Applied Mathematical Sciences, Vol. 3, 2009, no. 36, 1799 - 1810

On Certain Subclasses of Analytic Functions Defined by a Multiplier Transformation with Two Parameters

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Abstract

Let $\mathcal{A}$ denote the class of analytic functions with the normalization $f(0) = f'(0) - 1 = 0$ in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$, set

$$f_{b,\lambda}^n(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+b}{1+b}\right)^n \frac{(k+\lambda-1)!}{\lambda!(k-1)!} z^k$$
$$(n \in \mathbb{C},\, b \in \mathbb{C} \setminus \mathbb{Z}^-,\, \lambda > -1; z \in \mathbb{U}).$$

and define $(f_{b,\lambda}^n)^{(-1)}$ in terms of the Hadamard product

$$f_{b,\lambda}^n(z) * (f_{b,\lambda}^n)^{(-1)}(z) = \frac{z}{(1-z)^\mu} \qquad (\mu > 0; z \in \mathbb{U}).$$

In this paper, the authors introduce several new subclasses of analytic functions defined by means of the operator $\mathcal{I}_{b,\lambda,\mu}^n : \mathcal{A} \to \mathcal{A}$, given by

$$I_{b,\lambda,\mu}^n f(z) = (f_{b,\lambda}^n)^{(-1)} * f(z) \quad (f \in \mathcal{A}; n \in \mathbb{C},\, b \in \mathbb{C} \setminus \mathbb{Z}^-,\, \lambda > -1; \mu > 0).$$

Inclusion properties of these classes and the classes involving the generalized Libera integral operator are also considered

Mathematics Subject Classification: 30C45

Keywords: Analytic functions, Multiplier transformation, Subordination, Integral operator

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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the unit disk $\mathbb{U} = \{z : |z| < 1\}$. If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists Schwarz function w in $\mathbb{U}$ such that $f(z) = g(w(z))$. We denote by $S^*(\gamma)$, $C(\gamma)$ and $K(\gamma, \delta)$ the subclasses of $\mathcal{A}$ consisting of all analytic functions which are, respectively, starlike of order $\gamma (0 \le \gamma < 1)$, convex of order $\gamma (0 \le \gamma < 1)$ and close-to-convex of order δ type γ in $\mathbb{U}$.(see, e.g., Srivastava and Owa [1]).

For $n \in \mathbb{C}$, $b \in \mathbb{C} \setminus \mathbb{Z}^-$ and $\lambda > -1$, the authors [4] introduced the Multiplier transformation $\mathcal{D}^n_{b,\lambda}$ of functions $f \in \mathcal{A}$ by

$$\mathcal{D}^n_{b,\lambda} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+b}{1+b}\right)^n \frac{(k+\lambda-1)!}{\lambda!(k-1)!} a_k z^k.$$

Let $\mathcal{P}$ be the class of all functions ϕ which are analytic and univalent in $\mathbb{U}$ and for which $\phi(\mathbb{U})$ is convex with $\phi(0) = 1$ and $\mathrm{Re}\{\phi(z)\} > 0$ for $z \in \mathbb{U}$.

Making use of subordination principle between two analytic functions, we introduce the subclasses $\mathcal{S}^*(\gamma; \phi)$, $\mathcal{C}(\gamma; \phi)$ and $\mathcal{K}(\gamma, \delta; \phi; \psi)$ of the class $\mathcal{A}$ for $0 \le \gamma$, $\delta < 1$ and $\phi, \psi \in \mathcal{P}$ (cf., [2]), which are defined by

$$\mathcal{S}^*(\gamma; \phi) = \left\{ f : f \in \mathcal{A} \text{ and } \frac{1}{1-\gamma}\left(\frac{zf'(z)}{f(z)} - \gamma\right) \prec \phi(z) \text{ in } \mathbb{U} \right\},$$

$$\mathcal{C}(\gamma; \phi) = \left\{ f : f \in \mathcal{A} \text{ and } \frac{1}{1-\gamma}\left(1 + \frac{zf''(z)}{f'(z)} - \gamma\right) \prec \phi(z) \text{ in } \mathbb{U} \right\},$$

and

$$\mathcal{K}(\gamma, \delta; \phi, \psi) = \left\{ f : f \in \mathcal{A} \text{ and } \exists\ g \in \mathcal{S}^*(\gamma; \phi) s.t. \frac{1}{1-\delta}\left(\frac{zf'(z)}{g(z)} - \delta\right) \prec \psi(z) \text{ in } \mathbb{U} \right\}.$$

In particular, when $\gamma = \delta = 0$ we have the classes $\mathcal{S}^*(\phi)$, $\mathcal{C}(\phi)$, and $\mathcal{K}(\phi,\psi)$ investigated by Ma and Minda [12] and Kim et al. [13]. For suitable choices of ϕ and ψ we can easily gather the various subclasses of $\mathcal{A}$. For example :

$$\mathcal{S}^*(\gamma;\frac{1+z}{1-z}) = S^*(\gamma), \qquad \mathcal{C}(\gamma;\frac{1+z}{1-z}) = C(\gamma),$$
$$\mathcal{K}(\gamma,\delta;\frac{1+z}{1-z},\frac{1+z}{1-z}) = K(\gamma,\delta).$$

