Applied Mathematical Sciences, Vol. 2, 2008, no. 48, 2363 - 2376

Risk-Sensitive Control for Systems with Input Delays

Jun Yoneyama

Department of Electronics and Electrical Engineering
Aoyama Gakuin University
5-10-1 Fuchinobe, Sagamihara, Kanagawa 229-8558, Japan
yoneyama@ee.aoyama.ac.jp

Abstract

The partially observed risk-sensitive optimal stochastic control of a system with multiple time-delays in input, which minimizes the expected value of an exponential cost criterion, is considered. The change of measure technique is used to obtain the sufficient statistics of the cost criterion given the measurement history, which is called the information state. The information state helps to convert the partially observed problem into an equivalent fully observed one. The risk-sensitive controller is obtained via a set of differential Riccati equations. The result for the finite-time time-delay system is extended to obtain the infinite-time case. Furthermore, asymptotic behaviors of both small noise and small risk limits of the problem, which correspond to deterministic game and risk-neutral stochastic problem, respectively, are analyzed.

Mathematics Subject Classification: 93E20

Keywords: Risk-sensitive control; Time-delay systems; Asymptotic limits

1 Introduction

Since Jacobson first attempted to solve a risk-sensitive control problem for fully observed linear systems in [6], a risk-sensitive control problem for various classes of systems has been considered([1], [2], [3], [4], [11], [14], [15]). The relationship between the risk-sensitive control problem and differential game was first investigated by [6]. Ever since it has been shown that a risk-sensitive controller is equivalent to a game strategy in a stochastic differential game setting. It is also known that the controller reduces to an H_∞ controller when Wiener noises in the system are absent. The further investigation on the relationships between the risk-sensitive control problem, differential games and an H_∞ control problem have been made in [7], [8] and [9] where the viscosity solutions were used to establish the relationships.

In this paper, we consider the risk-sensitive stochastic optimal control problem for partially observed system with multiple time-delays in input. Delays often occur in the transmission of information or material between different parts of a system. Thus time-delay systems appear in various applications and system formulations, and hence the controller design for time-delay systems is important in many fileds. The standard H_∞ control problem for time-delay systems was solved in [12], [13] and reference therein. However, few works on robust stochastic problems for time-delay systems have appeared in the recent literature. In a stochastic time-delay system in this paper, we introduce an important parameter, called a small-noise parameter, which is multiplied by disturbances in the system so that the noise intensities are governed by the parameter. The cost criterion is the expected value of an generalized exponential function and has a significant parameter, called a risk-sensitivity parameter. This parameter generalizes a stochastic optimal control problem. Motivated by the idea of [1] and [2], we consider the sufficient statistics of the cost criterion given the measurement history, which satisfies an equation similar to a Zakai equation. Then we obtain the information state for our class of systems. The use of the information state converts a partially observed stochastic control problem into an equivalent fully observed stochastic control problem. Then we employ Dynamic Programming to solve the problem. The result is extended to the infinite-time problem. Moreover, asymptotic limits of the problem are analyzed as the small-noise and risk-sensitivity parameters tend to zero.

2 Risk-sensitive Optimal Control

In this section, we introduce a time-delay system and formulate a risk-sensitive optimal control problem. We then attempt to solve a risk-sensitive optimal control problem for the time-delay system, using the information state. The result of the problem is extended to the infinite-time problem.

2.1 Problem Formulation

On a probability space $(\Omega, \mathcal{A}, P)$ we consider a time-delay system described by

$$\left\{ \begin{array}{rcl} dx & = & (Ax(t) + \sum_{i=0}^{l} B_i u(t-h_i))dt + \sqrt{\varepsilon} D dw, \\ d\tilde{y} & = & Cx(t)dt + \sqrt{\varepsilon} dv, \\ x(0) & = & \hat{x}_0 + \sqrt{\varepsilon}\tilde{x}_0, \\ u(\beta) & = & \psi(\beta), \ -h \le \beta \le 0 \end{array} \right. \tag{1}$$

where $x(\cdot) \in \Re^n$ is the state, $w(\cdot) \in \Re^p$ is the process noise, $u(\cdot) \in \Re^m$ is the control and $y(\cdot) \in \Re^r$ is the measured output, and $v(\cdot) \in \Re^r$ is the measurement noise. $\hat{x}_0$ is a known vector and $\tilde{x}_0$ is a zero-mean random variable with covariance P_0. $\psi(\beta) \in L^2(-h, 0; \Re^n)$ is given. $h_i > 0,\ i = 1, \cdots, l$ are constant time-delays and $h_0 = 0$. ε is a positive scalar called a small-noise parameter. For the system (1), we assume the followings.

Assumption 2.1 *1) $x(0)$ is a random variable with gaussian probability law with mean x_0 and covariance matrix P_0.*
2) $w(\cdot)$ is a zero-mean Wiener process with covariance matrix W.
3) $v(\cdot)$ is a zero-mean Wiener process with covariance matrix $V > 0$.
4) The processes $w(\cdot)$, $v(\cdot)$ and the initial variable $x(0)$ are mutually independent.

