0.9359. \tag{22}$$

□

We now turn to consideration of uniformly distributed random variables.

3 Uniformly Distributed Random Variables

In this section we consider bounds for uniformly distributed random variables.

3.1 Discrete uniformly distributed random variables

Suppose X takes value on $B = \{0, 1, \ldots T-1\}$ with constant probability $P(X = i) = \frac{1}{T}$ for $i \in B$ (where we assume $T > 3$; the case $T = 2$ was covered in Section 2). The mean and the variance of X are $\mu = \frac{T-1}{2}$ and $\sigma^2 = \frac{T^2-1}{12}$, respectively and $\mathcal{U}_X \subset [0, T-1]$. In addition,

$$m_X(t) = \frac{1 - e^{Tt}}{1 - e^t}\frac{1}{T}. \tag{23}$$

Employing Theorem 3, we have

$$\begin{aligned} L_X(t) &= \frac{m_X\left(\frac{t-\mu}{\sigma^2}\right)}{m_N\left(\frac{t-\mu}{\sigma^2}\right)} = R_{X,N}\left(\frac{6(2t-T+1)}{T^2-1}\right) \\ &= \left(\frac{e^{-6T\frac{-2t+T-1}{T^2-1}} - 1}{e^{-6\frac{-2t+T-1}{T^2-1}} - 1}\right)\frac{e^{\frac{3}{2}\frac{(T-1)^2-4t^2}{T^2-1}}}{T} \\ &= \frac{\sum_{i=0}^{T-1} e^{-\frac{3(-2t+T-1)(-T+4i+1-2t)}{2(T^2-1)}}}{T}. \end{aligned} \tag{24}$$

For simplicity, here we will write $L(t, T)$ to represent $L_X(t)$.

We will prove the following

Theorem 5 *Suppose X is uniformly distributed on B. Then $L_X(t)$ is maximized at $t = \frac{T-1}{2}$ and minimized at $t = 0$. In addition, for all $t \in [0, T-1]$,*

$$\begin{aligned} 1 > L(t,T) &\geq L(0,T) = \left(\frac{e^{-\frac{6T}{T+1}} - 1}{e^{-\frac{6}{T+1}} - 1}\right)\frac{e^{\frac{3}{2}\frac{T-1}{T+1}}}{T} \\ &\geq \lim_{T\to\infty} L(0,T) = \frac{1-e^{-6}}{6}e^{3/2} \approx 0.745097\ldots . \end{aligned} \tag{25}$$

The next lemma will be useful in proving the final inequality in (25), for generalizations see [2].

Lemma 1 *Set $s = e^{-6}$, and define the function f via*

$$f(x) = \frac{1 - s^{1-x}}{1 - s^x}\frac{x}{1-x}s^{\frac{x}{2}}. \tag{26}$$

Then, $f(x)$ is monotonically increasing for $0 < x < 1/4$.

Proof: Taking the natural log of f in (26) gives

$$\ln f(x) = \ln(1 - s^{1-x}) - \ln(1 - s^x) + \ln x - \ln(1 - x) + \frac{x}{2} \ln s. \tag{27}$$

Differentiating, simplifying and noting that $s = e^{-6}$ and $0 < x < 1$, we have

$$\begin{aligned} \frac{d}{dx} \ln f(x) &= 3 + \frac{1}{1-x} - \left(\frac{6}{1-s^x} - \frac{1}{x}\right) - \frac{6}{s^{x-1} - 1} \\ &\geq 4 - \left(\frac{6}{1-s^x} - \frac{1}{x}\right) - \frac{6}{s^{x-1} - 1} \\ &= 4 - h(x) - k(x), \end{aligned} \tag{28}$$

say.

Since $k'(x) = 36s^{x-1}(s^{x-1} - 1)^{-2} \geq 0$, we have that $k(x)$ is increasing on $0 < x < 1/4$ and thus

$$k(x) \leq k(1/4) \leq 0.068. \tag{29}$$

Also,

$$h'(x) = \frac{-36x^2 s^x - 2s^x + 1 + s^{2x}}{(1 - s^x)^2 x^2} = \frac{r(x)}{(1 - s^x)^2 x^2}$$

where

$$r'(x) = 12s^x(1 - 6x + 18x^2 - e^{-6x}). \tag{30}$$

Since $e^{-6x} \leq 1 - 6x + 18x^2$ for all x, $r(x)$ is monotonically increasing, and $r(x) \geq r(0) = 0$. Thus, $h'(x) \geq 0$, and hence

