

# Some Important Results on Triangular Sum Graphs

**S. K. Vaidya**

Department of Mathematics, Saurashtra University  
Rajkot - 360005, India  
samirkvaidya@yahoo.co.in

**U. M. Prajapati**

St. Xavier's College, Ahmedabad - 380009, India  
udayan64@yahoo.com

**P. L. Vihol**

V.V.P. Engineering College, Rajkot-360005, India  
viholprakash@yahoo.com

## **Abstract**

Let  $G = (V, E)$  be a graph with  $p$  vertices and  $q$  edges. A graph  $G$  is said to admit a *triangular sum labeling* if its vertices can be labeled by non-negative integers such that induced edge labels obtained by the sum of the labels of end vertices are the first  $q$  triangular numbers. A graph  $G$  which admits a triangular sum labeling is called a triangular sum graph. In the present work we investigate some classes of graphs which does not admit a triangular sum labeling. Also we show that some classes of graphs can be embedded as an induced subgraph of a triangular sum graph. This work is a nice composition of graph theory and combinatorial number theory.

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**Keywords:** Triangular number, Triangular sum labeling

## **1. Introduction and Definitions**

We begin with simple, finite, connected, undirected and non-trivial graph  $G = (V, E)$ , where  $V$  is called the set of vertices and  $E$  is called the set of edges. For various graph theoretic notations and terminology we follow Gross and Yellen [3] and for number theory we follow Burton [1]. We will give brief summary of definitions which are useful for the present investigations.

**Definition 1.1** If the vertices of the graph are assigned values, subject to certain conditions is known as graph labeling.

For detail survey on graph labeling one can refer Gallian [2]. Vast amount of literature is available on different types of graph labeling and more than 1000 research papers have been published so far in last four decades. Most interesting labeling problems have three important ingredients.

- a set of numbers from which vertex labels are chosen.
- a rule that assigns a value to each edge.
- a condition that these values must satisfy.

The present work is aimed to discuss one such labeling known as triangular sum labeling.

**Definition 1.2** A *triangular number* is a number obtained by adding all positive integers less than or equal to a given positive integer  $n$ . If  $n^{\text{th}}$  triangular number is denoted by  $T_n$  then  $T_n = \frac{1}{2}n(n+1)$ . It is easy to observe that there does not exist consecutive integers which are triangular numbers.

**Definition 1.3** A *triangular sum labeling* of a graph  $G$  is a one-to-one function  $f : V \rightarrow N$  ( where  $N$  is the set of all non-negative integers) that induces a bijection  $f^+ : E(G) \rightarrow \{T_1, T_2, \dots, T_n\}$  of the edges of  $G$  defined by  $f^+(uv) = f(u) + f(v), \forall e = uv \in E(G)$ . The graph which admits such labeling is called a triangular sum graph. This concept was introduced by Hegde and Shankaran [4]. In the same paper they obtained a necessary condition for an Eulerian graph to admit a triangular sum labeling. Moreover they investigated some classes of graphs which can be embedded as an induced subgraph of a triangular sum graph. In the present work we investigate some classes of graphs which does not admit a triangular sum labeling.

**Definition 1.4** The *helm* graph  $H_n$  is the graph obtained from a wheel  $W_n = C_n + K_1$  by attaching a pendant edge at each vertex of  $C_n$ .

**Definition 1.5** The graph  $G = \langle W_n : W_m \rangle$  is the graph obtained by joining apex vertices of wheels  $W_n$  and  $W_m$  to a new vertex  $x$ . ( A vertex corresponding to  $K_1$  in  $W_n = C_n + K_1$  is called an apex vertex.)

**Definition 1.6** A chord of a cycle  $C_n$  is an edge joining two non-adjacent vertices of cycle  $C_n$ .

**Definition 1.7** Two chords of a cycle are said to be twin chords if they

form a triangle with an edge of the cycle  $C_n$ .

## 2. Main Results

**Lemma 2.1** In every triangular sum graph  $G$  the vertices with label 0 and 1 are always adjacent.

**Proof:** The edge label  $T_1 = 1$  is possible only when the vertices with label 0 and 1 are adjacent.

**Lemma 2.2** In any triangular sum graph  $G$ , 0 and 1 cannot be the label of vertices of the same triangle contained in  $G$ .