Now we define the cost criterion for the time-delay system;

$$J(u) = E \exp[\frac{\theta}{\varepsilon}\{\Phi(x, t_f) + \int_0^{t_f} L(x, u, t)dt\}] \tag{2}$$

where

$$\Phi(x, t) = \frac{1}{2}x(t)^T Q_f x(t),$$

$$L(x, u, t) = \frac{1}{2}(||Q_0^{1/2}x(t) + \sum_{i=1}^{l}\int_{-h_i}^{0} Q_i^{1/2}(\beta)u(t+\beta)d\beta||^2 + u^T(t)Ru(t)),$$

and $||\cdot||$ is the Euclidean norm, $Q_f \geq 0,\ Q_0 \geq 0,\ R > 0,\ Q_i(\beta),\ i = 1, \cdots, l$ are matrix functions whose elements are bounded continuous, and θ is a positive scalar called a risk-sensitivity parameter.

Then the risk-sensitive optimal control problem is to find u^* such that $J(u)$ is minimized; $J(u^*) \leq J(u)$ where u is restricted to a causal function of the measurement history $\mathcal{Y}^t$ defined by $\mathcal{Y}^t = \sigma(\tilde{y}(\tau), \tau \leq t)$. We write $\tilde{u}_i(t) = u(t - h_i)$ and the partial derivatives of the function $f(x, t)$ as follows;

$$D_x f(x, t) = \frac{\partial f}{\partial x},\ D_x^2 f(x, t) = \frac{\partial^2 f}{\partial x^2}.$$

Similar notation will be used hereafter.

2.2 Information State

For the partially observed stochastic systems, the state is not directly observed. In this case, we would look for the property that the state of the system at time t should be computable from the information available at time t, which is the measurement history $\mathcal{Y}^t$ and the control history $\mathcal{U}^t = \sigma(u(\tau), \tau \leq t)$. Kumar and Varaiya have showed in [10] that the conditional probability density function of the cost criterion given the measurement history $\mathcal{Y}^t$, called an information state, does not depend on a control law and it can be evaluated from the measurement history $\mathcal{Y}^t$ and control history $\mathcal{U}^t$. The information state can eventually convert a partially observed stochastic problem into an equivalent one with the fully observed stochastic system.

As noted in [5], the use of Girsanov's theorem allows us to get much simpler form of the sufficient statistics of the cost criterion given the measurement history, from which the information state is obtained. To this end, we define the change of probability measure by

$$\begin{aligned} \frac{dP}{d\tilde{P}}\Big|_{\mathcal{F}^t} &= \eta_t \\ &= \exp\{\frac{1}{\varepsilon}(\int_0^t x^T C^T V^{-1} d\tilde{y} - \frac{1}{2}\int_0^t x^T C^T V^{-1} C x d\tau)\}. \end{aligned}$$

Then, η_t is an $\mathcal{F}^t$ martingale and $\tilde{E}[\eta_t] = 1$. We note that η_t is a solution of

$$d\eta_t = \frac{1}{\varepsilon}\eta_t x^T C^T V^{-1} d\tilde{y},\ \eta_0 = 1. \tag{3}$$

By Girsanov's theorem under this change of probability measure, $\tilde{y}$ is a Wiener process with the covariance matrix εV on the probability space $(\Omega, \mathcal{A}, \tilde{P})$ (see [5].). Moreover, $x(0), w$ and $\tilde{y}$ remain mutually independent, $x(0), w$ keeping the same probability laws. Therefore, we convert the system (1) for a different probability space $(\Omega, \mathcal{A}, \tilde{P})$.

For the time-delay system (1), we now calculate the sufficient statistics of the cost criterion given the measurement history. Consider the cost criterion

$$\begin{aligned} J(u) &= E[\exp\{\frac{\theta}{\varepsilon}(\Phi(x,t_f) + \int_0^{t_f} L(x,u,t)dt\}] \\ &= \tilde{E}[\eta_{t_f}\exp\{\frac{\theta}{\varepsilon}(\Phi(x,t_f) + \int_0^{t_f} L(x,u,t)dt\}]. \end{aligned}$$

If we write

$$M_t = \exp(\frac{\theta}{\varepsilon}\int_0^t L(x,u,\tau)d\tau),$$

then M_t satisfies

$$dM_t = \frac{\theta}{\varepsilon}L(x,u,t)M_t dt,\ M_0 = 1. \tag{4}$$

For any function ϕ, we write

$$\sigma(\phi)_t = \tilde{E}[\eta_t M_t \phi(x)].$$

In case the measure defined by $\sigma(\phi)_t$ has a density $q(x,t)$, we have

$$\sigma(\phi)_t \stackrel{\triangle}{=} <\phi, q> = \int_{\Re^n} \phi(x) q(x,t) dx. \tag{5}$$

Now we obtain the conditional probability density function given the measurement history; a modified Zakai equation for $\sigma(\phi)_t$.

Lemma 2.2 *Suppose that ϕ is a C^2 function. Then*

$$\sigma(\phi)_t = \sigma(\phi)_0 + \frac{\theta}{\varepsilon}\int_0^t \sigma(L\phi)d\tau + \frac{1}{\varepsilon}\int_0^t \sigma(\phi x^T C^T V^{-1})d\tilde{y} + \int_0^t \sigma(T\phi)d\tau \quad (6)$$

where T is the operator defined by

$$T = (Ax + \sum_{i=0}^{l} B_i \tilde{u}_i)\frac{\partial}{\partial x} + \frac{\varepsilon}{2} trDWD^T \frac{\partial^2}{\partial x^2}$$