$$h(x) \leq h\,(1/4) \leq 3.724 \tag{31}$$

Therefore,

$$\frac{d}{dx} \ln f(x) \geq 4 - 0.068 - 3.724 > 0 \tag{32}$$

and the result is proven. □

Proof of Theorem 5. Differentiating $L_X(t)$ in (24) with respect to t, we have

$$\begin{aligned} \frac{d}{dt} L(t,T) &= \frac{12Te^{-\frac{3(2t-T+1)^2}{2(T^2-1)}}}{T^2 - 1} \sum_{i=0}^{T-1} (i - t) e^{\frac{3(2i-T+1)(2t-T+1)}{T^2-1}} \\ &= \frac{12Te^{-\frac{3(2t-T+1)^2}{2(T^2-1)}}}{T^2 - 1} K_T(t), \end{aligned} \tag{33}$$

say.

We will show $L(t,T)$ is increasing for $t \in (0, \frac{T-1}{2})$, attains its maximum at $v = \frac{T-1}{2}$, and is decreasing for $t \in \big(\frac{T-1}{2}, T-1\big)$.

Set $x = \frac{T-1}{2}$ and $c = \frac{12}{T^2-1}$. Expanding $K(t) = K_T(t)$ in a Taylor's series about $t = x$, we have

$$K(t) = \sum_{n=0}^{\infty} \frac{-\xi_n}{n!}(t-x)^n, \tag{34}$$

where

$$\begin{aligned} \xi_n &= \sum_{i=0}^{T-1} c^{n-1}(i-x)^{n-1}(n-c(i-x)^2) &(35)\\ &= nc^{n-1}\sum_{i=0}^{T-1}(i-x)^{n-1} - c^n \sum_{i=0}^{T-1}(i-x)^{n+1}. &(36) \end{aligned}$$

For $n = 1$, we have

$$\xi_1 = \sum_{i=0}^{T-1}(1-c(i-x)^2) = T - c\sum_{i=0}^{T-1}\left(i - \frac{T-1}{2}\right) = T - c\frac{T^3-T}{12} = 0. \tag{37}$$

As well, when n is even, by symmetry we have

$$nc^{n-1}\sum_{i=0}^{T-1}(i-x)^{n-1} = 0 = c^n\sum_{i=0}^{T-1}(i-x)^{n+1},$$

and hence $\xi_n = 0$.

When $n \geq 3$ is odd, we have

$$\begin{aligned} n - c(i-x)^2 &= n - \frac{12}{T^2-1}(i-x)^2 \geq n - \frac{12}{T^2-1}x^2 \\ &> n - 3 \geq 0, \end{aligned} \tag{38}$$

and hence applying (35), $\xi_n > 0$.

Summarizing, we have

$$\begin{aligned} \frac{d}{dt}L(t,T) &= \frac{12Te^{-\frac{3(2t-T+1)^2}{2(T^2-1)}}}{T^2-1}\sum_{n=0}^{\infty}(-1)\xi_n\left(t - \frac{T-1}{2}\right)^n \\ &= \frac{12Te^{-\frac{3(2t-T+1)^2}{2(T^2-1)}}}{T^2-1}\sum_{k=1}^{\infty}(-1)\xi_{2k+1}\left(t - \frac{T-1}{2}\right)^{2k+1}. \end{aligned} \tag{39}$$

Therefore, $L(t,T)$ is increasing for $t \in (0, \frac{T-1}{2})$, attains its maximum at $t = \frac{T-1}{2}$, and is decreasing for $t \in \left(\frac{T-1}{2}, T-1\right)$. The first and second inequalities in (25) then follow. For the third inequality note that

$$L(0,T) = \left(\frac{e^{-\frac{6T}{T+1}} - 1}{e^{-\frac{6}{T+1}} - 1}\right) \frac{e^{\frac{3}{2}\frac{T-1}{T+1}}}{T} = e^{\frac{3}{2}} \frac{1-s^{1-x}}{1-s^x} \frac{x}{1-x} s^{\frac{x}{2}}, \tag{40}$$

where $x = 1/(T+1)$ and $s = e^{-6}$. The result then follows from Lemma 1 upon noting that $T \geq 3$ and for f as defined in (26), $\lim_{x\to 0} f(x) = (s-1)/\ln(s)$. □

3.2 Continuous uniformly distributed random variables

Suppose that X possesses a uniform distribution on the interval $[a,b]$. The mean and the variance of X are $\mu = \frac{b-a}{2}$ and $\sigma^2 = \frac{(b-a)^2}{12}$, respectively and $\mathcal{U}_X \subset [a,b]$.

