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The Distributions of Sums, Products and Ratios of Inverted Bivariate Beta Distribution[1]

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Abstract

In this paper we derive the exact distributions of sums, products, and ratios of two random variables when they follow the bivariate inverted beta distribution. Forms of the probability density functions of these distributions are presented. The moments of theses distributions are derived. We provide extensive tabulation of the percentiles points associated with the distributions obtained.

Keywords: Bivariate inverted beta distributions, Sums, Ratios, Percentiles, Moments

1 Introduction

The statistics literature has seen many developments in the theory and applications of linear combinations, ratios and products of random variables. Nadarajah and Gupta have derived the distributions of sums and ratios of beta Stacy distribution see [1]. Olkin and Liu have presented a bivariate beta distribution that has support on the simplex $0 \leq x_i \leq 1$, $(i = 1, 2)$. This distribution has a positively likelihood ratio dependent and hence it has positive quadratic dependent [2]. El-Gohary and Sarhan have derived distributions for the sums, products, ratios and differences of Marshall-Olkin bivariate exponential distribution [4]. El-Gohary has obtained a new class of multivariate

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linear failure rate distribution of Marshall-Olkin type [5] and has discussed its applications in the reliability theory, see [7, 8].

This paper produces the distributions of the sums, rations and products of two dependent random variables follow a bivariate inverted beta distribution. Such these distribution are interesting in statistics and several applications, such as reliability engineering, industrial engineering and computer systems. The Dirichlet distribution is often used as a prior distribution for the parameters of a multinormal distribution. Univariate beta distributions are used extensively use in Bayesan statistics, since beta distributions provide a family of conjugate prior distributions for binomial distributions. Many applications in the mathematical, physical and engineering sciences, one random variable is functionally related to two or more different random variables. For example, the random signal S at the input of an amplifier consists of the sum of a random signal to which is add independent random noise. Therefore the random signal is a sum of two random variables. If X is a random variable what is the probability density function of S. Many signal processing systems use electronic multipliers to multiply two signals together. If X is the signal on one input and Y is the signal on the other input, what is the probability density function of the output $P = XY$.

Linear combination of two random variables is one of interesting problems in automation, control, computer science and related fields. The ratio of two random variables takes place in the stress-strength model in the context of reliability. For example, the model describes the lifetime of a component that has a random strength X and subjected to random stress Y. The components fails at the instant that the stress applied to it exceeds the strength and the component will function satisfactorily whenever $X > Y$. Therefore, $P(Y > X) = P(2Y/(X+Y) < 1)$ is a measure of components reliability.

In this paper we will derive the exact distributions of $S = X + Y$, $D = X - Y$, $P = XY$ and $R = X/(X+Y)$ and their moments when the random variables X and Y are correlated inverted bivariate beta distribution with the joint probability density function given by [3]

$$f(x,y) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \frac{x^{\theta_1-1}\, y^{\theta_2-1}}{(1+x+y)^{\theta_1+\theta_2+\theta_3}}, x > 0, y > 0, \theta_i > 0, i = 1, 2.3 \tag{1.1}$$

Also, moments for the distributions of sums, products and ratios will be derived. Further, we provide extensive tabulations of the associated percentage points, obtained by intensive computer power.

This paper is organized as follows. Section *2* presents the probability density functions and cumulative distribution functions of $S = X + Y$, $P = XY$

and $R = X/(X+Y)$. The moments of $S = X+Y$, $P = XY$ and $R = X/(X+Y)$ are given in Section *3*. Finally, in Section *4*, the tabulations of the percentiles of P are presented.

The calculations through out this paper involve the complete beta function and Euler hypergeometric function that defined by

$$B(a,b) = \int_0^\infty \frac{u^{a-1}}{(1+u)^{a+b}}\, du, \tag{1.2}$$

$${}_2F_1\Big(\mu, \lambda; 2\mu; 1 - k_2/k_1\Big) = \frac{k_1^\lambda}{B(\lambda,\, 2\mu-\lambda)} \int_0^\infty \frac{u^{\lambda-1}\, du}{\Big(1 + (k_1+k_2)u + k_1 k_2 u^2\Big)^\mu}, \tag{1.3}$$

for $4\mu > \lambda > 0$.

$${}_2F_1(a, 1-b; a+1; x) = a\, x^{-a} \int_0^x u^{a-1}(1-u)^{b-1}\, du, \tag{1.4}$$

and

$${}_pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p, \beta_1, \beta_2, \ldots, \beta_p) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k, (\alpha_2)_k \ldots, (\alpha_p)_k\, z^k}{(\beta_1)_k, (\beta_2)_k, \ldots (\beta_q)_k\, !k} \tag{1.5}$$

where $(\alpha_i)_k = \alpha_i(\alpha_i+1)(\alpha_i+2)\ldots(\alpha_i+k-1)$ is called a generalized hypergeometric series.

For the full properties of these special functions, we refer to Gradshteyn and Ryzhik (2000) [9].

2 Probability density functions

In this section, we derive the exact probability density functions and cumulative distribution functions of the sums, products, ratios and differences of two random variables that follow from Inverted bivariate beta distribution.