and

$$\sigma(\phi)_0 = E[\phi(x_0)] = \varepsilon \int_{\Re^n} \phi(x) P_0 dx.$$

Proof: Applying the Ito's formula, we have

$$d\phi(x) = (T\phi)(x)dt + \sqrt{\varepsilon} D_x \phi(x) D dw. \quad (7)$$

It follows from (3), (4) and (7) that

$$\begin{aligned} d(\eta_t M_t \phi(x)) &= (d\eta_t) M_t \phi(x) + \eta_t (dM_t)\phi(x) + \eta M_t (d\phi(x)) \\ &= \frac{1}{\varepsilon}\eta_t M_t \phi(x) x^T C^T V^{-1} d\tilde{y} + \frac{\theta}{\varepsilon}\eta_t L(x,u,t) M_t \phi(x) dt \\ &\quad + \eta_t M_t (T\phi)(x) dt + \sqrt{\varepsilon}\eta_t M_t D_x \phi(x) D dw. \end{aligned}$$

Hence integrating both sides, we have

$$\begin{aligned} M_t \phi(x) &= M_0 \phi(x_0) + \frac{1}{\varepsilon}\int_0^t \eta_\tau M_\tau \phi(x) x^T C^T V^{-1} d\tilde{y} \\ &\quad + \frac{\theta}{\varepsilon}\int_0^t \eta_\tau L(x,u,\tau) M_\tau \phi(x) d\tau + \int_0^t \eta_\tau M_\tau (T\phi)(x) d\tau \\ &\quad + \sqrt{\varepsilon}\int_0^t \eta_\tau M_\tau D_x \phi(x) D dw. \end{aligned}$$

Taking the conditional expectation in both sides on $\mathcal{Y}^t$, we obtain the desired result. ♢

The conditional probability density function of the cost criterion given the measurement history is readily derived from Lemma 2.2, as in [2].

Theorem 2.3 *If $\phi(\cdot)_t$ has a density $q(x,t)$, then it satisfies*

$$\begin{aligned} dq &= [-D_x q(x,t)(Ax + \sum_{i=0}^{l} B_i \tilde{u}_i) - q(x,t) trA + \frac{\theta}{\varepsilon} q(x,t) L(x,u,t) \\ &\quad + \frac{\varepsilon}{2} trDWD^T D_x^2 q(x,t)]dt + \frac{1}{\varepsilon} q(x,t) x^T C^T V^{-1} d\tilde{y}, \quad (8) \\ q(x,0) &= \frac{\exp\{-\frac{1}{2\varepsilon}(x-\hat{x}_0)^T P_0^{-1}(x-\hat{x}_0)\}}{(2\pi)^{n/2}|\varepsilon P_0|^{1/2}}. \quad (9) \end{aligned}$$

We call $q(x,t)$ the information state.

Proof: Using (5) and (6), we have

$$\begin{aligned}\int_{\Re^n}\phi(x)q(x,t)dx &= \int_{\Re^n}\phi(x)q(x,0)dx + \frac{\theta}{\varepsilon}\int_{\Re^n}\int_0^t L(x,u,\tau)\phi(x)q(x,\tau)d\tau dx \\ &\quad +\frac{1}{\varepsilon}\int_{\Re^n}\int_0^t \phi(x)x^TC^TV^{-1}q(x,\tau)d\tilde{y}dx \\ &\quad +\int_{\Re^n}\int_0^t (T\phi(x))q(x,\tau)d\tau dx.\end{aligned}$$

It follows that

$$\begin{aligned}q(x,t) &= q(x,0) + \frac{\theta}{\varepsilon}\int_0^t L(x,u,\tau)q(x,\tau)d\tau + \frac{1}{\varepsilon}\int_0^t x^TC^TV^{-1}q(x,\tau)d\tilde{y} \\ &\quad +\int_0^t T^*q(x,\tau)d\tau\end{aligned}$$

where T^* is the adjoint of T. Differentiating both sides, we get

$$\begin{aligned}dq &= \frac{\theta}{\varepsilon}L(x,u,t)q(x,t)dt + \frac{1}{\varepsilon}x^TC^TV^{-1}q(x,t)d\tilde{y} + T^*q(x,t)dt \\ &= [-D_xq(x,t)(Ax+\sum_{i=0}^{l}B_i\tilde{u}_i) - q(x,t)\mathrm{tr}A + \frac{\theta}{\varepsilon}q(x,t)L(x,u,t) \\ &\quad +\frac{\varepsilon}{2}\mathrm{tr}DWD^TD_x^2q(x,t)]dt + \frac{1}{\varepsilon}q(x,t)x^TC^TV^{-1}d\tilde{y}.\end{aligned}$$

$\diamondsuit$

The information state $q(x,t)$ can explicitly be expressed.

Theorem 2.4 *The solution to the partial differential equation (8) with (9) is given by*

$$q(x,t) = \frac{\nu(t)\exp\{-\frac{1}{2}(x(t)-\hat{x}(t))^TP^{-1}(t)(x(t)-\hat{x}(t))\}}{(2\pi)^{n/2}|P(t)|^{1/2}} \tag{10}$$

where $\hat{x}(t)$ *satisfies*

$$\begin{aligned}d\hat{x} &= (A\hat{x}(t)+\sum_{i=0}^{l}B_iu(t-h_i))dt + \frac{1}{\varepsilon}P(t)C^TV^{-1}(d\tilde{y}-C\hat{x}(t)dt) \\ &\quad +\frac{\theta}{\varepsilon}P(t)(Q_0\hat{x}(t)+Q_0^{1/2}\sum_{i=1}^{l}\int_{-h_i}^{0}Q_i^{1/2}(\beta)u(t+\beta)d\beta), \\ \hat{x}(0) &= \hat{x}_0,\end{aligned} \tag{11}$$

