Applied Mathematical Sciences, Vol. 3, 2009, no. 36, 1787 - 1797

Application of Hölder Inequality in Generalised Convolutions for Functions with Respect to k-Symmetric Points

K. Al-Shaqsi and *M. Darus

School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600 Selangor D. Ehsan, Malaysia
ommath@hotmail.com, *maslina@ukm.my

Abstract

Two classes of univalent functions with respect to k-symmetric points define on the unit disk satisfying the conditions:

$$\sum_{n=1}^{\infty}(nk+1-\alpha)|a_{nk+1}|+\sum_{n=2;n\neq lk+1}^{\infty}n|a_n|\leq 1-\alpha,$$

and

$$\sum_{n=1}^{\infty}(nk+1)(nk+1-\alpha)|a_{nk+1}|+\sum_{n=2;n\neq lk+1}^{\infty}n^2|a_n|\leq 1-\alpha$$

are given. The two inequalities of the functions belonging to these two classes are the starlike and convex functions with respect to k-symmetric points, respectively. Some interesting properties of generalisations of Hadamard product in these classes are given.

Mathematics Subject Classification: 30C45, 30C50, 30C55

Keywords: Analytic functions, k-symmetric points, Hadamard product

*-corresponding author

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all functions which are univalent in $\mathbb{U}$. Also let $\mathcal{T}$ denote the subclasses of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geq 0).$$

We denote by $S^*(\alpha)$ and $C(\alpha)$ for $0 \leq \alpha < 1$ the familiar subclasses of $\mathcal{A}$ consisting of functions which are, respectively, starlike and convex functions of order α. Thus by definition, we have

$$S^*(\alpha) = \left\{ f : f \in \mathcal{A} \text{ and } \operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad (0 \leq \alpha < 1; z \in \mathbb{U}) \right\},$$

and

$$C(\alpha) = \left\{ f : f \in \mathcal{A} \text{ and } \operatorname{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \quad (0 \leq \alpha < 1; z \in \mathbb{U}) \right\}.$$

Also, denote by $\mathcal{T}S^*(\alpha)$ and $\mathcal{T}C(\alpha)$ the subclasses of $\mathcal{T}$ where

$$\mathcal{T}S^*(\alpha) = S^*(\alpha) \cap \mathcal{T} \quad \text{and} \quad \mathcal{T}C(\alpha) = C(\alpha) \cap \mathcal{T}.$$

Let $f_j(z) \in \mathcal{A}$, $(j = 1, 2, \cdots, m)$ be given by

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n.$$

Then the Hadamard product (or convolution) is defined by:

$$f_1(z) * f_2(z) * \cdots * f_m(z) = (f_1 * f_2 * \cdots * f_m)(z) = z + \sum_{n=2}^{\infty} \left(\prod_{j=1}^{m} a_{n,j} \right) z^n.$$

Also the generalised Hadamard Product is defined here by

$$(f_1 \diamond f_2 \diamond \cdots \diamond f_m)(z) = z + \sum_{n=2}^{\infty} \left(\prod_{j=1}^{m} (a_{n,j})^{\frac{1}{p_j}} \right) z^n.$$

where $\sum_{j=1}^{m} \frac{1}{p_j} = 1$, $p_j > 1$ and $j = 1, 2, \cdots m$.

Let $F_j(z) \in \mathcal{T}(j = 1, 2, \cdots, m)$ be given by

$$F_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \quad (a_{n,j} \geq 0).$$

Then the modified Hadamard product is defined by

$$(F_1 * F_2 * \cdots * F_m)(z) = z - \sum_{n=2}^{\infty} \left(\prod_{j=1}^{m} a_{n,j} \right) z^n \quad (a_{n,j} \geq 0).$$

Also the generalised modified Hadamard Product is defined here by

$$(F_1 \diamond F_2 \diamond \cdots \diamond F_m)(z) = z - \sum_{n=2}^{\infty} \left(\prod_{j=1}^{m} (a_{n,j})^{\frac{1}{p_j}} \right) z^n \quad (a_{n,j} \geq 0).$$

where $\sum_{j=1}^{m} \frac{1}{p_j} = 1$, $p_j > 1$ and $j = 1, 2, \cdots m$.