The following theorem give the probability density function for $S = X+Y$ in exact form when the random variables X and Y are distributed according to (1.1)

Theorem 2.1 *If X and Y are jointly distributed according to (1.1), then the probability density function and cumulative distribution function of $S = X+Y$ are given, respectively, by*

$$f_S(s) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1+\theta_2)\Gamma(\theta_3)} \frac{s^{\theta_1+\theta_2-1}}{(1+s)^{\theta_1+\theta_2+\theta_3}},\ s > 0. \tag{2.1}$$

and

$$F_S(s) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1+\theta_2+1)\Gamma(\theta_3)} s^{\theta_1+\theta_2+1} {}_2F_1(\theta_1+\theta_2+\theta_3, \theta_1+\theta_2; \theta_1+\theta_2+1; s), s > 0 \quad (2.2)$$

Proof. The proof of this theorem can be achieved Firstly, by deriving the joint pdf of $(S, R) = (X+Y,\ X/(X+Y))$. Using the joint pdf of (X,Y), given by (1.1), we can get the joint pdf of $(S,\ R)$ as

$$g(s,r) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \frac{r^{\theta_1-1}(1-r)^{\theta_2-1} s^{\theta_1+\theta_2-1}}{(1+s)^{\theta_1+\theta_2+\theta_3}}, s > 0,\ 0 < r < 1 \quad (2.3)$$

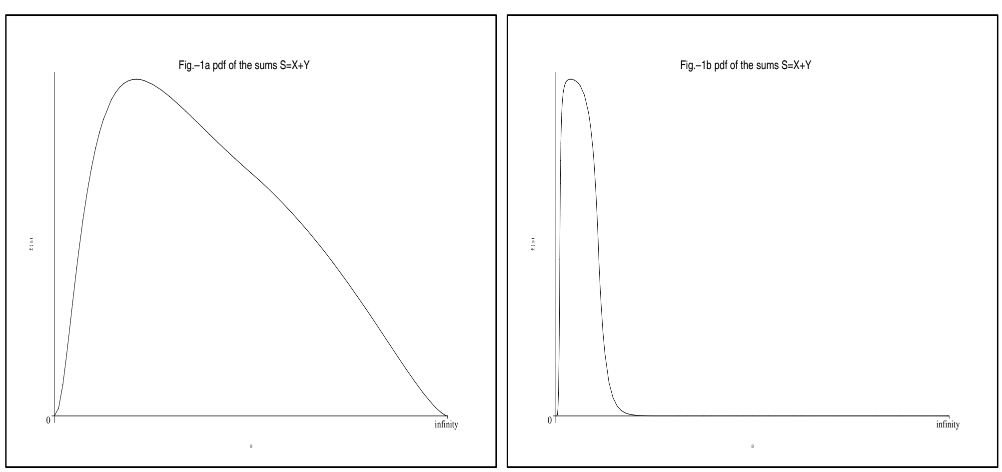
Therefore, the marginal pdf of S can be derived from $g(s,r)$ as follows

$$f_S(s) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \frac{s^{\theta_1+\theta_2-1}}{(1+s)^{\theta_1+\theta_2+\theta_3}} \int_0^1 r^{\theta_1-1}(1-r)^{\theta_2-1} dr$$

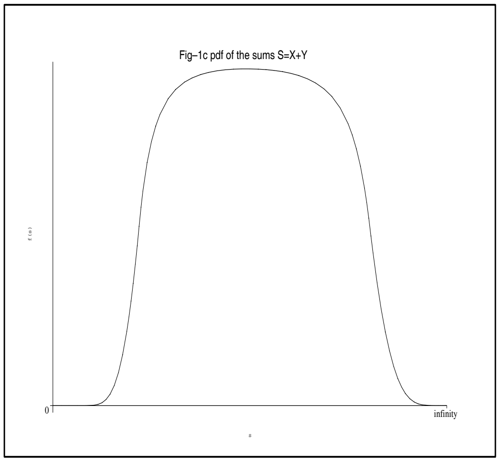
Solving the integrals in the above relations using Gradshteyn and Ryzhik (2000) the results obtained, one gets $f_S(s)$ as given in (2.1). Using the well known relation between *pdf* and *cdf* and (2.1), one can derive the cdf of s as given by (2.2), which completes the proof. □

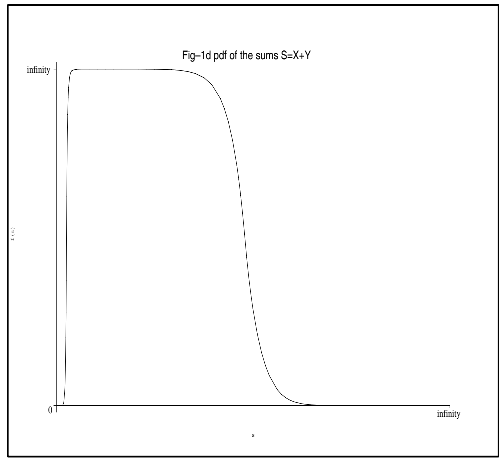
The distribution of sums of two correlated random variables has many several applications. For example, the distribution of the sums of two random variables gives the distribution of income of a person who has two different covariant incomes.