Setting

$$f^n_{b,\lambda}(z) = z + \sum_{k=2}^{\infty}\Big(\frac{k+b}{1+b}\Big)^n \frac{(k+\lambda-1)!}{\lambda!(k-1)!}z^k$$
$$(n \in \mathbb{C},\ b \in \mathbb{C}\setminus\mathbb{Z}^-,\ \lambda > -1; z \in \mathbb{U}),$$

we define a new function $(f^n_{b,\lambda})^{(-1)}$ in terms of the Hadamard product (or convolution)

$$f^n_{b,\lambda}(z) * (f^n_{b,\lambda})^{(-1)}(z) = \frac{z}{(1-z)^{\mu}} \qquad (\mu > 0; z \in \mathbb{U}).$$

Then, motivated essentially by the Choi-Sagio-Srivastava operator [2] (see also [3] and [5]), we now introduce the operator $\mathcal{I}^n_{b,\lambda,\mu} : \mathcal{A} \to \mathcal{A}$, which are defined here by

$$\mathcal{I}^n_{b,\lambda,\mu} f(z) = (f^n_{b,\lambda})^{(-1)}(z) * f(z) \tag{1.2}$$
$$(f \in \mathcal{A}; n \in \mathbb{C},\ b \in \mathbb{C}\setminus\mathbb{Z}^-,\ \lambda > -1,\ \mu > 0).$$

Note that $\mathcal{I}^0_{0,\lambda,\mu}$ the Choi-Sagio-Srivastava operator was introduced and studied by Choi et al. [2], and $\mathcal{I}^0_{0,\lambda,2}$ $(\lambda \in \mathbb{N}_0)$ the Noor integral operator was introduced by Noor [5]. (see also [3]). Also the operator $\mathcal{I}^n_{b,0,\mu}$ $(n \in \mathbb{R}; b > -1)$ was studied by Cho and Kim [6]. In particular, we note that $\mathcal{I}^0_{0,0,2} = zf'(z)$ and $\mathcal{I}^1_{0,0,2} = \mathcal{I}^0_{0,1,2} = f(z)$. In view of (1.1) and (1.2), for $n \in \mathbb{C}$, $b \in \mathbb{C}\setminus\mathbb{Z}^-$, $\lambda > -1$, $\mu > 0$, we obtain the following relations:

$$z(\mathcal{I}^{n+1}_{b,\lambda,\mu} f(z))' = (b+1)\mathcal{I}^n_{b,\lambda,\mu} f(z) - b\mathcal{I}^{n+1}_{b,\lambda,\mu} f(z), \tag{1.3}$$

$$z(\mathcal{I}^n_{b,\lambda+1,\mu} f(z))' = (\lambda+1)\mathcal{I}^n_{b,\lambda,\mu} f(z) - \lambda\mathcal{I}^n_{b,\lambda+1,\mu} f(z), \tag{1.4}$$

and

$$z(\mathcal{I}^n_{b,\lambda,\mu} f(z))' = \mu\mathcal{I}^n_{b,\lambda,\mu+1} f(z) - (\mu-1)\mathcal{I}^n_{b,\lambda,\mu} f(z), \tag{1.5}$$

Next, by using the operator $\mathcal{I}^n_{b,\lambda,\mu}$, we introduce the following classes of analytic functions for $\phi,\ \psi \in \mathcal{P}, n \in \mathbb{C},\ b \in \mathbb{C} \setminus \mathbb{Z}^-,\ \lambda > -1,\ \mu > 0$ and $0 \le \gamma,\ \delta < 1$:

$$\mathcal{S}^n_{b,\lambda,\mu}(\gamma;\phi) = \Big\{ f : f \in \mathcal{A} \quad \text{and} \quad \mathcal{I}^n_{b,\lambda,\mu} f(z) \in \mathcal{S}^*(\gamma;\phi) \Big\},$$
$$\mathcal{C}^n_{b,\lambda,\mu}(\gamma;\phi) = \Big\{ f : f \in \mathcal{A} \quad \text{and} \quad \mathcal{I}^n_{b,\lambda,\mu} f(z) \in \mathcal{C}(\gamma;\phi) \Big\},$$

and

$$\mathcal{K}^n_{b,\lambda,\mu}(\gamma,\delta;\phi,\psi) = \Big\{ f : f \in \mathcal{A} \quad \text{and} \quad \mathcal{I}^n_{b,\lambda,\mu} f(z) \in \mathcal{K}(\gamma,\delta;\phi,\psi) \Big\}.$$