Sakaguchi [1] once introduced a class S_S^* of functions starlike with respect to symmetric points, which consists of functions $f \in \mathcal{S}$ satisfying the inequality

$$\mathrm{Re}\left\{ \frac{zf'(z)}{f(z) - f(-z)} \right\} > 0, \qquad z \in \mathbb{U}.$$

Many different authors have studied the work of Sakaguchi [1] and have discussed extensively about this class and its subclasses (see[2-9]). In 1979 Chand and Singh [5] introduced the classes $S_S^{(k)}(\alpha)$ of functions starlike with respect to k-symmetric points of order α, and $C_S^{(k)}(\alpha)$ of functions convex with respect to k-symmetric points of order α which are the special classes corresponding to the ones defined in [9], which satisfy the following:

$$S_S^{(k)}(\alpha) = \left\{ f : f \in \mathcal{S} \text{ and } \mathrm{Re}\left\{ \frac{zf'(z)}{f_k(z)} \right\} > \alpha \quad (0 \leq \alpha < 1; z \in \mathbb{U}) \right\},$$

and

$$C_S^{(k)}(\alpha) = \left\{ f : f \in \mathcal{S} \text{ and } \mathrm{Re}\left\{ \frac{(zf'(z))'}{f_k'(z)} \right\} > \alpha \quad (0 \leq \alpha < 1; z \in \mathbb{U}) \right\},$$

where $k \geq 1$ is positive integer and $f_k(z)$ is defined by the following equality

$$f_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^{\nu} z), \qquad (\varepsilon = \exp(2\pi i/k); z \in \mathbb{U}).$$

Note that the function $f(z) \in \mathcal{A}$ is in the class $C_S^{(k)}(\alpha)$ if and only if $zf'(z) \in S_S^{(k)}(\alpha)$.

Finally, denote by $\mathcal{T}S_S^{(k)}(\alpha)$ and $\mathcal{T}C_S^{(k)}(\alpha)$ the subclasses of $\mathcal{T}$ where

$$\mathcal{T}S_S^{(k)}(\alpha) = S_S^{(k)}(\alpha) \cap \mathcal{T} \quad \text{and} \quad \mathcal{T}C_S^{(k)}(\alpha) = C_S^{(k)}(\alpha) \cap \mathcal{T}.$$

Now we state the results due to [9] as a special case when $\lambda = 0$, which we will use throughout this paper.

Theorem 1.1 *Let* $0 \leq \alpha < 1$, $k \geq 1$. *If*

$$\sum_{n=1}^{\infty} (nk+1-\alpha)|a_{nk+1}| + \sum_{\substack{n=2 \\ n \neq lk+1}}^{\infty} n|a_n| \leq 1-\alpha, \tag{1.1}$$

then $f(z) \in S_S^{(k)}(\alpha)$. *Condition (1.1) is also necessary if* $f(z) \in \mathcal{T}S_S^{(k)}(\alpha)$.

Theorem 1.2 *Let* $0 \leq \alpha < 1$, $k \geq 1$. *If*

$$\sum_{n=1}^{\infty} (nk+1)(nk+1-\alpha)|a_{nk+1}| + \sum_{\substack{n=2 \\ n \neq lk+1}}^{\infty} n^2|a_n| \leq 1-\alpha, \tag{1.2}$$

then $f(z) \in C_S^{(k)}(\alpha)$. *Condition (1.2) is also necessary if* $f(z) \in \mathcal{T}C_S^{(k)}(\alpha)$.

In the present paper, we shall make use of the generalised Hadamard product with a view of Theorems 1.1 and 1.2 to prove interesting characterisation theorems involving the classes $S_S^{(k)}(\alpha)$, $C_S^{(k)}(\alpha)$, $\mathcal{T}S_S^{(k)}(\alpha)$ and $\mathcal{T}C_S^{(k)}(\alpha)$.

