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A New Method for Solving Fuzzy Linear Programming by Solving Linear Programming

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Abstract

Since many real-world engineering systems are too complex to be defined in precise terms, imprecision is often involved in any engineering design process. Fuzzy linear programming problems have an essential role in fuzzy modeling, which can formulate uncertainty in actual environment. One of the most practicable subjects in recent studies is based on LR fuzzy number, which was defined and used by Dubois and Prade with some useful and easy approximation arithmetic operators on them. We use some vector computations on fuzzy vectors, where a fuzzy vector appears as a vector of triangular fuzzy numbers. Here, our main scope is finding some nonnegative fuzzy vector $\tilde{x}$ in which maximizes the objective function $\tilde{z} = c\tilde{x}$ so that $A\tilde{x} = \tilde{b}$, where A and $\tilde{b}$ are a real matrix and a fuzzy vector respectively, and $c_{1\times n}$ is a real vector too.

Keywords: Fuzzy arithmetic, Fuzzy linear programming, Fuzzy number

1 Introduction

Fuzzy set theory has been applied to many disciplines such as control theory and management sciences, mathematical modeling and industrial applications. The concept of fuzzy linear programming (FLP) on general level was first proposed by Tanaka et al. [6]. Afterwards, many authors considered various types of the FLP problems and proposed several approaches for solving these problems. In particular, the most convenient methods are based on the concept of comparison of fuzzy numbers by use of ranking functions [1,4,5]. Usually in such methods authors define a crisp model which is equivalent to the FLP problem and then use optimal solution of the model as the optimal solution of the FLP problem. In [4], by using a general linear ranking function we introduced a dual simplex algorithm for solving linear programming problem with fuzzy variables and its dual, fuzzy number linear programming problem directly. In this paper, we consider a linear programming problem with triangular fuzzy numbers. Our main contribution here is the establishment of a new method for solving the FLP problems without using any ranking function. Moreover, we illustrate our method with an example.

2 Preliminary

In this section we review some necessary backgrounds of the fuzzy theory in which will be used in this paper. Below, we give definitions and notations taken from [2].

2.1. Fuzzy numbers

Definition2.1. A fuzzy number $\tilde{A}$ is a convex normalized fuzzy set on the real line R such that:

1) There exists at least one $x_0 \in R$ with $\mu_{\tilde{A}}(x_0)=1$.
2) $\mu_{\tilde{A}}(x)$ is piecewise continuous.

Let us assume that the membership function of any fuzzy number $\tilde{A}$ is as follows:

$$\mu_{\tilde{A}}(x)=\begin{cases} 1-\dfrac{m^A-x}{\alpha^A}, & m^A-\alpha^A \le x < m^A \\ 1-\dfrac{x-m^A}{\beta^A}, & m^A \le x \le m^A+\beta^A \\ 0, & otherwise \end{cases}$$

where m^A is the mean value of $\tilde{A}$ and α^A and β^A are left and right spreads, respectively and it is termed as **triangular fuzzy number**. We show any triangular fuzzy number by $\tilde{A}=(m^A,\alpha^A,\beta^A)$. Let $F(R)$ be the set of all triangular fuzzy numbers.

Definition2.2. A fuzzy number $\tilde{A}=\{(x,\mu_{\tilde{A}}(x)) \mid x \in R\}$ is nonnegative if and only if $\mu_{\tilde{A}}(x) = 0$ for all $x < 0$ Then a triangular fuzzy number $\tilde{A} = (m^A, \alpha^A, \beta^A)$ is nonnegative if $m^A - \alpha^A \geq 0$.

Definition2.3. Two triangular fuzzy numbers $\tilde{A} = (m^A, \alpha^A, \beta^A)$ and $\tilde{B} = (m^B, \alpha^B, \beta^B)$ are said to be equal if and only if $m^A = m^B, \alpha^A = \alpha^B$ and $\beta^A = \beta^B$.

Definition2.4. A fuzzy number $\tilde{A} = (m^A, \alpha^A, \beta^A)$ is called symmetric, if $\alpha^A = \beta^A$.