Plots of the sums density function (2.1) for some selected values of distribution parameters are shown in figure 1.



Figures 1a and 1b display the graph of pdf of S at $\theta_1 = 1.5$, $\theta_2 = 2, \theta = 0.5$, and $\theta_1 = 5.5$, $\theta_2 = 2, \theta_3 = 20.5$ respectively.





Figures 1c and 1d display the graph of pdf of S at $\theta_1 = 1.5$, $\theta_2 = 25, \theta_3 = 5.5$, and $\theta_1 = 15.5$, $\theta_2 = 5, \theta_3 = 20.5$ respectively.

Theorem 2.2 *If X and Y are jointly distributed according to (1.1), the probability density function and cumulative distribution function of the rations $R = X/S$ are given, respectively, by*

$$f_R(r) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)}\, B(\theta_1+\theta_2, \theta_3-2)\, r^{\theta_1-1}(1-r)^{\theta_2},\ \theta_3 > 2,\ 0 < r < 1 \tag{2.4}$$

and

$$F_R(r) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)B(\theta_1+\theta_2, \theta_3-2)}{\Gamma(\theta_1+1)\Gamma(\theta_2\Gamma(\theta_3)}\, r^{\theta_1}{}_2F_1(\theta_1, -\theta_2; \theta_1+1; r),\ \theta_3 > 2,\ r < 1 \tag{2.5}$$

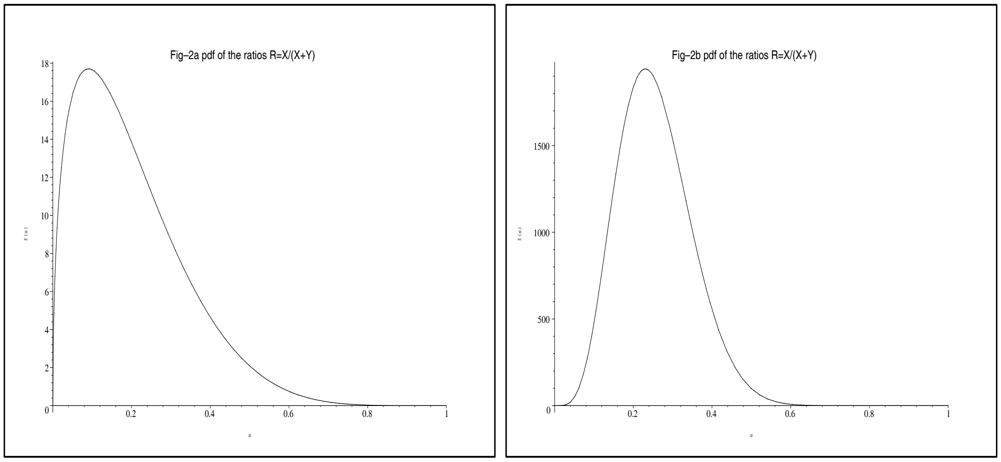
Proof. From (2.3), we have

$$f_R(r) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} r^{\theta_1-1}(1-r)^{\theta_2-1} \int_0^\infty \frac{s^{\theta_1+\theta_2-1}}{(1+s)^{\theta_1+\theta_2+\theta_3}} ds \tag{2.6}$$

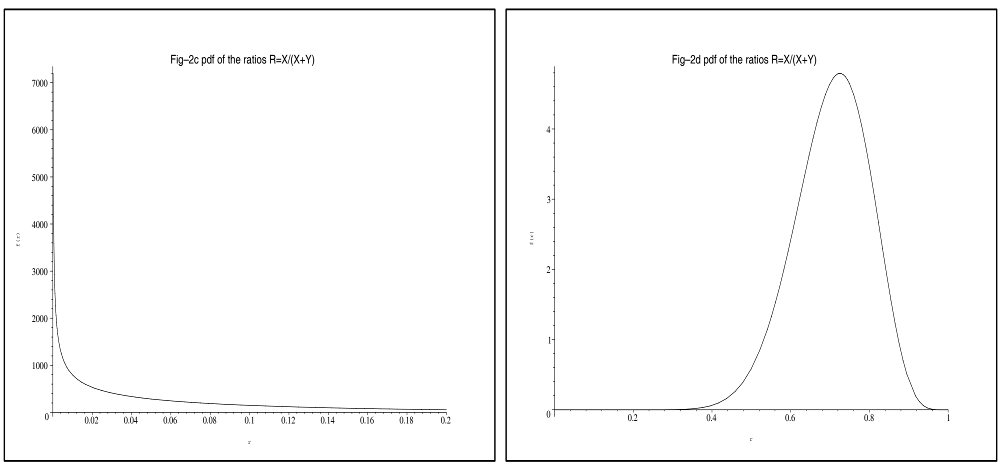
Solving the integrals in (2.6), using (1.2) one gets $f_R(r)$ as given in (2.4). Then, using (2.4) together with the well known relation between the pdf and cdf, one gets F_R as given by (2.5), which completes the proof of the theorem. □

Plots of the sums density function (2.4) for some selected values of distribution parameters are shown in figures 2.