2 Generalised convolution properties of functions in the classes $S_S^{(k)}(\alpha)$, $\mathcal{T}S_S^{(k)}(\alpha)$

We state our first theorem as follows:

Theorem 2.1 *If* $f_j \in S_S^{(k)}(\alpha_j)$, $(j = 1, 2, \cdots m)$, *then*

$$(f_1 \diamond f_2 \diamond \cdots \diamond f_m)(z) \in S_S^{(k)}(\beta, \mu),$$

where

$$\beta \le \min_{n\ge 2}\left\{1-\frac{nk}{\prod_{j=1}^{m}\left(\frac{nk+1-\alpha_j}{1-\alpha_j}\right)^{\frac{1}{p_j}}-1}\right\},$$

and

$$\mu \le \min_{\substack{n\ge 2\\ n\ne lk+1}}\left\{1-\frac{n}{\prod_{j=1}^{m}\left(\frac{n}{1-\alpha_j}\right)^{\frac{1}{p_j}}}\right\}.$$

for $\sum_{j=1}^{m}\frac{1}{p_j}=1$, $p_j>1$.

Proof. Let $f_j(z)\in S_S^{(k)}(\alpha_j)$, by using Theorem 1.1 we have:

$$\sum_{n=1}^{\infty}\frac{nk+1-\alpha_j}{1-\alpha_j}|a_{nk+1,j}|+\sum_{\substack{n=2\\ n\ne lk+1}}^{\infty}\frac{n}{1-\alpha_j}|a_{n,j}|\le 1,\qquad (j=1,2,\cdots m).$$

Moreover,

$$\prod_{j=1}^{m}\left(\sum_{n=1}^{\infty}\left\{\left(\frac{nk+1-\alpha_j}{1-\alpha_j}\right)^{\frac{1}{p_j}}|a_{nk+1,j}|^{\frac{1}{p_j}}\right\}^{p_j}\right)^{\frac{1}{p_j}} +\prod_{j=1}^{m}\left(\sum_{\substack{n=2\\ n\ne lk+1}}^{\infty}\left\{\left(\frac{n}{1-\alpha_j}\right)^{\frac{1}{p_j}}|a_{n,j}|^{\frac{1}{p_j}}\right\}^{p_j}\right)^{\frac{1}{p_j}}\le 1.$$

By using the Hölder inequality, we have

$$\sum_{n=1}^{\infty}\left\{\prod_{j=1}^{m}\left(\frac{nk+1-\alpha_j}{1-\alpha_j}\right)^{\frac{1}{p_j}}|a_{nk+1,j}|^{\frac{1}{p_j}}\right\} \le\prod_{j=1}^{m}\left(\sum_{n=1}^{\infty}\left\{\left(\frac{nk+1-\alpha_j}{1-\alpha_j}\right)^{\frac{1}{p_j}}|a_{nk+1,j}|^{\frac{1}{p_j}}\right\}^{p_j}\right)^{\frac{1}{p_j}},$$

and

$$\sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}\left\{\prod_{j=1}^{m}\left(\frac{n}{1-\alpha_j}\right)^{\frac{1}{p_j}}|a_{n,j}|^{\frac{1}{p_j}}\right\}$$

$$\leq \prod_{j=1}^{m}\left(\sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}\left\{\left(\frac{n}{1-\alpha_j}\right)^{\frac{1}{p_j}}|a_{n,j}|^{\frac{1}{p_j}}\right\}^{p_j}\right)^{\frac{1}{p_j}}.$$

Then, we have

$$\sum_{n=1}^{\infty}\left\{\prod_{j=1}^{m}\left(\frac{nk+1-\alpha_j}{1-\alpha_j}\right)^{\frac{1}{p_j}}|a_{nk+1,j}|^{\frac{1}{p_j}}\right\}$$

$$+\sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}\left\{\prod_{j=1}^{m}\left(\frac{n}{1-\alpha_j}\right)^{\frac{1}{p_j}}|a_{n,j}|^{\frac{1}{p_j}}\right\}\leq 1.$$