2.2. Arithmetic on triangular fuzzy numbers

Let $\tilde{A} = (m^A, \alpha^A, \beta^A)$ and $\tilde{B} = (m^B, \alpha^B, \beta^B)$ be two triangular fuzzy numbers, then arithmetic on them is defined as [2]:

Addition: $\tilde{A} \oplus \tilde{B} = (m^A + m^B, \alpha^A + \alpha^B, \beta^A + \beta^B)$

Scalar multiplication: For any scalar λ, we have

$$\lambda \tilde{A} = \lambda(m^A, \alpha^A, \beta^A) = \begin{cases} (\lambda m^A, \lambda \alpha^A, \lambda \beta^A), & if \ \lambda \geq 0 \\ (\lambda m^A, \lambda \beta^A, \lambda \alpha^A), & if \ \lambda \leq 0 \end{cases}$$

Subtraction: $\tilde{A} - \tilde{B} = (m^A - m^B, \alpha^A + \beta^B, \beta^A + \alpha^B)$

2.3. Nonnegative matrix and nonnegative fuzzy vector

Definition2.5. A matrix A is called nonnegative and denoted by $A \geq 0$ if each element of A be a nonnegative number.

Definition2.6. A fuzzy vector $\tilde{b} = (\tilde{b}_i)_{m \times 1}$ is called nonnegative and denoted by $\tilde{b} \geq 0$, if each element of $\tilde{b}$ be a nonnegative fuzzy, that is $\tilde{b}_i \geq 0$.

3 Fuzzy Linear System of Equations

Definition3.1. Consider the $m \times n$ linear system as:

$$A\tilde{x} = \tilde{b}, \qquad (1)$$

where $A = [a_{ij}]_{m \times n}$ is a nonnegative crisp matrix and $\tilde{x} = (\tilde{x}_j)$, $\tilde{b} = (\tilde{b}_i)$ are nonnegative fuzzy vectors and $\tilde{x}_j, \tilde{b}_j \in F(R)$ for all $1 \leq j \leq n,\ 1 \leq i \leq m$, is called a fuzzy linear system with nonnegative triangular numbers.

Definition3.2. We say a nonnegative fuzzy vector $\tilde{x}$ is the solution of $A\tilde{x} = \tilde{b}$, where A and $\tilde{b}$ are defined in (1), if $\tilde{x}$ satisfies in system.

Now since $\tilde{x} \in F^n(R)$ and $\tilde{b} \in F^m(R)$, we may let $\tilde{x} = (x^m, x^\alpha, x^\beta)$ $\tilde{b} = (b^m, b^\alpha, b^\beta)$ where $x^m, x^\alpha, x^\beta \in R^n$ and $b^m, b^\alpha, b^\beta \in R^m$ Then, we may rewrite the system $A\tilde{x} = \tilde{b}$ as:

$$A(x^m, x^\alpha, x^\beta) = (b^m, b^\alpha, b^\beta), \quad x^m - x^\alpha \geq 0. \qquad (2)$$

In other hand, $\tilde{b}$ and $\tilde{x}$ are two nonnegative fuzzy vectors, hence by use of Definition 2.3 and arithmetic on nonnegative triangular fuzzy number, it is enough to solve the following crisp system:

$$Ax^m = b^m, \quad Ax^\alpha = b^\alpha, Ax^\beta = b^\beta. \qquad (3)$$

Note that if we use from the symmetric triangular fuzzy numbers, then last system $Ax^\beta = b^\beta$ is not necessary to solved, because it is equal to system $Ax^\alpha = b^\alpha$.

3 Fuzzy Linear Programming

Definition 3.1. Consider the following linear programming problem:

$$\max \quad \tilde{z} = c\tilde{x}$$
$$s.t. \quad \begin{cases} A\tilde{x} = \tilde{b} \\ \tilde{x} \geq 0 \end{cases} \qquad (4)$$

where the coefficient matrix $A = [a_{ij}]_{m\times n}$ and the vector $c = (c_1, ..., c_n)$ are a nonnegative crisp matrix and vector respectively, and $\tilde{x} = (\tilde{x}_j)$, $\tilde{b} = (\tilde{b}_i)$, are nonnegative fuzzy vectors such that $\tilde{x}_j, \tilde{b}_i \in F(R)$ for all $1 \leq j \leq n,\ 1 \leq i \leq m$, is called a fuzzy linear programming (FLP) problem.

Definition3.2. We say that a fuzzy vector $\tilde{x}$ is a fuzzy feasible solution of $A\tilde{x} = \tilde{b}, \tilde{x} \geq 0$ where A and $\tilde{b}$ are defined in (4), if $\tilde{x}$ satisfies in system.