Figure 2 displays the pdf of $R = X/(X+Y)$, at different sets of the parameters θ_1, θ_2 and θ.



Figures 2a and 2b display the graph of pdf of S at $\theta_1 = 5$, $\theta_2 = 15, \theta_2 = 2.5$, and $\theta_1 = \theta_2 = \theta = 1$ respectively.



Figures 2c and 2d display the graph of pdf of S at $\theta_1 = 1.5$, $\theta_2 = 5$, $\theta_3 = 5.5$, and $\theta_1 = 15$, $\theta_2 = 5.5$, $\theta_3 = 20$ respectively.

Theorem 2.3 *If X and Y are jointly distributed according to (1.1), then the probability density function of the products $P = XY$ takes the following form*

$$f_P(p) = \frac{2^{2\theta_1+\theta_3}\,\Gamma(\theta_1+\theta_2+\theta_3)B(2\theta_1+\theta_3, 2\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)\,(1+\sqrt{1-4p})^{2\theta_1+\theta_3}}\, p^{\theta_1-1} \times$$
$${}_2F_1\Big(\theta_1+\theta_2+\theta_3, 2\theta_1+\theta_3; 2\theta_1+2\theta_2+2\theta_3, \frac{2\sqrt{1-4p}}{1+\sqrt{1-4p}}\Big),\ p > 0 \quad (2.7)$$

Proof. Starting with the joint pdf of $(X, P) = (X, XY)$ as in the following form

$$g(x,p) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \frac{x^{2\theta_1+\theta_3-1}\,p^{\theta_2-1}}{(p+x+x^2)^{\theta_1+\theta_2+\theta_3}},\ x > 0, p > 0, \theta_i > 0\,(i = 1,2,3) \quad (2.8)$$

Therefore, the marginal pdf of P can be derived from $g(x,p)$ as follows

$$f_P(p) = \int_0^\infty g(x,p)\,dx \tag{2.9}$$

Substituting from the above relations (2.8) into (2.9) and using the relation (1.3), one gets $f_P(p)$ as given by (2.7), which completes the proof of the theorem. □

Next, we derive exacts forms for the moments of sums, rations and products.

3 Moments of sums, ratios and products

In this section we derive the moments of $S = X + Y$, $P = XY$ and $R = X/(X+Y)$ when X and Y are distributed according to (1.1). Now we establish the following lemma.

Lemma 3.1 *If X and Y are jointly distributed according to (1.1) then*

$$E(X^nY^m) = \frac{\Gamma(\theta_1+\theta_2+\theta)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)}\,B\Big(\theta_1+m,\theta_2+\theta_3-m\Big)\,B\Big(\theta_2+n,\theta_3-m-n\Big),\ \theta_3 > m+n \tag{3.1}$$

Proof. Starting with the following well known relation

$$E(X^nY^m) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^ny^mf(x,y)dxdy \tag{3.2}$$

Substituting from (1.1) into (3.2), one gets

$$E(X^nY^m) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)}\int_0^\infty\int_0^\infty \frac{x^{\theta_1+m-1}y^{\theta+n-1}}{(1+x+y)^{\theta_1+\theta_2+\theta_3}}\,dx\,dy. \tag{3.3}$$

Using the definition of the beta function (1.3) and after some algebraic manipulation one gets the formula (3.1) of the moments.

Theorem 3.1 *If X and Y are jointly distributed according to (1.1), then*

$$E(S^m) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)\Gamma(\theta_1+\theta_2+\theta_3-2m)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)\Gamma(\theta_1-m)\Gamma(\theta_2+\theta_2-m)}\sum_{k=0}^{m} B(\theta_2+m-k,\theta_3+k-2m), \tag{3.4}$$

for $\theta_3 > 2m-k$.

Proof. Starting with the definition of expectation and using the binomial expansion, one gets

$$E(S^m) = E((X+Y)^m) = \sum_{k=0}^{m}\binom{m}{k}E(X^k\,Y^{m-k})$$

Applying lemma (3.1), we get (3.5), which completes the proof of the lemma. □.

Theorem 3.2 *If X and Y are jointly distributed according to (1.1), then*

$$E(P^m) = \frac{\Gamma\big(\theta_1+\theta_2+\theta_3\big)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} B\big(\theta_1+m, \theta_2+\theta_3-m\big) B\big(\theta_2+m, \theta_3-2m\big),\ \theta_3 > 2m \quad (3.5)$$

Proof. Starting by setting $m = n$ in the relation (3.1) we get the formula (3.5) which completes the proof. □.

Theorem 3.3 *If X and Y are jointly distributed according to (1.1), then*

$$E(R^m) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)\Gamma(\theta_1+\theta_2+\theta_3-2)}{\Gamma(\theta_1)\Gamma(\theta_2\Gamma(\theta_3)\Gamma(\theta_1+\theta_2)\Gamma(\theta_3-2)} B\big(\theta_2+m, \theta_2+1\big),\ \theta_3 > 2 \quad (3.6)$$

for $n \geq 1$.