and $P(t)$ *satisfies the Riccati equation*

$$\begin{aligned}\dot{P}(t) &= AP(t)+P(t)A^T+\varepsilon DWD^T + P(t)(\frac{\theta}{\varepsilon}Q_0 - C^TV^{-1}C)P(t), \\ P(0) &= \varepsilon P_0,\end{aligned} \tag{12}$$

and $\nu(t)$ *is given by*

$$\begin{aligned}\nu(t) = \ & \exp[\frac{1}{\varepsilon}(\int_0^t \hat{x}^T C^T V^{-1} d\tilde{y} - \frac{1}{2}\int_0^t \hat{x}^T C^T V^{-1} C\hat{x} d\tau) \\ & + \frac{\theta}{\varepsilon}\int_0^t (L(\hat{x}, u, \tau) + \frac{1}{2} trP(\tau)Q_0) d\tau].\end{aligned} \tag{13}$$

Proof: Differentiating $q(x,t)$ defined by (10), (11), (12) and (13), and substituting in (8) verify the result. $\diamondsuit$

Remark 2.5 *There exists a backward adjoint process* $p(x,t)$ *such that*

$$< p(x, t_1),\ q(x, t_1) >=< p(x, t_2),\ q(x, t_2) >$$

for all $t_1, t_2 \in [0,\ t_f]$*. Actually, the process* $p(x,t)$ *satisfies the parabolic equation adjoint to (8) and (9), namely,*

$$\begin{aligned} -dp &= [D_x p(x,t)(Ax + \sum_{i=0}^{l} B_i \tilde{u}_i) + \frac{\theta}{\varepsilon} p(x,t) L(x,u,t) \\ &\quad + \frac{\varepsilon}{2} trDWD^T D_x^2 p(x,t)]dt + \frac{1}{\varepsilon} p(x,t) x^T C^T V^{-1} d\tilde{y}, \\ p(x, t_f) &= \exp(\frac{\theta}{2\varepsilon} x^T(t_f) Q_f x(t_f)). \end{aligned}$$

2.3 Solution to Risk-sensitive Control Problem

Now we solve the risk-sensitive control problem. The cost criterion (2) can be written using the information state.

Theorem 2.6 *The cost criterion can be written as*

$$J(u) = E[\exp\{\frac{\theta}{2\varepsilon}(\hat{\phi}(\hat{x}(t_f), P(t_f)) + \int_0^{t_f} (L(\hat{x}, u, t) + \frac{1}{2} trP(t)Q_0) dt)\}] \tag{14}$$

where

$$\begin{aligned}\hat{\phi}(x, P) = \ & \frac{2\varepsilon}{\theta} \log |I - \frac{\theta}{\varepsilon} PQ_f| + x^T \hat{Q}_f x, \\ \hat{Q}_f = \ & \frac{1}{2}[(I - \frac{\theta}{\varepsilon} Q_f P)^{-1} Q_f + Q_f (I - \frac{\theta}{\varepsilon} PQ_f)^{-1}].\end{aligned}$$

Proof: Using Theorem 2.4, the cost criterion can be written as

$$\begin{aligned} J(u) &= E[\exp\{\frac{\theta}{\varepsilon}(\Phi(x, t_f) + \int_0^{t_f} L(x,u,t)dt\}] \\ &= \tilde{E}[\int_{\Re^n} p(x, t_f) q(x, t_f) dx] \\ &= E[\int_{\Re^n} \frac{\exp\{\frac{\theta}{2\varepsilon} x^T Q_f x - \frac{1}{2}(x - \hat{x})^T P^{-1} (x - \hat{x})\}}{(2\pi)^{n/2} |P|^{1/2}} dx \\ &\qquad \times \exp[\frac{\theta}{\varepsilon}\{\int_0^{t_f} (L(\hat{x}, u, t) + \frac{1}{2}\mathrm{tr}P(t)Q_0) dt\}] \\ &= E[\exp\{\frac{\theta}{2\varepsilon}\hat{\phi}(\hat{x}(t_f), P(t_f)) + \frac{\theta}{\varepsilon}\int_0^{t_f} (L(\hat{x}, u, t) + \frac{1}{2}\mathrm{tr}P(t)Q_0) dt\}]. \end{aligned}$$

$\diamondsuit$

Thus our original problem is restated as follows; minimize the cost criterion $J(u)$ defined by (14), subject to dynamics (11) and (12). Suppose that $X(\hat{x},P,t)$ is differentiable in $t,\hat{x}$ and P. Since the factor $\frac{2\varepsilon}{\theta}\log|I-\frac{\theta}{\varepsilon}PQ_f|+\int_0^{t_f}\mathrm{tr}P(t)Qd\tau$ of the cost criterion is independent of minimization, the Hamilton-Jacobi-Bellman equation for this altered problem can be written as

$$\left\{\begin{array}{rl} -\dfrac{\partial X}{\partial t}= & D_{\hat{x}}X\{A\hat{x}(t)+\displaystyle\sum_{i=0}^{l}B_iu(t-h_i) \\ & +\dfrac{\theta}{\varepsilon}P(t)(Q_0\hat{x}(t)+Q_0^{1/2}\displaystyle\sum_{i=1}^{l}\int_{-h_i}^{0}Q_i^{1/2}(\beta)u(t+\beta)d\beta)\}, \\ & +\dfrac{\theta}{\varepsilon}X(L(\hat{x},u,t)+\dfrac{1}{2}\mathrm{tr}P(t)Q_0)+\dfrac{1}{2\varepsilon}\mathrm{tr}D_{\hat{x}}^2XP(t)C^TV^{-1}CP(t), \\ X(\hat{x},P,t_f)= & \exp(\dfrac{\theta}{2\varepsilon}\hat{x}^T(t_f)\hat{Q}_f\hat{x}(t_f)). \end{array}\right. \tag{15}$$