**Proof:** Let  $a_0, a_1$ , and  $a_2$  be the vertices of a triangle. If  $a_0$  and  $a_1$  are labeled with 0 and 1 respectively and  $a_2$  is labeled with some  $x \in N$ , where  $x \neq 0, x \neq 1$ . Such vertex labeling will give rise to edge labels with 1,  $x$ , and  $x + 1$ . In order to admit a triangular sum labeling,  $x$  and  $x + 1$  must be triangular numbers. But it is not possible as we have mentioned in Definition 1.2

**Lemma 2.3** In any triangular sum graph  $G$ , 1 and 2 cannot be the labels of vertices of the same triangle contained in  $G$ .

**Proof:** Let  $a_0, a_1, a_2$  be the vertices of a triangle. Let  $a_0$  and  $a_1$  are labeled with 1 and 2 respectively and  $a_2$  is labeled with some  $x \in N$ , where  $x \neq 1, x \neq 2$ . Such vertex labeling will give rise to edge labels 3,  $x + 1$ , and  $x + 2$ . In order to admit a triangular sum labeling,  $x + 1$  and  $x + 2$  must be triangular numbers, which is not possible due to the fact mentioned in Definition 1.2.

**Theorem 2.4** The Helm graph  $H_n$  is not a triangular sum graph.

**Proof:** Let us denote the apex vertex as  $c_1$ , the consecutive vertices adjacent to  $c_1$  as  $v_1, v_2, \dots, v_n$ , and the pendant vertices adjacent to  $v_1, v_2, \dots, v_n$  as  $u_1, u_2, \dots, u_n$  respectively. If possible  $H_n$  admits a triangular sum labeling  $f : V \rightarrow N$ , then we consider following cases:

**Case 1:**  $f(c_1) = 0$ .

Then according to Lemma 2.1, we have to assign label 1 to exactly one of the vertices from  $v_1, v_2, \dots, v_n$ . Then there is a triangle having the vertices with labels 0 and 1 as adjacent vertices, which contradicts the Lemma 2.2.

**Case 2:** Any one of the vertices from  $v_1, v_2, \dots, v_n$  is labeled with 0. Without loss of generality let us assume that  $f(v_1) = 0$ . Then one of the vertices from  $c_1, v_2, v_n, u_1$  must be labeled with 1. Note that each of the vertices from  $c_1, v_2, v_n, u_1$  is adjacent to  $v_1$ .

**Subcase 1:** If one of the vertices from  $c_1, v_2, v_n$  is labeled with 1. In

each possibility there is a triangle having two of the vertices with labels 0 and 1, which contradicts the Lemma 2.2.

**Subcase 2:** If  $f(u_1) = 1$  then the edge label  $T_2 = 3$  can be obtained by vertex labels 0, 3 or 1, 2. The vertex with label 1 and the vertex with label 2 cannot be adjacent as  $u_1$  is a pendant vertex having label 1 and it is adjacent to the vertex with label 0. Therefore one of the vertices from  $v_2, v_n, c_1$  must receive the label 3. Thus there is a triangle whose two of the vertices are labeled with 0 and 3. Let the third vertex be labeled with  $x$ , with  $x \neq 0$  and  $x \neq 3$ . To admit a triangular sum labeling  $3, x, x + 3$  must be distinct triangular numbers. i.e.  $x$  and  $x + 3$  are two distinct triangular numbers other than 3 having difference 3, which is not possible.

**Case 3:** Any one of the vertices from  $u_1, u_2, \dots, u_n$  is labeled with 0. Without loss of generality we may assume that  $f(u_1) = 0$ . Then according to Lemma 2.1,  $f(v_1) = 1$ . The edge labels  $T_2 = 3$  can be obtained by vertex labels 0, 3 or 1, 2. The vertex with label 0 and the vertex with label 3 cannot be the adjacent vertices as  $u_1$  is a pendant vertex having label 0 and it is adjacent to the vertex with label 1. Therefore one of the vertices from  $v_2, v_n, c_1$  must be labeled with 2. Thus we have a triangle having vertices with labels 1 and 2 which contradicts the Lemma 2.3.

Thus in each of the possibilities discussed above,  $H_n$  does not admits a triangular sum labeling.

**Theorem 2.5** If every edge of a graph  $G$  is an edge of a triangle then  $G$  is not a triangular sum graph.

**Proof:** If  $G$  admits a triangular sum labeling then according to Lemma 2.1 there exists two adjacent vertices having labels 0 and 1 respectively. So there is a triangle having two of the vertices labeled with 0 and 1, which contradicts the Lemma 2.2. Thus  $G$  does not admit a triangular sum labeling.

Following are the immediate corollaries of the previous result.

