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A New Approach to the Gas Dynamics Equation: An Application of the Variational Iteration Method

Hossein Jafari [a,1], **Hassan Hosseinzadeh** [b] **and Elham Salehpoor**[b]

[a]Department of Mathematics, University of Mazandaran, Babolsar, Iran
jafari_h@math.com

[b]Department of Mathematics, Islamic Azad University,
Ghaemshahr Branch, P.O. Box 163, Ghaemshahr, Iran
elham_salehpor61@yahoo.com

Abstract

In this paper, He's variational iteration method (VIM) has been used to obtain solution of nonlinear gas dynamics equation. This method is based on Lagrange multipliers for identification of optimal values of parameters in a functional. Using this method creates a sequence which tends to the exact solution of the problem.

Keywords: Variational iteration method; Nonlinear gas dynamics equation; Partial differential equation

1 Introduction

Analytical methods commonly used to solve nonlinear equations are very restricted and numerical techniques involving discretization of the variables on the other hand gives rise to rounding off errors.

Recently introduced variational iteration method by He[3, 4, 5], which gives rapidly convergent successive approximations of the exact solution if such a solution exists, has proved successful in deriving analytical solutions of linear and nonlinear differential equations. This method is preferable over numerical methods as it is free from rounding off errors and neither requires large computer power/memory. He [4, 5] has applied this method for obtaining analytical solutions of autonomous ordinary differential equation, nonlinear partial differential equations with variable coefficients and integro-differential equations. The variational iteration method was successfully applied to Seventh order Sawada-Kotera equations [7]. Linear and nonlinear wave equations,

[1]Author for correspondence, jafari@umz.ac.ir, Tel:+98-1125342385

KdV, K(2,2), Burgers, and cubic Boussinesq equations have been solved by Wazwaz [9] using the variational iteration method.
In the present paper we employ VIM method for solving following equations. Consider the homogeneous gas dynamics equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = u(1-u)\,, \qquad 0 \leq x \leq 1, \quad t \geq 0 \tag{1}$$

with initial condition $u(x,0) = g(x)$. Further we compare the result with given solutions using ADM [1, 2].

2 He's variational iteration method

For the purpose of illustration of the methodology to the proposed method, using VIM, we begin by considering a differential equation in the formal form,

$$Lu + Nu = g(x,t), \tag{2}$$

where L is a linear operator, N a nonlinear operator and $g(x,t)$ is the source inhomogeneous term. According to the variational iteration method, we can construct a correction functional as follow

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(Lu_n(\xi) + N\tilde{u}(\xi) - g(\xi)\right)d\xi, \quad n \geq 0, \tag{3}$$

where λ is a general Lagrangian multiplier [6], which can be identified optimally via the variational theory, the subscript n denotes the nth order approximation, $\tilde{u}_n$ is considered as a restricted variation [5, 6] i.e., $\delta\tilde{u}_n = 0$.

So, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(x,t)$, $n \geq 0$ of the solution $u(x,t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. Consequently, the solution

$$u(x,t) = \lim_{n\longrightarrow\infty} u_n(x,t). \tag{4}$$

For the convergence of VIM we refer the reader to Dehghan's work [5, 8].

3 Applying VIM for gas dynamics equation

In this section, we applay VIM to nonlinear gas dynamics equation(Eq.(1)). Its correction variational functional in t-direction to obtain the solution of gas dynamics equation (1) by variational iteration method can be expressed as follows:

$$u_{n+1}(x,t) \;=\; u_n(x,t) + \int_0^t \lambda(\xi)\left[\frac{\partial}{\partial \xi}(u_n) - u_n + (N(\tilde{u}_n))\right] d\xi, \quad n \geq 0, (5)$$

where $N\widetilde{u}_n(x,\xi) = \frac{1}{2}\widetilde{u}^2_{nx}(x,\xi) + \widetilde{u}^2(x,\xi))$. taking variation with respect to the independent variable u_n noticing that $\delta N\widetilde{u}_n(x,\xi) = 0$

$$\begin{aligned}\delta u_{n+1}(x,t) &= \delta u_n(x,t) + \delta \int_0^t \lambda(\xi)\left[\frac{\partial}{\partial \xi}(u_n) - u_n + (N(\tilde{u}_n))\right] d\xi \\ &= \delta u_n(x,t) + \lambda\,\delta u_n|_{\xi=t} - \int_0^t \lambda'(\xi)\,\delta u_n + \lambda(\xi)\,\delta u_n d\xi = 0, \quad (6)\end{aligned}$$

This yields the stationary conditions

$$1 + \lambda(\xi) = 0, \qquad \lambda(\xi) + \lambda'(\xi)|_{\xi=t} = 0. \tag{7}$$

This in turn gives $\lambda(\xi) = -e^{t-\xi}$. Substituting this value of the Lagrange multiplier into the functional (5) gives the iteration formula

$$u_{n+1}(x,t) \;=\; u_n(x,t) - \int_0^t e^{t-\xi}\left[\frac{\partial}{\partial \xi}(u_n) - u_n + (N(u_n))\right] d\xi, \quad n \geq 0, (8)$$

thus, we can obtain approximation or exact solution for $u(x,t)$.

4 Illustrative Example

To demonstrate the effectiveness of the method we consider here Eqs. (1) with given initial condition.

Example 4.1 *Consider the nonlinear gas dynamics equations (1) with the I.C.*

$$u(x,0) = e^{-x}. \tag{9}$$

Substituting (9) into Eq.(8) we obtain the following successive approximations

$$\begin{aligned}u_1(x,t) &= u_0 - \int_0^t e^{t-\xi}\left[\frac{\partial}{\partial \xi}(u_0) - u_0 + \frac{1}{2}u^2_{0x}(x,\xi) + u^2_0(x,\xi)\right] d\xi = e^{t-x}, \\ u_2(x,t) &= u_1 - \int_0^t e^{t-\xi}\left[\frac{\partial}{\partial \xi}(u_1) - u_1 + \frac{1}{2}u^2_{1x}(x,\xi) + u^2_1(x,\xi)\right] d\xi = e^{t-x},\end{aligned}$$

finally, $u_n(x,t) = e^{t-x}$, then using Eq. (4) we have $u(x,t) = e^{t-x}$which is exact solution.

Remark 1: Nonlinear gas dynamics equation has been solved by ADM by Evans and Bulut [2]. they obtained exact solution after some iteration but we obtain exact solution after 2 iteration.

5 Conclusion

Variational iteration method is a powerful tool which is capable of handling linear/nonlinear partial differential equations. The method has been successfully applied to nonlinear gas dynamics equation. Also, comparisons were made between He's variational iteration method and Adomian decomposition method (ADM) for nonlinear gas dynamics equation.

The VIM reduces the volume of calculations without requiring to compute the Adomian polynomials. He's variational iteration method facilitates the computational work and gives the solution rapidly if compared with Adomian method.

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