Now we obtain the main result. For simplicity, we let $B_i=0,\ Q_i(\cdot)=0,\ i=2,\cdots,l$, and denote $h\triangleq h_1,\ \bar{Q}_1(\beta)\triangleq Q_0^{1/2}Q_1^{1/2}(\beta),\ \bar{Q}_2(\alpha,\beta)\triangleq Q_1^{1/2}(\alpha)Q_1^{1/2}(\beta)$.

Theorem 2.7 *Consider the risk-sensitive control problem for the time - delay system. Suppose that the following conditions hold;*
(i) There exists a solution $P(t)$ of the Riccati equation

$$\left\{\begin{array}{rcl} \dot{P}(t) & = & AP(t)+P(t)A^T+\varepsilon DWD^T+P(t)(\dfrac{\theta}{\varepsilon}Q_0-C^TV^{-1}C)P(t), \\ P(0) & = & \varepsilon P_0. \end{array}\right.$$

(ii) For the solution $P(t)$ in (i), there exists a triplet $\Pi=(\Pi_0(t),\ \Pi_1(\beta,t),\ \Pi_2(\alpha,\beta,t))$ of the Riccati equations

$$\left\{\begin{array}{l} -\dfrac{\partial}{\partial t}\Pi_0(t)+(A+\dfrac{\theta}{\varepsilon}P(t)Q_0)^T\Pi_0(t)+\Pi_0(t)(A+\dfrac{\theta}{\varepsilon}P(t)Q_0)+Q_0 \\ \qquad -\{\Pi_0(t)B_0+\Pi_1(0,t)B_1\}R^{-1}\{B_0^T\Pi_0(t)+B_1^T\Pi_1^T(0,t)\} \\ \qquad +\dfrac{\theta}{\varepsilon^2}\Pi_0(t)P(t)C^TV^{-1}CP(t)\Pi_0(t)=0, \\ -(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial\beta})\Pi_1(\beta,t)+(A+\dfrac{\theta}{\varepsilon}P(t)Q_0)^T\Pi_1(\beta,t)+\bar{Q}_1(\beta) \\ \qquad -\{\Pi_0(t)B_0+\Pi_1(0,t)B_1\}R^{-1}\{B_0^T\Pi_1(\beta,t)+B_1^T\Pi_2^T(0,\beta,t)\} \\ \qquad +\dfrac{\theta}{\varepsilon^2}\Pi_0(t)P(t)C^TV^{-1}CP(t)\Pi_1(\beta,t)+\dfrac{\theta}{\varepsilon}\Pi_0(t)P(t)\bar{Q}_1(\beta)=0, \\ -(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial\alpha}+\dfrac{\partial}{\partial\beta})\Pi_2(\alpha,\beta,t)+\bar{Q}_2(\alpha,\beta)+\dfrac{\theta}{\varepsilon}\Pi_1^T(\alpha,t)P(t)\bar{Q}_1(\beta) \\ \qquad +\dfrac{\theta}{\varepsilon}\bar{Q}_1^T(\alpha)P(t)\Pi_1(\beta,t)+\dfrac{\theta}{\varepsilon^2}\Pi_1^T(\alpha,t)P(t)C^TV^{-1}CP(t)\Pi_1(\beta,t) \\ \qquad -\{\Pi_1^T(\alpha,t)B_0+\Pi_2(\alpha,0,t)B_1\}R^{-1}\{B_0^T\Pi_1(\beta,t)+B_1^T\Pi_2^T(0,\beta,t)\}=0 \end{array}\right. \tag{16}$$

with the boundary conditions

$$\Pi_0(t_f) = \hat{Q}_f,\ \Pi_0(t) = \Pi_0^T(t),\ \Pi_1(-h,t) = \Pi_0(t),$$
$$\Pi_2(-h,\beta,t) = \Pi_1(\beta,t),\ \Pi_2(\alpha,\beta,t) = \Pi_2^T(\beta,\alpha,t),\ -h \le \alpha,\ \beta \le 0.$$

Then the risk-sensitive controller is given by

$$\begin{cases} u^*(t) = & -R^{-1}[\{B_0^T\Pi_0(t) + B_1^T\Pi_1^T(0,t)\}x(t) \\ & +\int_{-h}^0 \{B_0^T\Pi_1(\beta,t) + B_1^T\Pi_2(0,\beta,t)\}B_1u(t+\beta)d\beta], \\ d\hat{x} = & (A\hat{x}(t) + B_0u(t) + B_1u(t-h))dt + \frac{1}{\varepsilon}P(t)C^TV^{-1}(d\tilde{y} - C\hat{x}(t)dt) \\ & +\frac{\theta}{\varepsilon}P(t)(Q_0\hat{x}(t) + \int_{-h}^0 \bar{Q}_1(\beta)u(t+\beta)d\beta), \\ \hat{x}(0) = & \hat{x}_0, \\ u(\beta) = & \psi(\beta). \end{cases}$$