We note that

$$f(z) \in \mathcal{C}^n_{b,\lambda,\mu}(\gamma;\phi) \Leftrightarrow zf'(z) \in \mathcal{S}^n_{b,\lambda,\mu}(\gamma;\phi). \tag{1.6}$$

In particular, we set

$$\mathcal{S}^n_{b,\lambda,\mu}\left(\gamma; \Big(\frac{1+Az}{1+Bz}\Big)^{\alpha}\right) = \mathcal{S}^n_{b,\lambda,\mu}(\gamma;A,B;\alpha), \qquad (0 < \alpha \le 1;\ -1 \le B < A \le 1),$$

and

$$\mathcal{C}^n_{b,\lambda,\mu}\left(\gamma; \Big(\frac{1+Az}{1+Bz}\Big)^{\alpha}\right) = \mathcal{C}^n_{b,\lambda,\mu}(\gamma;A,B;\alpha), \qquad (0 < \alpha \le 1;\ -1 \le B < A \le 1).$$

In this paper, we investigate several inclusion properties for the classes $\mathcal{S}^n_{b,\lambda,\mu}(\gamma;\phi)$, $\mathcal{C}^n_{b,\lambda,\mu}(\gamma;\phi)$ and $\mathcal{K}^n_{b,\lambda,\mu}(\gamma,\delta;\phi,\psi)$ associated with the operator $\mathcal{I}^n_{b,\lambda,\mu}$. Some applications involving these and other families of operator are also obtained.

2 Inclusion properties involving $\mathcal{I}^n_{b,\lambda,\mu}$

To derive our results we need the following lemmas:

Lemma 2.1 *[7]. Let $\beta,\ \nu$ be complex numbers. Let $\phi \in \mathcal{P}$ be convex univalent in $\mathbb{U}$ with $\phi(0) = 1$ and $\Re\ [\beta\phi(z) + \nu] > 0,\ z \in \mathbb{U}$. If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is analytic in $\mathbb{U}$ with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \nu} \prec \phi(z) \Rightarrow p(z) \prec \phi(z), \qquad (z \in \mathbb{U}).$$

Lemma 2.2 *[11]. Let $\phi \in \mathcal{P}$ be convex univalent in $\mathbb{U}$ and w be analytic in $\mathbb{U}$ with $\Re\ w(z) \ge 0,\ z \in \mathbb{U}$ If p is analytic in $\mathbb{U}$ with $p(0) = \phi(0)$, then*

$$p(z) + w(z)zp'(z) \prec \phi(z) \Rightarrow p(z) \prec \phi(z), \qquad (z \in \mathbb{U}).$$

At first, with the help of Lemma 2.1, we obtain the following:

Theorem 2.3

$$\mathcal{S}^{n}_{b,\lambda,\mu+1}(\gamma;\phi) \subset \mathcal{S}^{n}_{b,\lambda,\mu}(\gamma;\phi) \subset \mathcal{S}^{n+1}_{b,\lambda,\mu}(\gamma;\phi),$$

for $n \in \mathbb{C}$, $b \in \mathbb{C} \setminus \mathbb{Z}^{-}$, $\lambda \geq 0$, $\mu \geq 1$, $0 \leq \gamma < 1$ *and* $\phi \in \mathcal{P}$.

Proof. First, we will show that

$$\mathcal{S}^{n}_{b,\lambda,\mu+1}(\gamma;\phi) \subset \mathcal{S}^{n}_{b,\lambda,\mu}(\gamma;\phi).$$

Let $f \in \mathcal{S}^{n}_{b,\lambda,\mu+1}(\gamma;\phi)$ and set

$$p(z) = \frac{1}{1-\gamma}\left(\frac{z(\mathcal{I}^{n}_{b,\lambda,\mu}f(z))'}{\mathcal{I}^{n}_{b,\lambda,\mu}f(z)} - \gamma\right), \tag{2.1}$$

where p analytic in $\mathbb{U}$ with $p(0) = 1$. Applying (1.5) and (2.1)

$$\mu\frac{\mathcal{I}^{n}_{b,\lambda,\mu+1}f(z)}{\mathcal{I}^{n}_{b,\lambda,\mu}f(z)} = (1-\gamma)p(z) + \gamma + \mu - 1. \tag{2.2}$$

Taking the logarithmic differentiation on both sides of (2.2) and multiplying by z, we have

$$\frac{1}{1-\gamma}\left(\frac{z(\mathcal{I}^{n}_{b,\lambda,\mu+1}f(z))'}{\mathcal{I}^{n}_{b,\lambda,\mu+1}f(z)} - \gamma\right) = p(z) + \frac{zp'(z)}{(1-\gamma)p(z) + \gamma + \mu - 1} \; (z \in \mathbb{U}). \tag{2.3}$$

Applying Lemma 2.1 to (2.3), it follows that $p \prec \phi$, that is $f \in \mathcal{S}^{n}_{b,\lambda,\mu}(\gamma,\phi)$. To prove the second part, let $f \in \mathcal{S}^{n}_{b,\lambda,\mu}(\gamma,\phi)$ and put

$$h(z) = \frac{1}{1-\gamma}\left(\frac{z(\mathcal{I}^{n+1}_{b,\lambda,\mu}f(z))'}{\mathcal{I}^{n+1}_{b,\lambda,\mu}f(z)} - \gamma\right),$$

where h is analytic in $\mathbb{U}$ with $h(0) = 1$. Then, by using the arguments similar to those detailed above with (1.3), it follows that $h \prec \phi$ in $\mathbb{U}$, which implies that $f \in \mathcal{S}^{n+1}_{b,\lambda,\mu}(\gamma;\phi)$. Therefore, we complete the proof of Theorem 2.1.