The moment generating function of X is given by

$$m_X(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}. \tag{41}$$

We recall the following elementary result from [1].

Lemma 2 *Suppose $c, d \in \mathbb{R}$ and $c \neq 0$, then*

$$L_{cX+d}(t) = L_X\left(\frac{t-d}{c}\right). \tag{42}$$

□

Note that Lemma 2 reduces our consideration through scaling to $(a,b) = (0,1)$.

In light of Theorem 3, we then have, in this case,

$$\begin{aligned} L_X(t) &= \frac{m_X\left(\frac{t-\mu}{\sigma^2}\right)}{m_N\left(\frac{t-\mu}{\sigma^2}\right)} = R_{X,N}\left(12t-6\right) \\ &= \frac{e^{12t-6} - 1}{(12t-6)e^{\frac{(12t-6)^2}{24} + 6t - 3}}. \end{aligned} \tag{43}$$

For simplicity, we will write $L(t)$ to represent $L_X(t)$.

We will prove the following theorem.

Theorem 6 *Suppose X is a continuous uniform $[0,1]$ random variable. Then,*

$$\begin{aligned} 1 = L(1/2) &\geq L_X(t) = L(t) \\ &\geq L(0) = L(1) \\ &= \frac{1-e^{-6}}{6} e^{3/2} \approx 0.7450967\ldots . \end{aligned} \tag{44}$$

Proof. Differentiating in (43) gives

$$\begin{aligned} L^{'}(t) &= -\frac{12t^2-6t+1}{3(2t-1)^2 e^{\frac{3(2t-1)(2t+1)}{2}}} \left(\frac{(12t^2-18t+7)e^{12t-6}}{12t^2-6t+1} - 1 \right) \\ &= -\frac{12t^2-6t+1}{3(2t-1)^2 e^{\frac{3(2t-1)(2t+1)}{2}}} g(t), \end{aligned} \tag{45}$$

say.

Since $g'(t) = \frac{108e^{12t-6}(2t-1)^4}{(12t^2-6t+1)^2} \geq 0$ and $g(1/2) = 0$, we have that $g(t) < 0$ for $0 < t < \frac{1}{2}$ and $g(t) > 0$ for $\frac{1}{2} < t < 1$.

Returning to (45), we have $L^{'}(t) > 0$ for $0 < t < \frac{1}{2}$ and $L^{'}(t) < 0$ for $\frac{1}{2} < t < 1$, and the result follows. □

3.3 Uniform distributions on non-evenly spaced points

Lemma 2 and Theorem 4 imply that the minimum value of

$$\{L_X(t) : t \in \mathcal{U}_X\}$$

over all possible distributions on two points is attained when each is assigned probability 1/2. One might ask what pertains on support sets with a larger number of points. Perhaps surprisingly over distributions on three points the minimum value of $L_X(t)$ is not attained at equally spaced points as is seen in the following example.

Example. Consider a random variable X with uniformly distributed mass on the three points 0, x and 1, where $0 < x < 1$. We have $E[X] = (x+1)/3$, $Var(X) = 2/9(1-x+x^2)$, $\mathcal{U}_X \subset [0,1]$ and upon applying Theorem 3,

$$L_X(t) = 1/3 \left(1 + e^{-3/2 \frac{(-3t+1+x)x}{1+x^2-x}} + e^{-3/2 \frac{-3t+1+x}{1+x^2-x}}\right) e^{1/4 \frac{(-3t+1+x)(1+x+3t)}{1+x^2-x}} \tag{46}$$

For simplicity, here we will write $L(t,x)$ to represent $L_X(t)$.

Note that Theorem 5, for $T = 3$, implies that $\min_{0\leq t\leq 1} L(t,1/2) = L(0,1/2) \approx 0.8982552642$, while $L(0,1/3) = 1/3 \left(1 + e^{-6/7} + e^{-\frac{18}{7}}\right) e^{4/7} \approx 0.8858691762$.

It would be interesting to know the relative configuration of n points with uniformly distributed mass that minimizes $L_X(t)$ over the associated set $\mathcal{S}$. By Lemma 2, it suffices to restrict attention to points $0 = x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = 1$ and $\mathcal{U}_X \subset [0, 1]$. □

4 Distributions with unbounded support

In this section we consider some families of random variables with unbounded support.

4.1 Exponential random variables

Suppose X is an exponential random variable with parameter $\lambda > 0$, i.e. $p(x) = \lambda e^{-\lambda x}$ for $x > 0$. Then $\mu = 1/\lambda$, $\sigma^2 = 1/\lambda^2$, $m_X(t) = \lambda/(\lambda - t)$ for $t < \lambda$ and $\mathcal{U}_X \subset [0, \infty)$. As well, $L_X(t)$ is finite only for $t < 2/\lambda$.