Proof: (15) can be solved by assuming a solution in the form

$$\begin{aligned} X(\hat{x},\Pi,t) &= \gamma(t)\exp[\frac{\theta}{2\varepsilon}\{\hat{x}^T(t)\Pi_0(t)\hat{x}(t) + 2\hat{x}(t)\int_{-h}^0 \Pi_1(\beta,t)B_1u(t+\beta)d\beta \\ &\quad + \int_{-h}^0\int_{-h}^0 u^T(t+\alpha)B_1^T\Pi_2(\alpha,\beta,t)B_1u(t+\beta)d\alpha d\beta\}]. \end{aligned} \tag{17}$$

Substitution of (17) into (15) and minimization with respect to the control verify that $\Pi(t)$ satisfies (16). ◇

2.4 The Infinite-time Problem

In this section, we consider the infinite-time problem for the time-delay system where the terminal time $t_f \to 0$ and $Q_f = 0$ in the cost criterion (2);

$$J(u) = \lim_{t_f\to\infty}\frac{1}{t_f}E\exp[\frac{\theta}{2\varepsilon}\int_0^{t_f}(||Q_0^{1/2}x(t) + \int_{-h}^0 Q_1^{1/2}(\beta)u(t+\beta)||^2 + u^T(t)Ru(t))dt]$$

We make the following assumptions on the system (1). They give stabilizability and detectability on the system (1).

Assumption 2.8 *Define* $\Delta(s) = sI - A$. *Then we have*

$$rank[\Delta(s)\quad B + Be^{-sh}] = n,\ rank[\Delta^T(s)\quad C^T] = n,$$
$$rank[\Delta(s)\quad D] = n,\ rank[\Delta^T(s)\quad Q_0^{1/2}] = n,$$

for all s *in the right-half plane including the imaginary axis such that*

$$\det\Delta(s) = 0.$$

The following result is deduced from Theorem 2.7.

Theorem 2.9 *Suppose that the following conditions hold;*
(i) There exists a nonnegative definite solution P of the Riccati equation

$$AP + PA^T + \varepsilon DWD^T + P(\frac{\theta}{\varepsilon}Q_0 - C^TV^{-1}C)P = 0. \tag{18}$$

(ii) For the solution P in (i), there exists a nonnegative definite triplet $\Pi = (\Pi_0,\ \Pi_1(\beta),\ \Pi_2(\alpha,\beta))$ of the Riccati equations

$$\begin{cases} (A+\frac{\theta}{\varepsilon}PQ_0)^T\Pi_0 + \Pi_0(A+\frac{\theta}{\varepsilon}PQ_0) + Q_0 - \{\Pi_0B_0 + \Pi_1(0)B_1\}R^{-1} \\ \quad \times\{B_0^T\Pi_0 + B_1^T\Pi_1^T(0)\} + \frac{\theta}{\varepsilon^2}\Pi_0PC^TV^{-1}CP\Pi_0 = 0, \\ -\frac{\partial}{\partial\beta}\Pi_1(\beta) + (A+\frac{\theta}{\varepsilon}PQ_0)^T\Pi_1(\beta) + \bar{Q}_1(\beta) + \frac{\theta}{\varepsilon}\Pi_0P\bar{Q}_1(\beta) \\ \quad -\{\Pi_0B_0 + \Pi_1(0)B_1\}R^{-1}\{B_0^T\Pi_1(\beta) + B_1^T\Pi_2^T(0,\beta)\} \\ \quad +\frac{\theta}{\varepsilon^2}\Pi_0PC^TV^{-1}CP\Pi_1(\beta) = 0, \\ -(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta})\Pi_2(\alpha,\beta) + \bar{Q}_2(\alpha,\beta) + \frac{\theta}{\varepsilon^2}\Pi_1^T(\alpha)PC^TV^{-1}CP\Pi_1(\beta) \\ \quad +\frac{\theta}{\varepsilon}\Pi_1^T(\alpha)P\bar{Q}_1(\beta) + \frac{\theta}{\varepsilon}\bar{Q}_1^T(\alpha)P\Pi_1(\beta) \\ \quad -\{\Pi_1^T(\alpha)B_0 + \Pi_2(\alpha,0)B_1\}R^{-1}\{B_0^T\Pi_1(\beta) + B_1^T\Pi_2^T(0,\beta)\} = 0 \end{cases} \tag{19}$$

with the boundary conditions

$$\begin{array}{l} \Pi_1(-h) = \Pi_0,\ \Pi_2(-h,\beta) = \Pi_1(\beta), \\ \Pi_0 = \Pi_0^T,\ \Pi_2(\alpha,\beta) = \Pi_2^T(\beta,\alpha),\ -h \le \alpha,\ \beta \le 0. \end{array} \tag{20}$$

Then the risk-sensitive controller is given by

$$\begin{cases} u^*(t) &=& -R^{-1}[\{B_0^T\Pi_0 + B_1^T\Pi_1^T(0)\}x(t) \\ && + \int_{-h}^0 \{B_0^T\Pi_1(\beta) + B_1^T\Pi_2(0,\beta)\}B_1u(t+\beta)d\beta], \\ d\hat{x} &=& (A\hat{x}(t) + B_0u(t) + B_1u(t-h))dt + \frac{1}{\varepsilon}PC^TV^{-1}(d\tilde{y} - C\hat{x}(t)dt), \\ && +\frac{\theta}{\varepsilon}P(Q_0\hat{x}(t) + \int_{-h}^0 \bar{Q}_1(\beta)u(t+\beta)d\beta), \\ \hat{x}(0) &=& \hat{x}_0, \\ u(\beta) &=& \psi(\beta). \end{cases}$$