Theorem 2.4

$$\mathcal{S}^{n}_{b,\lambda,\mu+1}(\gamma;\phi) \subset \mathcal{S}^{n}_{b,\lambda,\mu}(\gamma;\phi) \subset \mathcal{S}^{n}_{b,\lambda+1,\mu}(\gamma;\phi),$$

for $n \in \mathbb{C}$, $b \in \mathbb{C} \setminus \mathbb{Z}^{-}$, $\lambda \geq 0$, $\mu \geq 1$, $0 \leq \gamma < 1$ *and* $\phi \in \mathcal{P}$.

Proof. We proved the first part in Theorem 2.3. To prove the second part let $f \in \mathcal{S}^n_{b,\lambda,\mu}(\gamma, \phi)$ and put

$$p(z) = \frac{1}{1-\gamma}\left(\frac{z(\mathcal{I}^n_{b,\lambda+1,\mu}f(z))'}{\mathcal{I}^n_{b,\lambda+1,\mu}f(z)} - \gamma\right),$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$. Then, by using the arguments similar to those detailed in Theorem 2.3 with (1.4), it follows that $p \prec \phi$ in $\mathbb{U}$, which implies that $f \in \mathcal{S}^n_{b,\lambda+1,\mu}(\gamma; \phi)$.

Theorem 2.5 *Let* $n \in \mathbb{C}$, $b \in \mathbb{C} \setminus \mathbb{Z}^-$, $\lambda \geq 0$ *and* $\mu \geq 1$. *Then*

$$\mathcal{C}^n_{b,\lambda,\mu+1}(\gamma; \phi) \subset \mathcal{C}^n_{b,\lambda,\mu}(\gamma; \phi) \subset \mathcal{C}^{n+1}_{b,\lambda,\mu}(\gamma; \phi) \qquad (0 \leq \gamma < 1; \phi \in \mathcal{P}).$$

Proof. Applying (1.6) and Theorem 2.3, we observe that

$$\begin{aligned}
f(z) \in \mathcal{C}^n_{b,\lambda,\mu+1}(\gamma;\phi) &\Leftrightarrow \mathcal{I}^n_{b,\lambda,\mu+1}f(z) \in \mathcal{C}(\gamma;\phi)\\
&\Leftrightarrow z(\mathcal{I}^n_{b,\lambda,\mu+1}f(z))' \in \mathcal{S}^*(\gamma;\phi)\\
&\Leftrightarrow \mathcal{I}^n_{b,\lambda,\mu+1}(zf'(z)) \in \mathcal{S}^*(\gamma;\phi)\\
&\Leftrightarrow zf'(z) \in \mathcal{S}^n_{b,\lambda,\mu+1}(\gamma;\phi)\\
&\Rightarrow zf'(z) \in \mathcal{S}^n_{b,\lambda,\mu}(\gamma;\phi)\\
&\Leftrightarrow \mathcal{I}^n_{b,\lambda,\mu}(zf'(z)) \in \mathcal{S}^*(\gamma;\phi)\\
&\Leftrightarrow z(\mathcal{I}^n_{b,\lambda,\mu}f(z))' \in \mathcal{S}^*(\gamma;\phi)\\
&\Leftrightarrow \mathcal{I}^n_{b,\lambda,\mu}f(z) \in \mathcal{C}(\gamma;\phi)\\
&\Leftrightarrow f(z) \in \mathcal{C}^n_{b,\lambda,\mu}(\gamma;\phi),
\end{aligned}$$

and

$$\begin{aligned}
f(z) \in C^n_{b,\lambda,\mu}(\gamma,\phi) &\Leftrightarrow zf'(z) \in \mathcal{S}^n_{b,\lambda,\mu}(\gamma;\phi)\\
&\Rightarrow zf'(z) \in \mathcal{S}^{n+1}_{b,\lambda,\mu}(\gamma;\phi)\\
&\Leftrightarrow z(\mathcal{I}^{n+1}_{b,\lambda,\mu}f(z))' \in \mathcal{S}^*(\gamma;\phi)\\
&\Leftrightarrow \mathcal{I}^{n+1}_{b,\lambda,\mu}f(z) \in \mathcal{C}(\gamma;\phi)\\
&\Leftrightarrow f(z) \in \mathcal{C}^{n+1}_{b,\lambda,\mu}(\gamma;\phi),
\end{aligned}$$

which evidently proves Theorem 2.5.