Proof. Using (2.4), one gets

$$E(R^n) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)\Gamma(\theta_1+\theta_2+\theta_3-2)}{\Gamma(\theta_1)\Gamma(\theta_2\Gamma(\theta_3)\Gamma(\theta_1+\theta_2)\Gamma(\theta_3-2)} \int_0^1 r^{\theta_1+m-1}(1-r)^{\theta_2}\, dr, \quad (3.7)$$

Using the complete beta function definition yields (3.7), which completes the proof □.

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4 Percentiles points for sums, ratios and products

In this section, we provide extensive tabulations of the percentiles of the distributions of S, R and P. This percentiles are computed numerically by solving the following equations, with respect to s_q, r_q and p_q respectively,

$$\int_0^{s_q} f_S(t)dt = q,\ \int_0^{r_q} f_R(t)dt = q,\ \int_0^{p_q} f_P(p)dp = q \quad (4.1)$$

where the probability density function $f_S(s)$, $f_R(r)$ and $f_P(p)$ are given by (2.1), (2.6) and (2.7) respectively. In fact this function involves computation of Euler hypergeometric, gamma and functions and routines for these are widely available.

In the other hand the percentiles of the distributions of S, and R can be easily computed by using the expressions for their cumulative distribution functions $F_S(s)$, and $F_R(r)$, respectively. Inverting the equations $F_S(s_q) = q, F_R(r) = q$, $F_p(p_q) = q$, one gets.

The percentage point for the sums $S_q = X+Y$ distribution can be obtained by solving the equation, with respect to s_q,

$$\frac{\Gamma\big(\theta_1+\theta_2+\theta_3\big)}{\Gamma\big(\theta_1+\theta_2\big)\Gamma\big(\theta_3\big)} s_q^{\theta_1+\theta_2+1}\,{}_2F_1\big(\theta_1+\theta_2+\theta_3, \theta_1+\theta_2; \theta_1+\theta_2+\theta_3+1, s_q\big) = q \tag{4.2}$$

The percentage point for the ratios $R_q = X/(X+Y)$ distribution can be obtained by solving the equation, with respect to r_q,

$$\frac{\Gamma\big(\theta_1+\theta_2+\theta_3\big)B(\theta_1+\theta_2, \theta_3-2)}{\Gamma\big(\theta_1+1\big)\Gamma\big(\theta_2\big)\Gamma\big(\theta_3\big)} r_q^{\theta_1}\,{}_2F_1\big(\theta_1, -\theta_2; \theta_1+1, r_q\big) = q \tag{4.3}$$

Finally, percentage points for $P_q = XY$ can be obtained by solving the equation, with respect to p_q,

$$\frac{2^{2\theta_1+\theta_3}\,\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} B(2\theta_1+\theta_3, 2\theta_2+\theta_3) \int_0^{p_q} \left\{ \frac{t^{\theta_1-1}}{(1+\sqrt{1-4t})^{2\theta_1+\theta_3}} \times \right.$$

$$\left. {}_2F_1\left(\theta_1+\theta_2+\theta_3, 2\theta_1+\theta_3; 2\theta_1+2\theta_2+2\theta_3, \frac{2\sqrt{1-4t}}{1+\sqrt{1-4t}}\right) \right\} dt = q, \tag{4.4}$$

The following tables give the numerical solution (4.2), (4.3) and (4.4) for different values of the parameters θ_1, θ_2 and θ_3.

Table 1. Displays the numerical values of the percentiles of the sums $S = X + Y$ corresponding the probabilities $0.9, 0.95, 0.975, 0.99, 0.995, 0.999$ for different values of the parameters θ_1, θ_2 and θ_3.