Remark 2.10 *The nonnegativity of Π is defined as follows;*

$$z^T\Pi_0z + 2\int_{-h}^0 z^T\Pi_1(\beta)B_1w(\beta)d\beta + \int_{-h}^0\int_{-h}^0 w^T(\alpha)B_1^T\Pi_2(\alpha,\beta)B_1w(\beta)d\alpha d\beta \ge 0$$

for all $z \in \Re^n$ and $w \in L^2(-h,0;R^m)$.

3 Asymptotic Analysis of Risk-sensitive Controller

We investigate asymptotic limits of the risk-sensitive controllers given in Theorem 2.9 when ε and θ tend to zero. It has been shown in [8] and [9] that the small noise and small risk limits of the controllers are equivalent to H_∞ controller and risk-neutral optimal controller, respectively.

3.1 Small Noise Limit

We consider the small noise limit when ε goes to zero, and obtain the risk-sensitive optimal controllers in the small noise limit.

First consider the Riccati equations (18) and (19). Taking $\varepsilon \to 0$, we have

$$\frac{P}{\varepsilon} \to P.$$

provided $\tilde{y} \to \int_0^t y ds$. Thus we have the following result from Theorem 2.9.

Corollary 3.1 *Suppose that the following conditions hold;*
(i) There exists a nonnegative definite solution P to the Riccati equation

$$AP + PA^T + DWD^T + P(\theta Q_0 - C^T V^{-1} C)P = 0.$$

(ii) For the solution P in (i), there exists a nonnegative definite triplet $\Pi = (\Pi_0,\ \Pi_1(\beta),\ \Pi_2(\alpha,\beta))$ to the Riccati equations

$$\begin{cases} (A+\theta PQ_0)^T\Pi_0 + \Pi_0(A+\theta PQ_0) + Q_0 + \theta\Pi_0 PC^TV^{-1}CP\Pi_0 \\ \qquad -\{\Pi_0 B_0 + \Pi_1(0)B_1\}R^{-1}\{B_0^T\Pi_0 + B_1^T\Pi_1^T(0)\} = 0, \\ -\dfrac{\partial}{\partial\beta}\Pi_1(\beta) + (A+\theta PQ_0)^T\Pi_1(\beta) + \bar{Q}_1(\beta) + \theta\Pi_0 P\bar{Q}_1(\beta) \\ \qquad +\theta\Pi_0 PC^TV^{-1}CP\Pi_1(\beta) \\ \qquad -\{\Pi_0 B_0 + \Pi_1(0)B_1\}R^{-1}\{B_0^T\Pi_1(\beta) + B_1^T\Pi_2^T(0,\beta)\} = 0, \\ -(\dfrac{\partial}{\partial\alpha} + \dfrac{\partial}{\partial\beta})\Pi_2(\alpha,\beta) + \bar{Q}_2(\alpha,\beta) + \theta\Pi_1^T(\alpha)P\bar{Q}_1(\beta) \\ \qquad +\theta\bar{Q}_1^T(\alpha)P\Pi_1(\beta) + \theta\Pi_1^T(\alpha)PC^TV^{-1}CP\Pi_1(\beta) \\ \qquad -\{\Pi_1^T(\alpha)B_0 + \Pi_2(\alpha,0)B_1\}R^{-1}\{B_0^T\Pi_1(\beta) + B_1^T\Pi_2^T(0,\beta)\} = 0 \end{cases}$$

with the boundary conditions (20). Then the risk-sensitive optimal controller *in the small noise limit is given by*

$$\left\{\begin{array}{rl} u^*(t) = & -R^{-1}[\{B_0^T\Pi_0 + B_1^T\Pi_1^T(0)\}x(t) \\ & +\int_{-h}^0 \{B_0^T\Pi_1(\beta) + B_1^T\Pi_2(0,\beta)\}B_1u(t+\beta)d\beta], \\ \dot{\hat{x}}(t) = & A\hat{x}(t) + B_0u(t) + B_1u(t-h) + PC^TV^{-1}(y(t) - C\hat{x}(t)) \\ & +\theta P(Q_0\hat{x}(t) + \int_{-h}^0 \bar{Q}_1(\beta)u(t+\beta)d\beta), \\ \hat{x}(0) = & \hat{x}_0, \\ u(\beta) = & \psi(\beta). \end{array}\right. \tag{21}$$

Remark 3.2 *The controller $u^*(t)$ in (21) is equivalent to an H_∞ controller for the time-delay system(See [13] for example.).*

3.2 Small Risk Limit

We next consider the small risk limit when θ goes to zero, and obtain the risk-sensitive optimal controllers in the small risk limit.

Consider the Riccati equations (18) and (19) again. Taking $\theta \to 0$, we have the following result from Theorem 2.9.