By applying (1.6) and Theorem 2.4 and using the same methods to prove Theorem 2.5 we can prove the follow:

Theorem 2.6 *Let* $n \in \mathbb{C}$, $b \in \mathbb{C} \setminus \mathbb{Z}^-$, $\lambda \geq 0$ *and* $\mu \geq 1$. *Then*

$$\mathcal{C}^n_{b,\lambda,\mu+1}(\gamma; \phi) \subset \mathcal{C}^n_{b,\lambda,\mu}(\gamma; \phi) \subset \mathcal{C}^n_{b,\lambda+1,\mu}(\gamma; \phi) \qquad (0 \leq \gamma < 1; \phi \in \mathcal{P}).$$

Taking

$$\phi(z) = \left(\frac{1+Az}{1+Bz}\right)^{\alpha} \qquad (-1 \leq B < A \leq 1; 0 < \alpha \leq 1; z \in \mathbb{U}) \tag{2.4}$$

in Theorem 2.3 and Theorem 2.5, we have the following

Corollary 2.7 *Let $n \in \mathbb{C}$, $b \in \mathbb{C} \setminus \mathbb{Z}^-$, $\lambda \geq 0$ and $\mu \geq 1$. Then*

$$\mathcal{S}^n_{b,\lambda,\mu+1}(\gamma; A, B; \alpha) \subset \mathcal{S}^n_{b,\lambda,\mu}(\gamma; A, B; \alpha) \subset \mathcal{S}^{n+1}_{b,\lambda,\mu}(\gamma; A, B; \alpha)$$
$$(0 \leq \gamma < 1; \ -1 \leq B < A \leq 1; 0 < \alpha \leq 1),$$

and

$$\mathcal{C}^n_{b,\lambda,\mu+1}(\gamma; A, B; \alpha) \subset \mathcal{C}^n_{b,\lambda,\mu}(\gamma; A, B; \alpha) \subset \mathcal{C}^{n+1}_{b,\lambda,\mu}(\gamma; A, B; \alpha)$$
$$(0 \leq \gamma < 1; \ -1 \leq B < A \leq 1; 0 < \alpha \leq 1).$$

Also, by taking (2.4) in Theorem 2.4 and Theorem 2.6, we have the following

Corollary 2.8 *Let $n \in \mathbb{C}$, $b \in \mathbb{C} \setminus \mathbb{Z}^-$, $\lambda \geq 0$ and $\mu \geq 1$. Then*

$$\mathcal{S}^n_{b,\lambda,\mu+1}(\gamma; A, B; \alpha) \subset \mathcal{S}^n_{b,\lambda,\mu}(\gamma; A, B; \alpha) \subset \mathcal{S}^{n}_{b,\lambda+1,\mu}(\gamma; A, B; \alpha)$$
$$(0 \leq \gamma < 1; \ -1 \leq B < A \leq 1; 0 < \alpha \leq 1),$$

and

$$\mathcal{C}^n_{b,\lambda,\mu+1}(\gamma; A, B; \alpha) \subset \mathcal{C}^n_{b,\lambda,\mu}(\gamma; A, B; \alpha) \subset \mathcal{C}^{n}_{b,\lambda+1,\mu}(\gamma; A, B; \alpha)$$
$$(0 \leq \gamma < 1; \ -1 \leq B < A \leq 1; 0 < \alpha \leq 1).$$

Next, by using Lemma 2.2, we obtain the following inclusion relation for the class $\mathcal{K}^n_{b,\lambda,\mu}(\gamma, \delta; \phi, \psi)$.

Theorem 2.9 *Let $n \in \mathbb{C}$, $b \in \mathbb{C} \setminus \mathbb{Z}^-$, $\lambda > -1$ and $\mu \geq 1$. Then*

$$\mathcal{K}^n_{b,\lambda,\mu+1}(\gamma, \delta; \phi, \psi) \subset \mathcal{K}^n_{b,\lambda,\mu}(\gamma, \delta; \phi, \psi) \subset \mathcal{K}^{n+1}_{b,\lambda,\mu}(\gamma, \delta; \phi, \psi)$$
$$(0 \leq \gamma, \ \delta < 1; \phi, \psi \in \mathcal{P}).$$

Proof. We begin by proving that

$$\mathcal{K}^n_{b,\lambda,\mu+1}(\gamma, \delta; \phi, \psi) \subset \mathcal{K}^n_{b,\lambda,\mu}(\gamma, \delta; \phi, \psi).$$