Now since $\tilde{x} \in F^n(R)$ and $\tilde{b} \in F^m(R)$, we may let $\tilde{x} = (x^m, x^\alpha, x^\beta)$ $\tilde{b} = (b^m, b^\alpha, b^\beta)$, where $x^m, x^\alpha, x^\beta \in R^n$ and $b^m, b^\alpha, b^\beta \in R^m$. Then, we may rewrite the system $A\tilde{x} = \tilde{b}$ as:

$$A(x^m, x^\alpha, x^\beta) = (b^m, b^\alpha, b^\beta). \qquad (5)$$

In other hand, $\tilde{b}$ and $\tilde{x}$ are two nonnegative fuzzy vectors, hence by use of Definition 2.3 and arithmetic on nonnegative triangular fuzzy numbers, it is equivalent to the following crisp system:

$$Ax^m = b^m, \quad Ax^\alpha = b^\alpha, Ax^\beta = b^\beta, \quad x^m - x^\alpha \geq 0. \qquad (6)$$

Now we define an operator "max" for a fuzzy linear function which is defined as:

$$\tilde{z} = f(\tilde{x}_1, ..., \tilde{x}_n) = \sum_{j=1}^{n} c_j \tilde{x}_j = c_1 \tilde{x}_1 \oplus ... \oplus c_n \tilde{x}_n,$$

where $c_j, j = 1,...,n$, are real numbers and $\tilde{x}_j \in F(R), j = 1,...,n.$

Definition3.3. Let $\tilde{x} = (\tilde{x}_1, ..., \tilde{x}_n)^T$ and $\tilde{b} = (\tilde{b}_1, ..., \tilde{b}_m)^T$ be two nonnegative fuzzy vectors, where $\tilde{x}_j = (x_j, \underline{x_j}, \overline{x_j})$ and $\tilde{b}_j = (b_i, \underline{b_i}, \overline{b_i}) \in F(R)$. A fuzzy vector $\tilde{x}$ maximizes the linear function $\tilde{z} = f(\tilde{x}_1, ..., \tilde{x}_n)$, such that

$$\begin{aligned} &A\tilde{x} = \tilde{b} \\ &\tilde{x} \geq 0, \end{aligned} \qquad (7)$$

where $\tilde{x}_j = (x_j, \underline{x_j}, \overline{x_j}), j = 1,...,n,$ and $x_j, \underline{x_j}, \overline{x_j}, j = 1,...,n,$ are nonnegative real numbers, if and only if $x = (x_1, \underline{x_1}, \overline{x_1}, x_2, \underline{x_2}, \overline{x_2}, ..., x_n, \underline{x_n}, \overline{x_n}) \in R^{3n}$ maximizes the below real function:

$$z = \sum_{c_j > 0} c_j (x_j + \frac{1}{2}(\overline{x_j} - \underline{x_j})) + \sum_{c_j < 0} c_j (x_j + \frac{1}{2}(\underline{x_j} - \overline{x_j})) \qquad (8)$$

such that

$$Ax = b, A\underline{x} = \underline{b}, A\overline{x} = \overline{b}, x - \underline{x} \geq 0, x, \underline{x}, \overline{x} \geq 0. \qquad (9)$$

Here we give an illustrate example.

Example 4.1. Assume that a company makes two products. Product $P1$ has a 40$ per unit profit and product $P2$ has a 30$ per unit profit. Each unit of product $P1$ requires twice as many labor hours as each product $P2$. The total available labor hours are somewhat close to 500 hours per day, and may possibly be changed due to special arrangements for overtime work. The supply of material is almost 400 units of both products, $P1$ and $P2$, per day, but may possibly be changed according to previous experience. The problem is how many units of products $P1$ and $P2$ should be made per day to maximize the total profit? Let

$\tilde{x}_1$ and $\tilde{x}_2$ denote the number of units of products $P1$ and $P2$ made in one day, respectively. Then the problem can be formulated as the following linear programming with triangular fuzzy variables problem.

$$\max \quad \tilde{z} = 40\tilde{x}_1 \oplus 30\tilde{x}_2$$

$$s.t. \quad \begin{cases} \tilde{x}_1 \oplus \tilde{x}_2 \leq \tilde{400}, \\ 2\tilde{x}_1 \oplus \tilde{x}_2 \leq \tilde{500}, \\ \tilde{x}_1, \tilde{x}_2 \geq 0. \end{cases} \qquad (10)$$