Here, we see that

$$\sum_{n=1}^{\infty}\left\{\left(\frac{nk+1-\beta}{1-\beta}\right)\prod_{j=1}^{m}|a_{nk+1,j}|^{\frac{1}{p_j}}\right\}$$

$$+\sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}\left\{\left(\frac{n}{1-\mu}\right)\prod_{j=1}^{m}|a_{n,j}|^{\frac{1}{p_j}}\right\}\leq 1$$

with

$$\beta\leq\min_{n\geq 2}\left\{1-\frac{nk}{\prod_{j=1}^{m}\left(\frac{nk+1-\alpha_j}{1-\alpha_j}\right)^{\frac{1}{p_j}}-1}\right\},$$

and

$$\mu\leq\min_{\substack{n\geq 2\\ n\neq lk+1}}\left\{1-\frac{n}{\prod_{j=1}^{m}\left(\frac{n}{1-\alpha_j}\right)^{\frac{1}{p_j}}}\right\}.$$

Thus, by Theorem 1.1, the proof of Theorem 2.1 is complete.

Next, we obtain our first corollary.

Corollary 2.2 *If $f_j(z) \in S_S^{(k)}(\alpha)$, $(j = 1, \cdots, m)$, then*

$$(f_1 \diamond f_2 \diamond \cdots \diamond f_m)(z) \in S_S^{(k)}(\alpha),$$

Proof. In view of Theorem 1.1, Corollary 2.2 follows readily from Theorem 2.1 for the special case when $\alpha_j = \alpha$.

Further, we obtain the following results:

Theorem 2.3 *If $F_j(z) \in \mathcal{T}S_S^{(k)}(\alpha_j)$, $(j = 1, \cdots, m)$, then*

$$(F_1 \diamond F_2 \diamond \cdots \diamond F_m)(z) \in \mathcal{T}S_S^{(k)}(\beta, \mu),$$

where β and μ given by conditions in Theorem 2.1 and for $\sum_{j=1}^{m} \frac{1}{p_j} = 1$, $p_j > 1$.

Proof. By using the same technique as in the proof of Theorem 2.1, the required result is obtained.

Theorem 2.4 *Let the function $f_j(z) \in S_S^{(k)}(\alpha_j)$, $(j = 1, \cdots, m)$, and let $t_m(z)$ be defined by*

$$t_m(z) = z + \sum_{n=1}^{\infty} \left(\sum_{j=1}^{m} (a_{nk+1,j})^p \right) z^n + \sum_{\substack{n=2 \\ n \neq lk+1}}^{\infty} \left(\sum_{j=1}^{m} (a_{n,j})^p \right) z^n. \quad (2.1)$$

Then

$$t_m(z) \in S_S^{(k)}(\delta, \gamma),$$

where

$$\delta = 1 - \frac{nk}{\frac{1}{m}\left(\frac{nk+1-\alpha}{1-\alpha}\right)^p - 1}, \qquad \gamma = 1 - \frac{n}{\frac{1}{m}\left(\frac{n}{1-\alpha}\right)^p}$$

and

$$\left(\frac{nk+1-\alpha}{1-\alpha}\right)^p; \left(\frac{n}{1-\alpha}\right)^p \geq mn, \; \alpha = \min_{1 \leq j \leq m} \alpha_j.$$

Proof. Since $f_j \in S_S^{(k)}(\alpha_j)$, using Theorem 1.1, we observe that

$$\sum_{n=1}^{\infty} \left(\frac{nk+1-\alpha_j}{1-\alpha_j}\right)^p |a_{nk+1,j}|^p + \sum_{\substack{n=2 \\ n \neq lk+1}}^{\infty} \left(\frac{n}{1-\alpha_j}\right)^p |a_{n,j}|^p$$

$$\leq \left(\sum_{n=1}^{\infty} \frac{nk+1-\alpha_j}{1-\alpha_j} |a_{nk+1,j}| \right)^p + \left(\sum_{\substack{n=2 \\ n \neq lk+1}}^{\infty} \frac{n}{1-\alpha_j} |a_{n,j}| \right)^p \leq 1. \quad (2.2)$$

It follows from (2.2) that

$$\sum_{n=1}^{\infty}\left\{\frac{1}{m}\sum_{j=1}^{m}\left(\frac{nk+1-\alpha_j}{1-\alpha_j}\right)^p|a_{nk+1,j}|^p\right\} + \sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}\left\{\frac{1}{m}\sum_{j=1}^{m}\left(\frac{n}{1-\alpha_j}\right)^p|a_{n,j}|^p\right\}\leq 1.$$