Let $f \in \mathcal{K}^n_{b,\lambda,\mu+1}(\gamma, \delta; \phi, \psi)$. Then, in view of the definition of the class $\mathcal{K}^n_{b,\lambda,\mu+1}(\gamma, \delta; \phi, \psi)$, there exists a function $g \in \mathcal{S}^n_{b,\lambda,\mu+1}(\gamma; \phi)$ such that

$$\frac{1}{1-\delta}\left(\frac{z(\mathcal{I}^n_{b,\lambda,\mu+1}f(z))'}{\mathcal{I}^n_{b,\lambda,\mu+1}g(z)} - \delta\right) \prec \psi(z) \qquad (z \in \mathbb{U}).$$

Now let

$$p(z) = \frac{1}{1-\delta}\left(\frac{z(\mathcal{I}^n_{b,\lambda,\mu}f(z))'}{\mathcal{I}^n_{b,\lambda,\mu}g(z)} - \delta\right)$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$. Using (1.5), we obtain

$$[(1-\delta)p(z)+\delta]\mathcal{I}^n_{b,\lambda,\mu}g(z) + (\mu-1)\mathcal{I}^n_{b,\lambda,\mu}f(z) = \mu\mathcal{I}^n_{b,\lambda,\mu+1}f(z). \tag{2.5}$$

Differentiating (2.5) and multiplying by z, we have

$$\begin{aligned}(1-\delta)zp'(z)\mathcal{I}^n_{b,\lambda,\mu}g(z) + [(1-\delta)p(z)+\delta]z(\mathcal{I}^n_{b,\lambda,\mu}g(z))' \\ = \mu z(\mathcal{I}^n_{b,\lambda,\mu+1}f(z))' - (\mu-1)z(\mathcal{I}^n_{b,\lambda,\mu}f(z))'.\end{aligned} \tag{2.6}$$

Since $g \in \mathcal{S}^n_{b,\lambda,\mu+1}(\gamma;\phi)$, by Theorem 2.3, we know that $g \in \mathcal{S}^n_{b,\lambda,\mu}(\gamma;\phi)$. Let

$$q(z) = \frac{1}{1-\gamma}\left(\frac{z(\mathcal{I}^n_{b,\lambda,\mu}g(z))'}{\mathcal{I}^n_{b,\lambda,\mu}g(z)} - \gamma\right).$$

Then, using (1.5) once again, we have

$$\mu\frac{\mathcal{I}^n_{b,\lambda,\mu+1}g(z)}{\mathcal{I}^n_{b,\lambda,\mu}g(z)} = (1-\gamma)q(z) + \mu + \gamma - 1. \tag{2.7}$$

¿From (2.6) and (2.7), we obtain

$$\frac{1}{1-\delta}\left(\frac{z(\mathcal{I}^n_{b,\lambda,\mu+1}f(z))'}{\mathcal{I}^n_{b,\lambda,\mu+1}g(z)} - \delta\right) = p(z) + \frac{zp'(z)}{(1-\gamma)q(z)+\mu+\gamma-1}.$$

Since $\mu \geq 1$ and $q \prec \phi$ in $\mathbb{U}$,

$$\operatorname{Re}\{(1-\gamma)q(z)+\mu+\gamma-1\} > 0 \qquad (z \in \mathbb{U}).$$

Hence applying Lemma 2.2, we can show that $p \prec \phi$, so that $f \in \mathcal{K}^n_{b,\lambda,\mu}(\gamma,\delta;\phi,\psi)$. For the second part, by using the arguments similar to those detailed above with (1.3), we obtain

$$\mathcal{K}^n_{b,\lambda,\mu}(\gamma,\delta;\phi,\psi) \subset \mathcal{K}^{n+1}_{b,\lambda,\mu}(\gamma,\delta;\phi,\psi).$$

Therefore, we complete the proof of Theorem 2.9.

Theorem 2.10 *Let $n \in \mathbb{C}$, $b \in \mathbb{C}\setminus\mathbb{Z}^-$, $\lambda \geq 0$ and $\mu \geq 1$. Then*

$$\mathcal{K}^n_{b,\lambda,\mu+1}(\gamma,\delta;\phi,\psi) \subset \mathcal{K}^n_{b,\lambda,\mu}(\gamma,\delta;\phi,\psi) \subset \mathcal{K}^n_{b,\lambda,\mu+1}(\gamma,\delta;\phi,\psi)$$
$$(0 \leq \gamma,\ \delta < 1; \phi,\psi \in \mathcal{P}).$$

Proof. We proved the first part in Theorem 2.9. For the second part, by using the arguments similar to those detailed in Theorem 2.9 with (1.4).