In light of Theorem 3, we have

$$\begin{aligned} L_X(t) &= \frac{m_X\left(\frac{t-1/\lambda}{1/\lambda^2}\right)}{m_N\left(\frac{t-1/\lambda}{1/\lambda^2}\right)} = R_{X,N}\left(\lambda^2 t - \lambda\right) \\ &= -\frac{e^{-\frac{1}{2}(t\lambda-1)(t\lambda+1)}}{t\lambda - 2}, \end{aligned} \tag{47}$$

for $t < 2/\lambda$. For simplicity, we will write $L(t, \lambda)$ to represent $L_X(t)$.

We will prove the following theorem.

Theorem 7 *Suppose X is an exponential random variable with parameter λ. Then, for all $0 < t < 2/\lambda$,*

$$L_X(t) = L(t, \lambda) \geq L(0, \lambda) = \frac{1}{2} e^{\frac{1}{2}} \approx 0.8243606355. \tag{48}$$

Proof. Differentiating in (47) gives

$$\frac{d}{dt} L_X(t) = \frac{e^{-\frac{1}{2}(t\lambda-1)(t\lambda+1)} \lambda \left(t\lambda - 1\right)^2}{\left(t\lambda - 2\right)^2} > 0, \tag{49}$$

and the result follows. □

4.2 Poisson random variables

Suppose X is a Poisson random variable with parameter $\lambda > 0$, i.e. $P(X = i) = e^{-\lambda}\lambda^i/i!$ for $i = 0, 1, 2, \ldots$. Then $\mu = \lambda$, $\sigma^2 = \lambda$, $m_X(t) = e^{\lambda(e^t-1)}$ and $\mathcal{U}_X \subset [0, \infty)$.

In light of Theorem 3, we have

$$\begin{aligned} L_X(t) &= \frac{m_X\left(\frac{t-\lambda}{\lambda}\right)}{m_N\left(\frac{t-\lambda}{\lambda}\right)} = R_{X,N}\left(t/\lambda - 1\right) \\ &= e^{1/2\left(-\lambda^2+2\lambda^2 e^{-\frac{-t+\lambda}{\lambda}}-t^2\right)\lambda^{-1}}, \end{aligned} \tag{50}$$

for $t > 0$. For simplicity, we will write $L(t, \lambda)$ to represent $L_X(t)$.

We will prove the following theorem.

Theorem 8 *Suppose X is a Poisson random variable with parameter λ. Then, for all $0 < t < \infty$,*

$$L_X(t) = L(t, \lambda) \geq L(0, \lambda) = e^{-\lambda\frac{e-2}{2e}} \approx e^{-.1321205588\lambda}. \tag{51}$$

Proof. Differentiating in (50) gives

$$\frac{d}{dt}L_X(t) = \left(\lambda e^{\frac{t}{\lambda}-1} - t\right) e^{1/2\left(-\lambda^2+2\,\lambda^2 e^{-\frac{-t+\lambda}{\lambda}}-t^2\right)\lambda^{-1}}\lambda^{-1}. \tag{52}$$

Noting that $e^{x-1} > x$ for all $x \geq 0$ (see for instance Mitrinović [7], page 266), we have that $\frac{d}{dt}L_X(t) \geq 0$ for $t > 0$, and the result follows. □

4.3 Geometric random variables

Suppose X is a geometric random variable with parameter $0 < p < 1$, i.e. $P(X = i) = p(1-p)^{i-1}$ for $i = 1, 2, 3, \ldots$. Then $\mu = 1/p$, $\sigma^2 = (1-p)/p^2$, $\mathcal{U}_X \subset [1, \infty]$, $m_X(t) = e^t p/(1 - e^t(1-p))$ for $t < \log(1/(1-p))$ and $L_X(t)$ is finite only for $t \in \mathcal{T}_p$, where

$$\begin{aligned} \mathcal{T}_p &= \left\{ t : \frac{t-\frac{1}{p}}{\frac{1-p}{p^2}} < \log\left(\frac{1}{1-p}\right) \right\} \\ &= \left\{ t : t < \frac{1}{p} + \log\left(\frac{1}{1-p}\right)\frac{1-p}{p^2} \right\}. \end{aligned} \tag{53}$$

In light of Theorem 3, we have

$$\begin{aligned} L_X(t) &= \frac{m_X\left(\frac{t-\mu}{\sigma^2}\right)}{m_N\left(\frac{t-\mu}{\sigma^2}\right)} = R_{X,N}\left(\frac{p(tp-1)}{1-p}\right) \\ &= \frac{e^{\frac{1}{2}\frac{(1-tp)(1-p(2-t))}{1-p}}p}{1-(1-p)e^{\frac{p(tp-1)}{1-p}}}, \end{aligned} \tag{54}$$

for $t \in \mathcal{T}_p$. For simplicity, we will write $L(t,p)$ to represent $L_X(t)$.