Putting $\alpha=\min\limits_{1\leq j\leq m}\alpha_j$, and by virtue of Theorem 1.1, we find that

$$\sum_{n=1}^{\infty}\frac{nk+1-\delta}{1-\delta}\sum_{j=1}^{m}|a_{nk+1,j}|^p+\sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}\frac{n}{1-\gamma}\sum_{j=1}^{m}|a_{n,j}|^p$$
$$\leq \sum_{n=1}^{\infty}\frac{1}{m}\left(\frac{nk+1-\alpha}{1-\alpha}\right)^p\sum_{j=1}^{m}|a_{nk+1,j}|^p+\sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}\frac{1}{m}\left(\frac{n}{1-\alpha}\right)^p\sum_{j=1}^{m}|a_{n,j}|^p$$
$$\leq \sum_{n=1}^{\infty}\left\{\frac{1}{m}\sum_{j=1}^{m}\left(\frac{nk+1-\alpha_j}{1-\alpha_j}\right)^p|a_{nk+1,j}|^p\right\} + \sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}\left\{\frac{1}{m}\sum_{j=1}^{m}\left(\frac{n}{1-\alpha_j}\right)^p|a_{n,j}|^p\right\}\leq 1,$$

if,

$$\delta=1-\frac{nk}{\frac{1}{m}\left(\frac{nk+1-\alpha}{1-\alpha}\right)^p-1},\qquad \gamma=1-\frac{n}{\frac{1}{m}\left(\frac{n}{1-\alpha}\right)^p}.$$

Now let

$$u(n)=1-\frac{nk}{\frac{1}{m}\left(\frac{nk+1-\alpha}{1-\alpha}\right)^p-1},\qquad v(n)=1-\frac{n}{\frac{1}{m}\left(\frac{n}{1-\alpha}\right)^p}.$$

Then $u'(n)$, $v'(n)\geq 0$ if $p\geq 2$. Hence

$$\delta\leq 1-\frac{nk}{\frac{1}{m}\left(\frac{nk+1-\alpha}{1-\alpha}\right)^p-1},\qquad \gamma\leq 1-\frac{n}{\frac{1}{m}\left(\frac{n}{1-\alpha}\right)^p}.$$

By $\left(\frac{nk+1-\alpha}{1-\alpha}\right)^p$; $\left(\frac{n}{1-\alpha}\right)^p\geq mn$, we see that $0\leq\delta<1$ and $0\leq\gamma<1$. Thus the proof of Theorem 2.4 is complete.

3 Generalised convolution properties of functions in the classes $C_S^{(k)}(\alpha)$, $\mathcal{T}C_S^{(k)}(\alpha)$

In this section, we give another set of results regarding the classes $C_S^{(k)}(\alpha)$ and $\mathcal{T}C_S^{(k)}(\alpha)$.

Theorem 3.1 *If the functions* $f_j \in C_S^{(k)}(\alpha_j)$, $(j = 1, \cdots, m)$, *then*

$$(f_1 \diamond f_2 \diamond \cdots \diamond f_m)(z) \in C_S^{(k)}(\beta, \mu),$$

where β *and* μ *given by conditions in Theorem 2.1 and for* $\sum_{j=1}^{m} \frac{1}{p_j} = 1$, $p_j > 1$.

Proof. Let $f_j \in C_S^{(k)}(\alpha_j)(j = 1, \cdots, m)$, by using Theorem 1.2, we have

$$\sum_{n=1}^{\infty} \frac{(nk+1)(nk+1-\alpha_j)}{1-\alpha_j} |a_{nk+1,j}| + \sum_{\substack{n=2 \\ n \neq lk+1}}^{\infty} \frac{n^2}{1-\alpha_j} |a_n, j| \leq 1.$$

Thus the proof of Theorem 3.1 is much akin to that of Theorem 2.1 already detailed, instead of Theorem 1.1, it uses Theorem 1.2.

Corollary 3.2 *If* $f_j(z) \in C_S^{(k)}(\alpha)$, $(j = 1, \cdots, m)$, *then*

$$(f_1 \diamond f_2 \diamond \cdots \diamond f_m)(z) \in C_S^{(k)}(\alpha).$$

Proof. In view of Theorem 1.2, Corollary 3.2 follows readily from Theorem 3.1 for special case when $\alpha_j = \alpha$.