3 Inclusion properties involving the integral operator F_c

In this section, we consider the generalized Libera integral operator F_c [8] (see also [9] and [10]) defined by

$$F_c(z) = F_c(f)(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt \qquad (f \in \mathcal{A}; c > -1). \tag{3.1}$$

¿From the definition of F_c defined by (3.1), we observe that

$$z(\mathcal{I}^n_{b,\lambda,\mu}F_c(z))' = (c+1)\mathcal{I}^n_{b,\lambda,\mu}f(z) - c\mathcal{I}^n_{b,\lambda,\mu}F_c(z). \tag{3.2}$$

Theorem 3.1 *Let* $c \geq 0$, $n \in \mathbb{C}$, $b \in \mathbb{C} \setminus \mathbb{Z}^-$, $\lambda > -1$ *and* $\mu > 0$. *If* $f \in \mathcal{S}^n_{b,\lambda,\mu}(\gamma;\phi)$, $(0 \leq \gamma < 1; \phi \in \mathcal{P})$, *then* $F_c(z) \in \mathcal{S}^n_{b,\lambda,\mu}(\gamma;\phi)$, $(0 \leq \gamma < 1; \phi \in \mathcal{P})$.

Proof. Let $f \in \mathcal{S}^n_{b,\lambda,\mu}(\gamma;\phi)$ and set

$$p(z) = \frac{1}{1-\gamma}\left(\frac{z(\mathcal{I}^n_{b,\lambda,\mu}F_c(z))'}{\mathcal{I}^n_{b,\lambda,\mu}F_c(z)} - \gamma\right), \tag{3.3}$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$.
By using (3.2)and (3.3), we obtain

$$(c+1)\frac{\mathcal{I}^n_{b,\lambda,\mu}f(z)}{\mathcal{I}^n_{b,\lambda,\mu}F_c(z)} = (1-\gamma)p(z) + c + \gamma \tag{3.4}$$

Taking the logarithmic differentiation on both sides of (3.4) and multiplying by z, we have

$$\frac{1}{1-\gamma}\left(\frac{z(\mathcal{I}^n_{b,\lambda,\mu}f(z))'}{\mathcal{I}^n_{b,\lambda,\mu}f(z)} - \gamma\right) = p(z) + \frac{zp'(z)}{(1-\gamma)p(z) + c + \gamma} \qquad (z \in \mathbb{U}).$$

Hence, by virtue of Lemma 2.1, we conclude that $p \prec \phi$ in $\mathbb{U}$, which implies that $F_c(z) \in \mathcal{S}^n_{b,\lambda,\mu}(\gamma;\phi)$.

Next, we derive an inclusion property involving F_c, which is given by the following.

Theorem 3.2 *Let* $c \geq 0$, $n \in \mathbb{C}$, $b \in \mathbb{C} \setminus \mathbb{Z}^-$, $\lambda > -1$ *and* $\mu > 0$. *If* $f \in \mathcal{C}^n_{b,\lambda,\mu}(\gamma;\phi)$, $(0 \leq \gamma < 1; \phi \in \mathcal{P})$, *then* $F_c(z) \in \mathcal{C}^n_{b,\lambda,\mu}(\gamma;\phi)$, $(0 \leq \gamma < 1; \phi \in \mathcal{P})$.

Proof. By applying Theorem 3.1, it follows that

$$\begin{aligned} f(z) \in \mathcal{C}^n_{b,\lambda,\mu}(\gamma;\phi) &\Leftrightarrow zf'(z) \in \mathcal{S}^n_{b,\lambda,\mu}(\gamma;\phi) \\ &\Rightarrow F_c(zf'(z)) \in \mathcal{S}^n_{b,\lambda,\mu}(\gamma;\phi) \\ &\Leftrightarrow z(F_c(z))' \in \mathcal{S}^n_{b,\lambda,\mu}(\gamma;\phi) \\ &\Leftrightarrow F_c(z) \in \mathcal{C}^n_{b,\lambda,\mu}(\gamma;\phi), \end{aligned}$$

which proves Theorem 3.2.
¿From Theorem 3.1 and Theorem 3.2, we have the following.

Corollary 3.3 *Let* $c \geq 0$, $n \in \mathbb{C}$, $b \in \mathbb{C} \setminus \mathbb{Z}^-$, $\lambda > -1$ *and* $\mu > 0$. *If* f *belongs to the class* $\mathcal{S}^n_{b,\lambda,\mu}(\gamma; A, B; \alpha)$ $(or \mathcal{C}^n_{b,\lambda,\mu}(\gamma; A, B; \alpha))(0 \leq \gamma < 1;\ -1 \leq B < A \leq 1;\ 0 < \alpha \leq 1)$, *the* $F_c(z)$ *belongs to the class* $\mathcal{S}^n_{b,\lambda,\mu}(\gamma; A, B; \alpha)$ $(or \mathcal{C}^n_{b,\lambda,\mu}(\gamma; A, B; \alpha))(0 \leq \gamma < 1;\ -1 \leq B < A \leq 1;\ 0 < \alpha \leq 1)$.

Finally, we prove the following.

Theorem 3.4 $c \geq 0, n \in \mathbb{C}, b \in \mathbb{C} \setminus \mathbb{Z}^-, \lambda > -1$ *and* $\mu > 0$. *If* $f \in \mathcal{K}^n_{b,\lambda,\mu}(\gamma, \delta; \phi, \psi)$, $(0 \leq \gamma < 1; \phi \in \mathcal{P})$, *then* $F_c(z) \in \mathcal{K}^n_{b,\lambda,\mu}(\gamma, \delta; \phi, \psi)$, $(0 \leq \gamma < 1; \phi \in \mathcal{P})$.