Table 1

θ_1	θ_2	θ_3	q 0.9	0.95	0.975	0.99	0.995	0.999
					s_q			
1.0	1.0	1.0	18.50	38.50	78.50	198.5	398.5	1998.5
2.0	1.0	1.0	28.10	58.10	118.0	301.0	598.1	2998.0
3.0	1.0	1.0	37.50	77.50	157.5	400.0	798.0	4012.0
4.0	1.0	1.0	47.20	97.10	197.0	500.0	998.0	4998.0
5.0	1.0	1.0	57.00	116.5	236.5	598.0	1198	6012.0
6.0	1.0	1.0	66.00	136.0	276.0	702.0	1399	6999.0
7.0	1.0	1.0	76.00	156.0	316.0	796.0	1596	8016.0
8.0	1.0	1.0	85.00	175.0	355.0	895.0	1795	9005.0
9.0	1.0	1.0	94.50	194.5	394.5	994.5	1994.5	9995.0
10.	1.0	1.0	104.0	214.0	434.0	1094	2194	10999
11.	1.0	1.0	114.0	234.0	473.5	1200	2399	11999
12.	1.0	1.0	123.0	253.0	513.0	1293	2593	12999
13.	1.0	1.0	132.0	272.0	552.0	1402	2795	13995
14.	1.0	1.0	142.0	292.0	592.0	1492	2992	15002
16.	1.0	1.0	152.0	312.0	632.0	1592	3192	15992
17.	1.0	1.0	161.0	331.0	671.0	1701	3391	17001
18.	1.0	1.0	172.0	351.0	711.0	1800	3599	18009
19.	1.0	1.0	181.0	370.0	750.0	1900	3790	18990
20.	1.0	1.0	190.0	390.0	790.0	1990	3990	19990
5.0	3.0	1.0	190.0	390.0	790.0	2000	3996	19996
5.0	4.0	1.0	76.00	156.0	316.0	802.0	1596	8056.0
5.0	5.0	1.0	95.00	195.0	396.0	996.0	1996	9996.0
5.0	6.0	1.0	114.0	234.0	474.0	1204	2394	12094
5.0	7.0	1.0	134.0	273.0	554.0	1394	2794	13994
5.0	8.0	1.0	152.0	312.0	632.0	1592	3192	15999
5.0	9.0	1.0	172.0	351.0	711.0	1800	3598	17999
5.0	11	1.0	190.0	390.0	790.0	1999	3990	19999
5.0	13	1.0	210.0	430.0	870.0	2200	4395	21999.0
5.0	15	1.0	239.0	239.0	990.0	2499	2499	24999.0
5.0	17	1.0	19.30	29.10	42.80	70.10	101.0	240.00
5.0	20	1.0	6.080	7.880	10.00	13.40	16.53	26.230
5.0	5.0	2.0	3.44	4.240	5.122	6.450	7.550	10.700
5.0	5.0	4.0	2.39	2.840	3.360	4.080	4.670	6.2500

Table 2. Displays the numerical values of the percentiles of the ratios $R = x/(X+Y)$ corresponding the probabilities $0.9, 0.95, 0.975, 0.99, 0.995, 0.999$ for different values of the parameters θ_1, θ_2 and θ_3.

Table 2.

θ_1	θ_2	θ_3	q 0.9	0.95	0.975	0.99	0.995	0.999
				r_q				
1.0	4.0	3.0	0.4380	0.6840	0.6024	0.6840	0.7350	0.8230
2.0	4.0	3.0	0.5850	0.6580	0.7170	0.7780	0.8150	0.8780
3.0	4.0	3.0	0.6670	0.7290	0.7770	0.8280	0.8560	0.9080
4.0	4.0	3.0	0.7220	0.7750	0.8160	0.8580	0.8820	0.9240
5.0	4.0	3.0	0.7610	0.8080	0.8430	0.8800	0.9002	0.9360
6.0	4.0	3.0	0.7910	0.8320	0.8630	0.8950	0.9140	0.9440
7.0	4.0	3.0	0.8130	0.8500	0.8790	0.9070	0.9230	0.9510
8.0	4.0	3.0	0.8310	0.8650	0.8908	0.9169	0.9315	0.9560
9.0	4.0	3.0	0.8462	0.8772	0.9008	0.9243	0.9376	0.9598
10.0	4.0	3.0	0.8588	0.8874	0.9091	0.9307	0.9431	0.9633
11.0	4.0	3.0	0.8695	0.8960	0.9162	0.9360	0.9475	0.9662
12.0	4.0	3.0	0.8786	0.9034	0.9222	0.9407	0.9513	0.9687
13.0	4.0	3.0	0.8862	0.9098	0.9274	0.9447	0.9546	0.9708
14.0	4.0	3.0	0.8932	0.9154	0.9319	0.9482	0.9575	0.9727
15.0	4.0	3.0	0.8994	0.9204	0.9359	0.9513	0.9600	0.9743
16.0	4.0	3.0	0.9050	0.9248	0.9395	0.9540	0.9624	0.9761
17.0	4.0	3.0	0.9099	0.9287	0.9427	0.9565	0.9643	0.9771
18.0	4.0	3.0	0.9144	0.9322	0.9456	0.9587	0.9661	0.9788
19.0	4.0	3.0	0.9184	0.9355	0.9482	0.9606	0.9677	0.9793
20.0	4.0	3.0	0.9220	0.9384	0.9505	0.9624	0.9692	0.9803
21.0	4.0	3.0	0.9255	0.9410	0.9527	0.9641	0.9706	0.9812
22.0	4.0	3.0	0.9285	0.9435	0.95463	0.9656	0.9718	0.9820
23.0	4.0	3.0	0.9313	0.9457	0.9565	0.9670	0.9730	0.9827
24.0	4.0	3.0	0.9339	0.9478	0.9581	0.9682	0.9740	0.9834
25.0	4.0	3.0	0.9364	0.9497	0.9597	0.9650	0.9750	0.9840