**Corollary 2.6** The wheel graph  $W_n$  is not a triangular sum graph.

**Corollary 2.7** The fan graph  $f_n = P_{n-1} + K_1$  is not a triangular sum graph.

**Corollary 2.8** The friendship graph  $F_n = nK_3$  is not a triangular sum graph.

**Corollary 2.9** The graph  $g_n$  (the graph obtained by joining all the vertices of  $P_n$  to two additional vertices) is not a triangular sum graph.

**Corollary 2.10** The flower graph (the graph obtained by joining all the pendant vertices of helm graph  $H_n$  with the apex vertex) is not a triangular sum graph.

**Corollary 2.11** The graph obtained by joining apex vertices of two wheel graphs and two apex vertices with a new vertex is not a triangular sum graph.

**Theorem 2.12** The graph  $\langle W_n : W_m \rangle$  is not a triangular sum graph.

**Proof:** Let  $G = \langle W_n : W_m \rangle$ . Let us denote the apex vertex of  $W_n$  by  $u_0$  and the vertices adjacent to  $u_0$  of the wheel  $W_n$  by  $u_1, u_2, \dots, u_n$ . Similarly denote the apex vertex of other wheel  $W_m$  by  $v_0$  and the vertices adjacent to  $v_0$  of the wheel  $W_m$  by  $v_1, v_2, \dots, v_m$ . Let  $w$  be the new vertex adjacent to apex vertices of both the wheels. If possible let  $f : V \rightarrow N$  be one of the possible triangular sum labeling. According to the Lemma 2.1, 0 and 1 are the labels of any two adjacent vertices of the graph  $G$ , we have the following cases:

**Case 1:** If 0 and 1 be the labels of adjacent vertices in  $W_n$  or  $W_m$ , then there is a triangle having two of the vertices labeled with 0 and 1. Which contradicts the Lemma 2.2.

**Case 2:** If  $f(w) = 0$  then according to Lemma 2.1 one of the vertex from  $u_0$  and  $v_0$  is labeled with 1. Without loss of generality we may assume that  $f(u_0) = 1$ . To have an edge label  $T_2 = 3$  we have the following possibilities:

**Subcase 2.1:** One of the vertices from  $u_1, u_2, \dots, u_n$  is labeled with 2. Without loss of generality assume that  $f(u_i) = 2$ , for some  $i \in \{1, 2, 3, \dots, n\}$ . In this situation we will get a triangle having two of its vertices are labeled with 1 and 2, which contradicts the Lemma 2.3.

**Subcase 2.2:** Assume that  $f(v_0) = 3$ . Now to get the edge label  $T_3 = 6$  we have the following subcases:

**Subcase 2.2.1:** Assume that  $f(u_i) = 5$ , for some  $i \in \{1, 2, 3, \dots, n\}$ . In this situation we will get a triangle with distinct vertex labels 1, 5 and  $x$ . Then  $x + 5$  and  $x + 1$  will be the edge labels of two edges with difference 4. It is possible only if  $x = 5$ , but  $x \neq 5$  as we have  $f(u_i) = 5$ .

**Subcase 2.2.2:** Assume that 2 and 4 are the labels of two adjacent vertices from one of the two wheels. So there exists a triangle whose vertex labels are either 1, 2, and 4 or 3, 2, and 4. In either of the situation will give rise to an edge label 5 which is not a triangular number.

**Case 3:** If  $f(w) = 1$  then one of the vertex from  $u_0$  and  $v_0$  is labeled with 0. Without loss of generality assume that  $f(u_0) = 0$ . To have an edge label 3 we have the following possibilities:

**Subcase 3.1:** If  $f(u_i) = 3$  for some  $i \in \{1, 2, 3, \dots, n\}$ . Then there is a triangle having vertex labels as 0, 3,  $x$ , with  $x \neq 3$ . Thus we

have two edge labels  $x + 3$  and  $x$  which are two distinct triangular numbers having difference 3. So  $x = 3$ , which is not possible as  $x \neq 3$ .

**Subcase 3.2:** Assume that  $f(v_0) = 2$ . Now to obtain the edge label  $T_3 = 6$  we have to consider the following possibilities:

- (i)  $6=6+0$ ;                      (ii)  $6=5+1$ ;                      (iii)  $6=4+2$ .
- (i) If  $6 = 6 + 0$  then one of the vertices from  $u_1, u_2, \dots, u_n$  must be labeled with 6. Without loss of generality we may assume that  $f(u_i) = 6$  for some  $i \in \{1, 2, 3, \dots, n\}$ . In this situation there are two distinct triangles having vertex labels  $0, 6, x$  and  $0, 6, y$ , for two distinct triangular numbers  $x$  and  $y$  each of which are different from 0 and 6. Then  $x + 6$  and  $x$  are two distinct triangular numbers having difference 6. This is possible only for  $x = 15$ . On the other hand  $y + 6$  and  $y$  are two distinct triangular numbers having difference 6. Then  $y = 15$ . (The  $x = y = 15$  which is not possible as  $f$  is one-one)
- (ii) If  $6 = 5 + 1$  and  $f(w) = 1$ , then in this situation label of one of the vertex adjacent to  $w$  must be 5. This is not possible as  $w$  is adjacent to the vertices whose labels are 0 and 2.
- (iii) If  $6 = 2 + 4$ . In this case one of the vertices from  $v_1, v_2, \dots, v_m$  is labeled with 4. Assume that  $f(v_i) = 4$ , for some  $i \in \{1, 2, 3, \dots, m\}$ . In this situation there is a triangle having vertex labels 2, 4 and  $x$  (where  $x$  is a positive integer with  $x \neq 2, x \neq 4$ ). Then  $4+x$  and  $2+x$  will be the edge labels of two edges i.e.  $4+x$  and  $2+x$  are two distinct triangular numbers with difference 2 which is not possible.

Thus we conclude that in each of the possibilities discussed above the graph  $G$  under consideration does not admit a triangular sum labeling.

#### 4. Embedding of some Triangular sum graphs

**Theorem 4.1** Every cycle can be embedded as an induced subgraph of a triangular sum graph.

**Proof:** Let  $G = C_n$  be a cycle with  $n$  vertices. We define labeling  $f : V(G) \rightarrow N$  as follows such that the induced function  $f^+ : E(G) \rightarrow \{T_1, T_2, \dots, T_q\}$  is bijective.

$$f(v_1) = 0$$

$$f(v_2) = 6$$

$$f(v_i) = T_{i+2} - f(v_{i-1}); 3 \leq i \leq n - 1$$

$$f(v_n) = T_{f(v_{n-1})-1}$$

Now let  $A = \{T_1, T_2 \dots T_r\}$  be the set of missing edge labels. i.e. Elements of set A are the missing triangular numbers between 1 and  $T_{f(v_{n-1})-1}$ . Now add  $r$  pendent vertices which are adjacent to the vertex with label 0 and label these new vertices with labels  $T_1, T_2 \dots T_r$ . This construction will give rise to edges with labels  $T_1, T_2, \dots T_r$  such that the resultant supergraph  $H$  admits triangular sum labeling. Thus we proved that every cycle can be embedded as an induced subgraph of a triangular sum graph.

**Example 4.2** In the following *Figure 4.1* embedding of  $C_5$  as an induced subgraph of a triangular sum graph is shown.

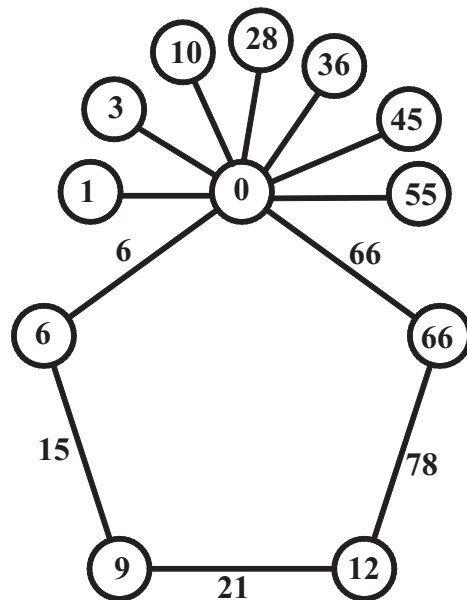


Figure 4.1

**Theorem 4.3** Every cycle with one chord can be embedded as an induced subgraph of a triangular sum graph.

**Proof:** Let  $G = C_n$  be the cycle with one chord. Let  $e = v_1v_k$  be the chord of cycle  $C_n$ . We define labeling as

We define labeling  $f : V(G) \rightarrow N$  as follows such that the induced function  $f^+ : E(G) \rightarrow \{T_1, T_2, \dots T_q\}$  is bijective.

$$f(v_1) = 0$$

$$f(v_2) = 6$$

$$f(v_i) = T_{i+2} - f(v_{i-1}); 3 \leq i \leq k - 1$$

$$f(v_k) = T_{f(v_{k-1})-1}$$

$$f(v_{k+i-1}) = T_{f(v_{k-1})-1+i} - f(v_{k+i-2}); 2 \leq i \leq n - k$$

$$f(v_n) = T_{f(v_{n-1})-1}$$

Now follow the procedure described in Theorem 4.1 and the resultant supergraph H admits triangular sum labeling. Thus we proved that every cycle with one chord can be embedded as an induced subgraph of a triangular sum graph.