Proof. Let $f \in \mathcal{K}^n_{b,\lambda,\mu}(\gamma, \delta; \phi, \psi)$. Then, in view of the definition of the class $\mathcal{K}^n_{b,\lambda,\mu}(\gamma, \delta; \phi, \psi)$, there exists a function $g \in \mathcal{S}^n_{b,\lambda,\mu}(\gamma; \phi)$ such that

$$\frac{1}{1-\delta}\left(\frac{z(\mathcal{I}^n_{b,\lambda,\mu}f(z))'}{\mathcal{I}^n_{b,\lambda,\mu}g(z)} - \delta\right) \prec \psi(z) \qquad (z \in \mathbb{U}).$$

Thus, we set

$$p(z) = \frac{1}{1-\delta}\left(\frac{z(\mathcal{I}^n_{b,\lambda,\mu}F_c(z))'}{\mathcal{I}^n_{b,\lambda,\mu}F_c(g)(z)} - \delta\right).$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$. Since $g \in \mathcal{S}^n_{b,\lambda,\mu}(\gamma; \phi)$, we see from Theorem 3.1 that $F_c(g) \in \mathcal{S}^n_{b,\lambda,\mu}(\gamma; \phi)$. Using (3.2), we have

$$(c+1)\frac{z(\mathcal{I}^n_{b,\lambda,\mu}f(z))'}{\mathcal{I}^n_{b,\lambda,\mu}F_c(z)} = [(1-\delta)p(z) + \delta][(1-\gamma)q(z) + c + \gamma] + (1-\delta)zp'(z),$$

where

$$q(z) = \frac{1}{1-\gamma}\left(\frac{z(\mathcal{I}^n_{b,\lambda,\mu}F_c(g)(z))'}{\mathcal{I}^n_{b,\lambda,\mu}F_c(g)(z)} - \gamma\right).$$

Hence, we have

$$\frac{1}{1-\delta}\left(\frac{z(\mathcal{I}^n_{b,\lambda,\mu}f(z))'}{\mathcal{I}^n_{b,\lambda,\mu}g(z)}-\delta\right)=p(z)+\frac{zp'(z)}{(1-\gamma)q(z)+c+\gamma}.$$

The remaining part of the proof in Theorem 3.4 is similar to that of Theorem 2.9 and so we omit it.

Acknowledgement The work presented here was supported by Fundamental Research Grant Scheme: UKM-ST-01-FRGS0055-2006, Malaysia.

References

[1] H. M. Srivastava and S. Owa (Editors), Current topics in analytic function theory, *World Scientific Publishing Company*, Singapore, New Jersey, London, and Hong Kong, 1992.

[2] J. H. Choi, M. Saigo and H. M. Srivastava, Some inclusion properties of a certain family of integral operators, *J. Math. Anal. Appl.* **276**, (2002), 432-445.

[3] J. -L. Liu, The Noor integral and strongly starlike functions, *J. Math. Anal. Appl.* **261**, (2001), 441-447.

[4] K. Al-Shaqsi and M. Darus, A multiplier transformation defined by convolution involving nth order polylogarithms functions. (Submitted)

[5] K. I. Noor, On new classes of integral operators, *J. Natur. Geom.* **16**, (1999), 71-80.

[6] N. E. Cho and J. A. Kim, Inclusion properties of certain subclasses of analytic functions defined by a multiplier transformation, *Comp. Math. Appl.* **52**, (2006), 323-330.

[7] P. Eenigenburg, S.S. Miller, P.T. Mocanu and M.O. Reade, On a Briot-Bouquet differential subordination, *General Inequal.* **3**, (1983), 339-348.

[8] R. J. Libera, Some classes of regular univalent functions, *Proc. Amer. Math. Soc.* **16**, (1965), 755-758.

[9] S. D. Bernardi, Convex and starlike univalent functions, *Trans. Amer. Math. Soc.* **35**, (1969), 429-446.

[10] S. Owa and H.M. Srivastava, Some applications of the generalized Libera integral operator, *Proc. Japan Acad. Set. A Math. Sci.* **62**, (1986), 125-128.

[11] S. S. Miller and P. T. Mocanu, Differential subordination and univalent functions, *Michigan Math. J.* **28**, (1981), 157-171.

[12] W. C. Ma and D. Minda, An internal geometric characterization of strongly starlike functions, *Ann. Univ. Mariae Curie-Sklodowska Sect.* A **45**, (1991), 89-97.

[13] Y. C. Kim, J. H. Choi and T. Sugawa, Coefficient bounds and convolution properties for certain classes of close-to-convex functions, *Proc. Japan Acad. Set. A Math. Sci.* **76**, (2000), 95-98.

Received: January, 2009