			q					
θ_1	θ_2	θ_3	0.9	0.95	0.975	0.99	0.995	0.999
26.0	4.0	3.0	0.9385	0.9515	0.9611	0.9705	0.97584	0.9846
27.0	4.0	3.0	0.9407	0.9532	0.9624	0.9716	0.9767	0.9851
28.0	4.0	3.0	0.9426	0.9547	0.9637	0.9725	0.9775	0.9856
29.0	4.0	3.0	0.9444	0.9562	0.9649	0.9734	0.9782	0.9862
30.0	4.0	3.0	0.9462	0.9576	0.9660	0.9742	0.9789	0.9865
31.0	4.0	3.0	0.9477	0.9589	0.9670	0.9750	0.9796	0.9869
32.0	4.0	3.0	0.9493	0.9601	0.9680	0.9758	0.9802	0.9873
33.0	4.0	3.0	0.9507	0.9612	0.9689	0.9765	0.9807	0.9877
34.0	4.0	3.0	0.9521	0.9623	0.96975	0.9771	0.9813	0.9880
35.0	4.0	3.0	0.9534	0.9633	0.97057	0.9777	0.98175	0.9884
36.0	4.0	3.0	0.9546	0.9643	0.97134	0.9783	0.98223	0.98864
37.0	4.0	3.0	0.9558	0.96513	0.97208	0.97884	0.98269	0.98895
38.0	4.0	3.0	0.95680	0.9660	0.97278	0.97939	0.98312	0.98921
39.0	4.0	3.0	0.9579	0.96683	0.97344	0.9799	0.98354	0.98947
40.0	4.0	3.0	0.9589	0.96761	0.97407	0.9804	0.9840	0.98975
41.0	1.0	3.0	0.9599	0.96835	0.97468	0.98085	0.98431	0.98998
43.0	2.0	3.0	0.96067	0.96907	0.97525	0.9813	0.98467	0.9902
45.0	3.0	3.0	0.96316	0.97104	0.97683	0.98245	0.98565	0.99084
47.0	4.0	3.0	0.96466	0.97222	0.97778	0.98318	0.98624	0.99121
50.0	4.0	3.0	0.96669	0.97382	0.97906	0.98415	0.98703	0.99173
3.0	1.0	3.0	0.96550	0.98305	0.99160	0.99666	0.99834	0.99967
3.0	3.0	3.0	0.7535	0.8108	0.8535	0.8948	0.9172	0.9525
3.0	5.0	3.0	0.59619	0.65875	0.70958	0.76380	0.79703	0.8565
3.0	7.0	3.0	0.4905	0.5497	0.6001	0.6564	0.6927	0.7614
3.0	9.0	3.0	0.4159	0.4701	0.5178	0.5726	0.6085	0.6795
3.0	11.0	3.0	0.35990	0.4101	0.4545	0.5066	0.5411	0.6113
3.0	13.0	3.0	0.3173	0.3635	0.4047	0.4536	0.4864	0.5542
3.0	15.0	3.0	0.28375	0.32629	0.364411	0.4102	0.4413	0.50604
3.0	17.0	3.0	0.2566	0.2959	0.3314	0.3743	0.40369	0.4655
3.0	19.0	3.0	0.23410	0.27058	0.3038	0.3439	0.3719	0.4309
3.0	21.0	3.0	0.21540	0.2493	0.28039	0.3185	0.34460	0.4006
3.0	23.0	3.0	0.1994	0.2311	0.26031	0.29594	0.32103	0.3745
3.0	25.0	3.0	0.1854	0.2153	0.2429	0.2768	0.30042	0.3520
3.0	27.0	3.0	0.1736	0.2016	0.2277	0.2597	0.28230	0.3309
3.0	29.0	3.0	0.1627	0.1895	0.21422	0.2449	0.26623	0.3128
3.0	31.0	3.0	0.1536	0.1788	0.2023	0.2314	0.2519	0.2965
3.0	33.0	3.0	0.1452	0.1692	0.1916	0.2195	0.2390	0.2816
3.0	35.0	3.0	0.1378	0.16055	0.20860	0.2084	0.22729	0.2684

Table 3. Displays the numerical values of the percentiles of the products $P = XY$ corresponding the probabilities $0.9, 0.95, 0.975, 0.99, 0.995, 0.999$ for different values of the parameters θ_1, θ_2 and θ_3.

