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The Generalized and Perturbed Lamé System

B. Merouani

Univ. F. ABBAS de Sétif, Algeria
mermaths@hotmail.com

M. Meflah

Univ. K. M. de Ouargla, Algeria

A. Boulaouad

Univ. F. ABBAS de Sétif, Algeria

Abstract

In this work, we study the existence, the uniqueness and the regularity of the solution for some boundary value problems gouverned by a perturbed and generalized Lamé system operator.

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1. Notations.- Ω denoted a bounded and connected open set of $\mathbb{R}^n$ $(n = 2, 3)$ with boundary $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$. Here, Γ is a lipschitzian manifold of dimension $n-1$, where $\Gamma_i \subset \Gamma$, $i = 1, 2$, with the measure of Γ_1 is strictly positive ($\text{mes}(\Gamma_1) > 0$) and $\Gamma_1 \cap \Gamma_2 = \phi$.

2. Problem statement.- We consider firstly the mathematical model of the perturbed Lamé system :

$$-L_p u + F(u),$$

where $F(u)$ is the perturbation and

$$L_p u = \mu \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) + (\lambda + \mu) \nabla (div(u)),$$

p, q are two real numbers such that $p \in]1, \infty[$ and $\frac{1}{p} + \frac{1}{q} = 1$, λ and μ are the Lamé coefficients subjected to the constraint $\lambda + \mu \geq 0$ and $\lambda > 0$, ν denotes the outgoing normal vector to Γ_2. For $p = 2$, we recover the classical Lamé system.

Given a function f and a matrix $\varphi = (\varphi_{i,j})_{1<i,j<n}$,such that $\varphi_{i,j} = \varphi_{j,i} \in C^{0,1}(\overline{\Omega})$ and $\varphi_{i,j}(x) > 0$, for all $x \in \Gamma_2$.We study the existence, the uniqueness and the regularity of the complex-valued solution $u = u(x)$, $x \in \Omega$, for the following problem :

$$(P) \begin{cases} -L_p u + F(u) = f, \text{ in } \Omega & (2.1) \\ u = 0, \text{ on } \Gamma_1 & (2.2) \\ \sigma(u).\nu + \varphi(x)\ u = 0, \text{ on } \Gamma_2 & (2.3) \end{cases}$$

Here $\sigma(u) = (\sigma_{ij}(u))_{1<i,j<n}$ is the matrix of the constraints tensor $\sigma_{ij}(u) = \lambda div(u)\delta_{ij} + 2\mu\ \varepsilon_{ij}(u)$, were $\varepsilon_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}), 1 \leq i, j \leq n$, are the components of the deformation tensor.

In this work, we consider the following cases:

· $F(u) = 0$,
· $F(u) = |u|^{\rho} u$ with $\rho = p - 2 > 0$,
· $F(u) = u^3$,
· $F(u) \equiv a(x,t)$.

We distinguish the cases:

· if $\Gamma_2 = \phi$, (P) becomes a Dirichlet problem,
· if $\Gamma_1 = \phi$, and $\varphi \equiv 0$ on Γ_2, (P) becomes a Neumann problem,
· if $\Gamma_2 \neq \phi$ and $\varphi \equiv 0$ on Γ_2, (P) becomes a mixed problem : Dirichlet-Neumann,
· if Γ_1 , $\Gamma_2 \neq \phi$ and $\varphi(x) \neq 0$ on Γ_2, (P) becomes a mixed problem : Dirichlet-(2.3).

Of course, when it is question of a Neumann problem ($\Gamma_1 = \phi$ and $\varphi \equiv 0$ on Γ_2), we suppose that the necessary condition of existence is satisfied, that is the data are orthogonal with respect to the rigid displacements:

$$\int_{\Omega} f.v dx = \int_{\Gamma} 0.v ds = 0,$$

for any v of the for

$$v(x,y)=\left\{\begin{array}{c} a+cy \\ b-cx \end{array}\right\},$$

with a,b,c are arbitrary real numbers.

3. Study of the case when $F(u)=|u|^{\rho}u$.- In the remaining part of this paper we study with details the last case when $F(u)=|u|^{\rho}u$.

The main result is:

Theorem 3.1.- We suppose that

$$f\in(W^{-1,q}(\Omega))^{n}).$$

Then, there exists a function $u=u(x)$ solution of the problem (P) with:

$$u\in(W_{0}^{1,p}(\Omega))^{n}),$$

Before giving the proof, we make the following remarks:

Remark 3.1.- The space $V=(H_{0}^{1}(\Omega))^{n}\cap(L^{p}(\Omega))^{n}$, where $p=\rho+2,$ is separable i.e., admits a countable dense subset.

In fact, V is identified, by the application

$$v\rightarrow\left\{v,\frac{\partial v}{\partial x_{1}},\frac{\partial v}{\partial x_{2}},...,\frac{\partial v}{\partial x_{n}}\right\},$$

to a closed subspace of

$$(L^{p}(\Omega))^{n}\times(L^{p}(\Omega))^{n}\times...\times(L^{p}(\Omega))^{n}$$

separable and uniformly convex, in such way that it possible to project a countable dense set on this subspace.