We will prove the following theorem.

Theorem 9 *Suppose X is a geometric random variable with parameter p. Then, for all $t \in \mathcal{U}_X$,*

$$\begin{aligned} L_X(t) &= L(t,p) \geq L(1,p) = \frac{e^{\frac{1}{2}(p-1)}p}{1-(1-p)e^{-p}} \\ &\geq L(1,0+) = \frac{1}{2}e^{-\frac{1}{2}} \approx 0.3032653298.... \end{aligned} \tag{55}$$

Proof. For the second inequality in (55), note that

$$\frac{d}{dp}L(1,p) = 1/2\, \frac{e^{-1/2\,p+1/2}\left(-2+p\right)\left(e^{-p}(p+1)-1\right)}{\left(1-e^{-p}+e^{-p}p\right)^2} > 0, \tag{56}$$

for $0 < p < 1$, since the function f defined by $f(p) = e^{-p}(p+1)-1$ is decreasing for p on $(0,1)$ and $f(0) = 0$.

For the first inequality, note that

$$\frac{d}{dt}L(t,p) = \frac{p^3 t e^{-\frac{1}{2}\frac{(pt-1)(-2p+pt+1)}{1-p}}\left((1-p)e^{\frac{(pt-1)p}{1-p}} - \frac{t-1}{t}\right)}{\left(1-(1-p)e^{\frac{(pt-1)p}{1-p}}\right)^2(1-p)}. \tag{57}$$

For $0 < p < 1$, set $f_p(t) = (1-p)e^{\frac{(pt-1)p}{1-p}}$, $g_p(t) = (t-1)/t$ and $U_p(t) = f_p(t) - g_p(t)$. We then have $f_p''(t) = p^4 e^{\frac{(pt-1)p}{1-p}}/(1-p) > 0$, $g_p''(t) = -2t^{-3} < 0$, $f_p'(1/p) = g_p'(1/p) = p^2$ and $f_p(1/p) = g_p(1/p) = 1-p$. Thus, $U_p''(t) > 0$ and $U_p'(1/p) = 0$. Hence, $U_p(t) > U_p(1/p) = 0$ and (57) gives that $\frac{d}{dt}L(t,p) \geq 0$ for $t \geq 1$ and $0 < p < 1$, and the result follows. □

Remark. The above considerations raise questions as to systematic understanding of variation in behavior of L_X according to summary properties of the variable X. Characteristics of L_X under various assumptions are currently under consideration.

We close with the following table which includes the limiting convolution ratio, $L_X(t)$, for some further common distributions with unbounded support.

Table 2: Limiting convolution ratio $L_X(t)$ for some further common distributions

Distribution	Prob. Function	Mean	Variance	M.G.F.	$L_X(t)$
Gamma	$\frac{e^{-x/\theta}x^{k-1}}{\theta^k\Gamma(k)}, x > 0$ $k, \theta > 0$	$k\theta$	$k\theta^2$	$\frac{1}{(1-\theta t)^k}$	$\frac{k^k\theta^k}{(2k\theta-t)^k}e^{-\frac{t^2-k^2\theta^2}{2k\theta^2}}$
Neg. Binomial	$\frac{\Gamma(r+k)}{k!\Gamma(r)}p^r(1-p)^k$ $0 < p < 1, k \in \mathbb{N}$	$\frac{r(1-p)}{p}, r > 0$	$\frac{r(1-p)}{p^2}$	$\left(\frac{p}{1-(1-p)e^t}\right)^r$	$\frac{p^r e^{-\frac{p^2t^2-r^2(1-p)^2}{2r(1-p)}}}{\left(1-(1-p)e^{\frac{p^2t-rp(1-p)}{r(1-p)}}\right)^r}$
Laplace	$\frac{1}{2b}e^{-\frac{\lvert x-\mu\rvert}{b}}$ $b > 0, \mu \in \mathbb{R}$	μ	$2b^2$	$\frac{e^{\mu t}}{1-b^2t^2}$	$\frac{4b^2}{4b^2-(t-\mu)^2}e^{-\frac{(t-\mu)^2}{4b^2}}$

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