Table 3

θ_1	θ_2	θ_3	q 0.9	0.95	0.975	0.99	0.995	0.999
			p_q					
0.5	2.0	3.0	00.72	01.51	02.93	06.52	11.35	38.35
1.0	2.0	3.0	01.53	03.06	05.72	12.25	21.11	69.75
1.5	2.0	3.0	02.34	04.59	08.47	18.01	30.68	100.8
2.0	2.0	3.0	03.15	06.12	11.21	23.56	40.20	130.9
2.5	2.0	3.0	03.99	07.65	13.94	29.18	49.65	161.0
3.0	2.0	3.0	04.80	09.18	16.67	34.90	59.10	191.5
3.5	2.0	3.0	05.60	10.70	19.50	40.50	68.55	221.5
4.0	2.0	3.0	06.40	12.25	22.12	46.25	77.99	252.0
4.5	2.0	3.0	07.25	13.75	24.85	52.50	87.40	282.4
5.0	2.0	3.0	08.05	15.28	27.57	57.57	96.90	31.57
5.5	2.0	3.0	08.90	16.80	30.30	62.90	106.5	342.5
6.0	2.0	3.0	09.72	18.35	33.02	68.52	115.7	372.7
6.5	2.0	3.0	10.48	19.85	35.75	74.50	125.2	403.9
7.0	2.0	3.0	11.30	21.40	38.46	79.60	134.6	432.9
7.5	2.0	3.0	12.10	22.95	41.19	85.19	143.9	470.0
8.0	2.0	3.0	12.90	24.45	43.91	90.91	153.5	492.5
8.5	2.0	3.0	13.80	26.00	46.63	97.00	163.0	513.0
9.0	2.0	3.0	14.60	27.50	49.35	102.0	172.2	552.7
9.5	2.0	3.0	15.40	29.00	52.15	107.9	183.2	600.0
10.	2.0	3.0	16.15	30.51	54.80	113.2	191.2	612.8
11.	2.0	3.0	17.85	33.60	60.25	124.5	209.9	675.9
12.	2.0	3.0	19.40	36.65	65.70	135.5	228.8	734.8
13.	2.0	3.0	21.15	39.70	71.13	146.8	247.5	798.5
14.	2.0	3.0	22.80	42.75	76.59	158.9	266.5	854.5
15.	2.0	3.0	24.35	45.75	82.03	170.1	285.1	915.8
16.	2.0	3.0	26.12	48.85	87.48	181.5	304.1	978.1
17.	2.0	3.0	27.75	51.85	92.91	194.2	322.9	1035
18.	2.0	3.0	29.15	54.98	98.35	202.7	341.7	1095
19.	2.0	3.0	30.85	57.95	103.8	214.1	360.8	1156

Table 3 continued

θ_1	θ_2	θ_3	q 0.9	0.95	0.975	0.99	0.995	0.999
25.	2.0	3.0	40.75	76.35	136.5	282.5	473.5	1516
26.	2.0	3.0	42.15	79.35	141.9	292.4	492.2	1575
27.	2.0	3.0	43.85	82.35	147.3	303.5	511.1	1635
28.	2.0	3.0	45.39	85.45	152.8	314.5	530.1	1695
29.	2.0	3.0	47.12	88.45	158.1	326.2	548.6	1756
30.	2.0	3.0	48.70	91.35	163.6	338.3	567.5	1819
31.	2.0	3.0	50.25	94.55	168.9	350.2	586.7	1876
32.	2.0	3.0	51.88	97.60	174.6	361.5	605.5	1937
33.	2.0	3.0	53.50	100.7	180.9	370.6	624.2	1997
34.	2.0	3.0	55.15	103.7	185.5	381.5	643.5	2060
35.	2.0	3.0	56.80	106.8	190.9	392.9	661.5	2125
36.	2.0	3.0	58.40	109.8	196.4	404.4	680.4	2178
37.	2.0	3.0	60.00	112.8	201.8	416.2	699.2	2240
38.	2.0	3.0	61.85	115.9	207.2	428.2	718.2	2300
39.	2.0	3.0	63.28	118.9	212.7	437.5	736.8	2353
40.	2.0	3.0	64.95	122.0	218.2	448.6	756.4	2417
41.	2.0	3.0	66.55	125.1	223.6	460.2	775.3	2478
42.	2.0	3.0	68.56	128.2	229.1	472.1	794.1	2538
43.	2.0	3.0	70.01	131.1	234.5	482.8	812.8	2598
44.	2.0	3.0	71.85	134.2	240.2	494.2	831.2	2658
45.	2.0	3.0	73.20	137.2	245.3	506.8	850.8	2723
46.	2.0	3.0	74.85	140.3	250.8	518.8	868.8	2780
47.	2.0	3.0	76.42	143.3	256.3	530.3	888.1	2840
48.	2.0	3.0	78.05	146.5	262.8	542.1	906.5	2898
49.	2.0	3.0	80.01	149.5	267.1	552.1	925.4	2958
50.	2.0	3.0	81.72	152.5	272.6	562.4	944.4	3020
2.0	1.0	4.0	0.665	1.222	2.096	3.999	6.272	16.62
2.0	2.0	4.0	1.339	2.382	3.981	7.410	11.48	29.79
2.0	3.0	4.0	2.016	3.532	5.844	10.77	16.62	42.82
2.0	4.0	4.0	2.687	4.679	7.702	14.19	21.72	55.82
2.0	5.0	4.0	3.382	5.821	9.557	17.49	26.82	68.75
2.0	6.0	4.0	4.041	6.975	11.41	20.85	31.90	81.45
2.0	7.0	4.0	4.720	8.112	13.27	24.16	36.99	94.45
2.0	8.5	4.0	5.750	9.829	16.04	29.19	44.65	113.5
2.0	9.0	4.0	6.085	10.40	16.97	31.05	47.16	120.1
2.0	10.	4.0	6.750	11.55	18.63	34.23	52.25	132.8
2.0	11.	4.0	7.420	12.70	20.67	37.67	57.36	145.6
2.0	13.	4.0	8.750	14.99	24.37	44.29	67.48	171.9

5 Conclusion

Finally, we conclude that the distributions of sums, ratios, and products of the bivariate distribution of X and Y follow from bivariate inverted beta distribution are obtained. Also the moments of $S = X + Y$, $R = X/(X + Y)$ and $P = XY$ are derived. Tabulation for the percentage points of, sums, ratios and products are derived.

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