Corollary 3.3 *Suppose that the following conditions hold;*
(i) There exists a nonnegative definite solution P to the Riccati equation

$$AP + PA^T + \varepsilon DWD^T - PC^TV^{-1}CP = 0.$$

(ii) For the solution P in (i), there exists a nonnegative definite triplet $\Pi = (\Pi_0,\ \Pi_1(\beta),\ \Pi_2(\alpha,\beta))$ to the Riccati equations

$$\left\{\begin{array}{l} A^T\Pi_0 + \Pi_0A + Q_0 - \{\Pi_0B_0 + \Pi_1(0)B_1\}R^{-1}\{B_0^T\Pi_0 + B_1^T\Pi_1^T(0)\} = 0, \\ -\frac{\partial}{\partial\beta}\Pi_1(\beta) + A^T\Pi_1(\beta) + \bar{Q}_1(\beta) \\ \qquad -\{\Pi_0B_0 + \Pi_1(0)B_1\}R^{-1}\{B_0^T\Pi_1(\beta) + B_1^T\Pi_2^T(0,\beta)\} = 0, \\ -(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta})\Pi_2(\alpha,\beta) + \bar{Q}_2(\alpha,\beta) \\ \qquad -\{\Pi_1^T(\alpha)B_0 + \Pi_2(\alpha,0)B_1\}R^{-1}\{B_0^T\Pi_1(\beta) + B_1^T\Pi_2^T(0,\beta)\} = 0 \end{array}\right.$$

with the boundary conditions (20). Then the risk-sensitive optimal controller in the small risk limit is given by

$$\left\{\begin{array}{rcl} u^*(t) & = & -R^{-1}[\{B_0^T\Pi_0 + B_1^T\Pi_1^T(0)\}x(t) \\ & & +\int_{-h}^0 \{B_0^T\Pi_1(\beta) + B_1^T\Pi_2(0,\beta)\}B_1u(t+\beta)d\beta], \\ d\hat{x} & = & (A\hat{x}(t) + B_0u(t) + B_1u(t-h))dt + \frac{1}{\varepsilon}PC^TV^{-1}(d\tilde{y} - C\hat{x}(t)dt), \\ \hat{x}(0) & = & \hat{x}_0, \\ u(\beta) & = & \psi(\beta). \end{array}\right. \tag{22}$$

Remark 3.4 *The controller $u^*(t)$ in (22) is equivalent to a risk-neutral optimal controller for the time-delay system.*

4 Conclusion

We have considered the risk-sensitive control problem for a class of time-delay systems. We have employed the information state to analyze the problem, and have explicitly obtained the risk-sensitive optimal controllers for both the finite-time and infinite-time problems. We have also investigated asymptotic limits of the controller, which correspond to H_∞ controller and risk-neutral optimal controller.

References

[1] A. Bensoussan and J.H. van Shuppen, Optimal control of partially observable stochastic systems with an exponential-of-integral performance index, *SIAM Journal of Control and Optimization,* **23** (1985), 599-613.

[2] A. Bensoussan. and R.J. Elliot, A finite dimensional risk sensitive control problem, *SIAM Journal of Control and Optimization,* **33** (1995), 1834-1846.

[3] A. Bensoussan. and R.J. Elliot, General finite-dimensional risk-sensitive problems and small noise limits, *IEEE Transactions on Automatic Control,* **41** (1996), 210-215.

[4] C.D. Charalambous, Partially observable nonlinear risk-sensitive control problems: dynamic programming and verification theorems, *IEEE Transactions on Automatic Control,* **42** (1997), 1130-1138.

[5] R.J. Elliot, *Stochastic Calculus and Applications,* New York: Springer-Verlag, 1982.

[6] D.H. Jacobson, Optimal stochastic linear systems with exponential criteria and their relation to deterministic differential games, *IEEE Transaction on Automatic Control,* **18** (1973), 124-131.

[7] M.R. James, Asymptotic analysis of nonlinear stochastic risk-sensitive control and differential games, *Math. Contr. Sign. Syst.,* **5** (1991), 401-417.

[8] M.R. James, J.S. Baras and R.J. Elliot, Output feedback risk-sensitive control and differential games for continuous-time nonlinear risk-sensitive

control and differential games for continuous-time nonlinear systems, *Proc. of 32rd IEEE Conference on Decision and Control,* (1993), pp. 3357-3360.

[9] M.R. James, J.S. Baras and R.J. Elliot, Risk-sensitive control and dynamic games for partially observed discrete-time nonlinear systems, *IEEE Transaction on Automatic Control,* **39** (1994), 780-792.

[10] P.R. Kumar and P. Varaiya, *Stochastic Systems - Estimation, Identification, and Adaptive Control,* Prentice Hall, 1986.

[11] J.L. Speyer, An adaptive terminal guidance scheme based on an exponential cost criterion with applications to homing missile guidance, *IEEE Transaction on Automatic Control,* **21** (1976), 1021-1032.

[12] G. Tadmor, The standard H^∞ problem in systems with a single input delay, *IEEE Transaction on Automatic Control,* **45** (2000), 382-397.

[13] K. Uchida, K. Ikeda, T. Azuma and A. Kojima, Finite-dimensional characterizations of H^∞ control for linear systems with delays in input and output, *International Journal of Robust and Nonlinear Control,* **13** (2003), 833-843.

[14] P. Whittle, Risk-sensitive linear/quadratic/gaussian control, *Adv. Appli. Prob.,***13** (1981), 764-777.

[15] J. Yoneyama, Risk-sensitive optimal control for jump systems with applications to sampled-data systems, *International Journal of Systems Science,* **8** (2001), 1021-1040.

Received: October 5, 2007