Theorem 3.3 *If* $F_j(z) \in \mathcal{T}C_S^{(k)}(\alpha_j)$, $(j = 1, \cdots, m)$, *then*

$$(F_1 \diamond F_2 \diamond \cdots \diamond F_m)(z) \in \mathcal{T}C_S^{(k)}(\beta, \mu),$$

where β *and* μ *given by conditions in Theorem 2.1 and for* $\sum_{j=1}^{m} \frac{1}{p_j} = 1$, $p_j > 1$.

Proof. By using the same technique as in the proof of Theorem 2.1, the required result is obtained.

Theorem 3.4 *Let the function* $f_j(z) \in C_S^{(k)}(\alpha_j)$, $(j = 1, \cdots, m)$, *and let* $t_m(z)$ *be given by (2.1). Then*

$$t_m(z) \in C_S^{(k)}(\delta, \gamma),$$

where

$$\delta = 1 - \frac{nk}{\frac{1}{m}(nk+1)^{p-1}\Big(\frac{nk+1-\alpha}{1-\alpha}\Big)^p - 1}, \qquad \gamma = 1 - \frac{n}{\frac{1}{m}n^{p-1}\Big(\frac{n}{1-\alpha}\Big)^p}$$

and

$$(nk+1)^{p-2}\Big(\frac{nk+1-\alpha}{1-\alpha}\Big)^p; \; n^{p-2}\Big(\frac{n}{1-\alpha}\Big)^p \geq m, \; \alpha = \min_{1\leq j\leq m} \alpha_j.$$

Proof. Since $f_j(z) \in C_S^{(k)}(\alpha_j)$, by using Theorem 1.2, we observe that

$$\sum_{n=1}^{\infty} \frac{(nk+1)(nk+1-\alpha_j)}{1-\alpha_j}|a_{nk+1,j}| + \sum_{\substack{n=2 \\ n\neq lk+1}}^{\infty} \frac{n^2}{1-\alpha_j}|a_n, j| \leq 1, \qquad (j = 1, \cdots, m).$$

Thus the proof of Theorem 3.4 using Theorem 1.2 is precisely in the same manner as the above proof of Theorem 2.4 using Theorem 1.1.

Acknowledgement The work here is fully supported by UKM-GUP-TMK-07-02-107: University Research Grant, UKM, Malaysia.

References

[1] K. Sakaguchi, On certain univalent mapping, *PJ. Math. Soc. Japan.* **11**, (1959),72-75.

[2] J. Sokół, A. Szpila and M. Szpila, On some subclass of starlike functions with respect to symmetric points, *Zeszyty Nauk. Politech. Rzeszowskiej Mat. Fiz.* **12**, (1991), 65-73.

[3] J. Stankiewicz, Some remarks on functions starlike with respect to symmetric points, *Ann. Univ. Mariae Curie- Skodowska Sect.* A**19**, (1970), 5359.

[4] J. Thangamani, On starlike functions with respect to symmetric points, *Indian J. Pure Appl. Math.* **11**, (1980), 392-405.

[5] R. Chand, P. Singh, On certain schlicht mappings, *Indian J. Pure Appl. Math.* **10**, (1979), 1167-1174.

[6] R. Parvatham, S. Radha, On α-starlike and α-close-to-convex functions with respect to n-symmetric points, *Indian J. Pure Appl. Math.* **17**, (1986), 1114-1122.

[7] T. V. Sudharsan, P. Balasubrahmanyam and K. G. Subramanian , On functions starlike with respect to symmetric and conjugate points, *Taiwanese J. Math.* **2**, (1998), 57-68.

[8] V. Ravichandran, Starlike and convex functions with respect to conjugate points, *Acta Math. Acad. Paedagog. N. (N.S.)* **20**, (2004), 31-37.

[9] Z. Wang, C. Gao , Some subclasses of close-to-convex and quasi-convex functions with respect to k-symmetric points, *Gen. Math.* **15**(4), (2007), 107–119.

Received: January, 2009