**Example 4.4** In the following *Figure 4.2* embedding of  $C_4$  with one chord as an induced subgraph of a triangular sum graph is shown.

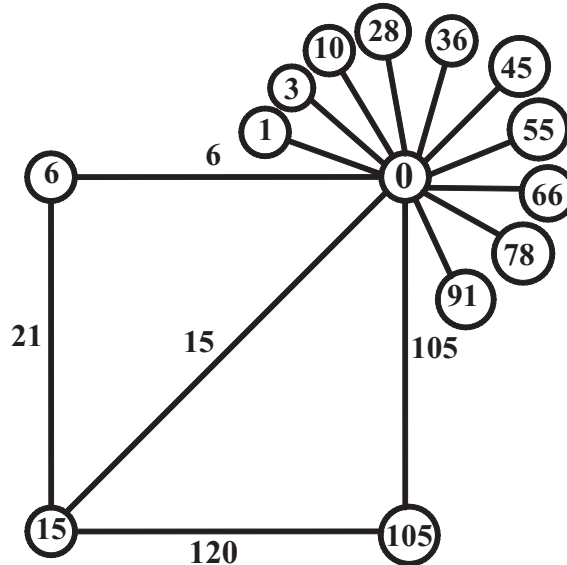


Figure 4.2

**Theorem 4.5** Every cycle with twin chords can be embedded as an induced subgraph of a triangular sum graph.

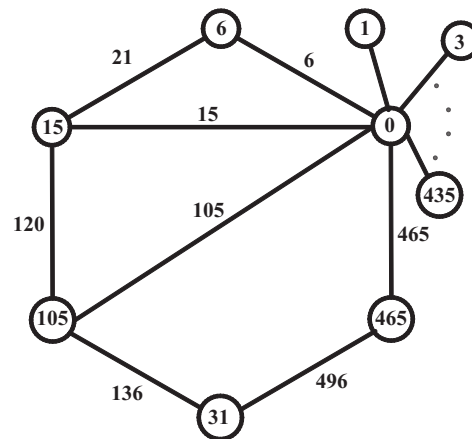
**Proof:** Let  $G = C_n$  be the cycle with twin chords. Let  $e_1 = v_1v_k$  and  $e_2 = v_1v_{k+1}$  be two chords of cycle  $C_n$ . We define labeling  $f : V(G) \rightarrow N$  as follows such that the induced function  $f^+ : E(G) \rightarrow \{T_1, T_2, \dots, T_q\}$  is bijective.

$$\begin{aligned}
 f(v_1) &= 0 \\
 f(v_2) &= 6 \\
 f(v_i) &= T_{i+2} - f(v_{i-1}); 3 \leq i \leq k - 1 \\
 f(v_k) &= T_{f(v_{k-1})-1} \\
 f(v_{k+1}) &= T_{f(v_k)-1} \\
 f(v_{k+i}) &= T_{f(v_k)-1+i} - f(v_{k+i-1}); 2 \leq i \leq n - k - 1 \\
 f(v_n) &= T_{f(v_{n-1})-1}
 \end{aligned}$$

Now following the procedure adapted in Theorem 4.1 the resulting supergraph H admits triangular sum labeling. i.e. every cycle with twin chords can be embedded as an induced sub graph of a triangular sum graph.



**Example 4.6** In the following *Figure 4.3* embedding of  $C_6$  with twin chord as an induced subgraph of a triangular sum graph is shown.



*Figure 4.3*

## 5. Concluding Remarks

As every graph does not admit a triangular sum labeling, it is very interesting to investigate classes of graphs which are not triangular sum graphs and to embed classes of graphs as an induced subgraph of a triangular sum graph. We investigate several classes of graphs which does not admit triangular sum labeling. Moreover we show that cycle, cycle with one chord and cycle with twin chords can be embedded as an induced subgraph of a triangular sum graph. This work contribute several new result to the theory of graph labeling.

## References

- [1] D M Burton, *Elementary Number Theory*, Brown Publishers, Second Edition(1990).
- [2] J A Gallian, *A Dynamic Survey of Graph Labeling The Electronic Journal of Combinatorics*, 16(2009), #DS6.
- [3] J. Gross and J Yellen, *Handbook of Graph Theory*, CRC Press.
- [4] S M Hegde and P Shankaran, *On Triangular Sum Labeling of Graphs* in: B D Acharya, S Arumugam, A Rosa Ed., *Labeling of Discrete Structures and Applications*, Narosa Publishing House, New Delhi(2008) 109-115.

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