Remark 3.3.- The application defined on $(L^{p}(\Omega))^{n}$ by $u\longrightarrow|u|^{p-2}u$, is $(L^{q}(\Omega))^{n}$-valued function, moreover it is continuous. To see that, if $u\in(L^{p}(\Omega))^{n}$, $|u|^{p-2}u$ is measurable and

$$\int_{\Omega}\left||u|^{p-2}u\right|^{q}dx=\int_{\Omega}|u|^{p}dx<\infty\Longrightarrow u\in(L^{q}(\Omega))^{n}.$$

We deduce that for all $u\in(W^{1,p}(\Omega))^{n}$, and for all $1\leq i\leq n$,

$$\left|\frac{\partial u}{\partial x_{i}}\right|^{p-2}\frac{\partial u}{\partial x_{i}}\in(L^{q}(\Omega))^{n}.$$

So, it is possible to define the real-valued mapping a_p from $\left((W_0^{1,p}(\Omega))^n\right)^2 \longrightarrow \mathbb{R}$ by:

$$(u,v) \longmapsto a_p(u,v)$$

where: .

$$a_p(u,v) = 2\,\mu \sum_{i,j=1}^{n} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i} dx + \lambda \int_\Omega div(u) div(v) dx.$$

For any u in $(W_0^{1,p}(\Omega))^n$, the mapping $v \to a_p(u,v)$ is a continuous linear form from $\left((W_0^{1,p}(\Omega))^n\right)^2 \longrightarrow \mathbb{R}$. By Riesz theorem there exists a unique element $A(u)$ of $(W_0^{-1,q}(\Omega))^n$, such that:

$$a(u,v)_p = \langle A(u), v\rangle\,, \forall v \in (W_0^{1,p}(\Omega))^n.$$

The mapping $(W_0^{1,p}(\Omega))^n \longrightarrow (W_0^{-1,q}(\Omega))^n$, $u \longrightarrow A(u)$, is noted:

$$-L_p u = \; -\mu \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) - (\lambda+\mu)\nabla(div(u)),$$

and is called a **p-Lamé operator.**

The following proposition gives some properties of $-L_p$:

Proposition 3.1.- The operator:

$$-L_p : (W_0^{1,p}(\Omega))^n \to (W_0^{-1,q}(\Omega))^n$$

is bounded, hemicontinuous, monotone and coercive

Proof: Using the expression of the norm in dual space dual and Lebesgue's dominated convergence theorem, we prove that $-L_p$ is bounded and hemicontinuous. From the convexity of the real application $t \longrightarrow |t|^p$, we deduce the monotonicity of $-L_p$.

Proposition 3.2.- The problem (P) and the variational problem $(P.V)$:

$$a_p(u,v) + (|u|^{p-2}\,u, v) = (f,v) + (-\varphi(x)(u,v), \forall v \in (W_0^{1,p}(\Omega))^n,$$

are equivalent.

Proof: Indeed, it suffices to observe that $u = 0$ on $\Gamma_1 = 0 \Leftrightarrow u \in (W_0^{1,p}(\Omega))^n$, and the variational equality is then equivalent to

$$-L_p u + |u|^{p-2}\,u = f \quad \text{in } \Omega,$$

because $(D(\Omega))^n$ is dense in $(W_0^{1,p}(\Omega))^n$.

Let us return to the proof of the **theorem 3.1.**

(i) Construction of approximated solutions :
We look for

$$u_m = \sum_{i=1}^{n} \lambda_i v_i$$

solution of the following problem (P_m):
for all $1 \leq j \leq m$:

$$a_p(u_m, v_j) + (|u_m|^{p-2} u_m, v_j) = (f, v_j) + (-\varphi(x)(u_m, v_j)$$

We obtain a second order nonlinear differential system. Let $F : \mathbb{R}^m \longrightarrow \mathbb{R}^m$, be the function defined by:

$$F(\lambda_1, ..., \lambda_m) = \left(\left\langle A(\sum_{i=1}^{n} \lambda_i v_i), v_j \right\rangle - ((f, v_j) + (-\varphi(x)(u_m, v_j)). \right)_{1 \leq j \leq m}$$

(ii) Establishment of priory estimates
· Of the coerciveness of to one deducts that $||u_m||$ is a bounded;
· The operator has a bounded $\Longrightarrow (A(u_m))_{\ m\in\mathbb{N}}$ is a bounded in $V\prime$;
· $\exists u \in V, \exists \chi \in V'$ such that:

$$\begin{cases} u_p \rightharpoonup u, \ \sigma(V, V'), \\ A(u_m) \rightharpoonup \chi, \ \sigma(V', V). \end{cases}$$

(iii) Passage to the limit via compactness
· The monotony and the hemicontinuous imply that $\chi = A(u)$.Thus completes the proof of the **theorem 3.1.**

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