The supply of material and the available labor hours are close to 400 and 500, and hence are modeled as $(400,5,5)$ and $(500,7,7)$, respectively. Now the current fuzzy linear programming model may be written in the standard form as follows:

$$\max \quad \tilde{z} = 40\tilde{x}_1 \oplus 30\tilde{x}_2$$

$$s.t. \quad \begin{cases} \tilde{x}_1 \oplus \tilde{x}_2 \oplus \tilde{x}_3 = (400,5,5), \\ 2\tilde{x}_1 \oplus \tilde{x}_2 \oplus \tilde{x}_4 = (500,7,7), \\ \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4 \geq 0. \end{cases} \qquad (11)$$

where $\tilde{x}_3$ and $\tilde{x}_4$ are two slack variables.

Hence, the equivalent fuzzy linear programming problem as follows:

$$\max \quad \tilde{z} = 40(x_1, \underline{x_1}, \overline{x_1}) \oplus 30(x_2, \underline{x_2}, \overline{x_2})$$

$$s.t. \quad \begin{cases} (x_1, \underline{x_1}, \overline{x_1}) \oplus (x_2, \underline{x_2}, \overline{x_2}) \oplus (x_3, \underline{x_3}, \overline{x_3}) = (400,5,5), \\ 2(x_1, \underline{x_1}, \overline{x_1}) \oplus (x_2, \underline{x_2}, \overline{x_2}) \oplus (x_4, \underline{x_4}, \overline{x_4}) = (500,7,7), \\ (x_1, \underline{x_1}, \overline{x_1}), (x_2, \underline{x_2}, \overline{x_2}), (x_3, \underline{x_3}, \overline{x_3}), (x_4, \underline{x_4}, \overline{x_4}) \geq 0. \end{cases}$$

Since the fuzzy numbers are symmetric, therefore it is enough to solve the following fuzzy linear programming problem:

$$\max \quad z = 40x_1 + 30x_2$$

$$s.t. \quad \begin{cases} (x_1, \underline{x_1}, \overline{x_1}) \oplus (x_2, \underline{x_2}, \overline{x_2}) \oplus (x_3, \underline{x_3}, \overline{x_3}) = (400,5,5), \\ 2(x_1, \underline{x_1}, \overline{x_1}) \oplus (x_2, \underline{x_2}, \overline{x_2}) \oplus (x_4, \underline{x_4}, \overline{x_4}) = (500,7,7), \\ (x_1, \underline{x_1}, \overline{x_1}), (x_2, \underline{x_2}, \overline{x_2}), (x_3, \underline{x_3}, \overline{x_3}), (x_4, \underline{x_4}, \overline{x_4}) \geq 0. \end{cases}$$

Now we can obtain an optimal fuzzy solution for problem (10) by solving the following linear programming:

$$\max \quad z = 40x_1 + 30x_2$$

$$s.t. \quad \begin{cases} x_1 + x_2 + x_3 = 400, \\ 2x_1 + x_2 + x_4 = 500, \\ \underline{x_1} + \underline{x_2} + \underline{x_3} = 5, \\ 2\underline{x_1} + \underline{x_2} + \underline{x_4} = 7, \\ x_i - \underline{x_i} \geq 0, x_i \geq 0, \underline{x_i} \geq 0, i = 1,\ldots,4. \end{cases}$$

The optimal solution of the above linear programming is: $x_1 = 100, x_2 = 300, x_3 = 0, x_4 = 0, x_1 = 2, x_2 = 3, x_3 = 0, x_4 = 0$. Therefore, the optimal fuzzy solution of the problem (10) is: $\tilde{x}_1^* = (100,2,2), \tilde{x}_2^* = (300,3,3), \tilde{x}_3^* = (0,0,0), \tilde{x}_4^* = (0,0,0)$, and the optimal fuzzy value of the objective function is: $\tilde{z}^* = 40\tilde{x}_1^* \oplus 30\tilde{x}_2^* = (13000,170,170) = \widetilde{13000}$.

5 Conclusion

In this paper, we proposed a new method for solving the FLP problems by solving the classical linear programming problems where we know how to solve them. The significance of this paper is providing a new method for solving the fuzzy linear programming without using any ranking function. Moreover, this paper will be useful for future works on fuzzy linear programming and will be useful to study on some fundamental concepts of the fuzzy linear programming and in particular, investigating on duality results and more important on sensitivity analysis.

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