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The Effect of a Thin Layer on a Nonlinear Thermoelastic Plate

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Abstract

We consider a model of dynamic nonlinear elasticity in the presence of thermal effects, for a plate surrounded by a thin layer. This model, which describes the nonlinear oscillations of a plate subjected to thermal effects is reffered as " full von Karman thermoelastic system". We apply the formal asymptotic expansions method to establish approximate boundary conditions that model the effect of the thin layer. The results obtained in this paper extend those obtained earlier in [10], [9] for a non linear model which does not account for the thermal effects.

Mathematics subject Classification: 74K20; 35C20

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1 Introduction

The mathematical modelling of elastic bodies covered with thin layers is a problem of outstanding practical importance. However, from a numerical point of view, such structures require the discretisation of the thin layer, which needs very thin meshes and may lead to very expansive and difficult computations. An alternative approach consists in deriving approximate boundary conditions, incorporating in an approximate way the effect of the thin layer. More precisely, we seek an approximate problem posed on the interior domain (i.e, not including the thin layer) but taking into account its effect via these new conditions. The idea of introducing this type of boundary conditions which can be substituted to the thin layer has been widely used in numerous studies (see [1], [3], [5], [8], [9], [10], [11]). The purpose of this paper is to identify approximate boundary conditions for a non linear elastic plate surrounded by a thin elastic layer and subjected to thermal effects. This study extends the results obtained by the author in [10], where a non linear model which does not account for the thermal effects was considered.

Here, we shall briefly outline the strategy to be followed. First, we write the problem in a variational form. Making a change of scaling along the thickness of the thin layer, in order to have a problem posed over a set that does not depend on δ, we obtain an equivalent variational problem for which we apply the formal asymptotic expansions method to establish approximate boundary conditions. The idea is to approximate the solution by the series given by its asymptotic expansion truncated at a given order. The conditions satisfied by this approximation on the common boundary of the plate and the layer give the desired boundary conditions. When considering the asymptotic expansion truncated at a order 0, we obtain a model where the effect of the thin layer is completely neglected. This approximation is not interesting since our aim is to obtain a model that incorporates this effect. So, we go further in the asymptotic expansion until order 1 and give a model where the effect of the thin layer is taken into account and is completely embodied by new boundary conditions. Indeed, these conditions depend on the layer's elastic and thermal characteristics and of its thickness δ.

2 Setting of the problem

Let Ω_+ be a bounded open subset of R^2. The boundary of Ω_+ consists of two disjoints parts, Γ_0 and Γ, assumed to be smooth. For $\delta > 0$ sufficiently small, the elastic layer Ω_-^δ derives from a uniform dilation of Γ_0 in the normal direction, with thickness δ:

$$\Omega_-^\delta = \{x + r\,\nu\,;\ x \in \Gamma_0 \text{ and } 0 < r < \delta\},$$

where ν denotes the normal vector at point x on Γ_0, outer from Ω_+; the external boundary of the domain Ω_-^δ is Γ_δ and the whole domain is $\Omega^\delta = \Omega_+ \cup \Gamma_0 \cup \Omega_-^\delta$ (see figure 1). We consider the thermoelastic full von Karman system (1)-(8) for the whole bidimensionel plate Ω^δ, which consists of an elastodynamic system coupled with the Kirchhoff-love equation and two heat equations (see [2],[7]) :

$$\rho u^{''} - div\{C[\varepsilon(u) + f(\nabla w)]\} + \lambda \nabla \phi = 0 \quad in\ \ \Omega^\delta \times (0, T\ ), \tag{1}$$

$$\rho[I - \Delta]w^{''} + D\Delta^2 w - div\{C[\varepsilon(u) + f(\nabla w)]\nabla w\} + \lambda \Delta \theta = 0 \quad in\ \Omega^\delta \times (0, T), \tag{2}$$

$$\rho \phi' - k\Delta\phi + \lambda div u' = 0 \qquad in\ \Omega^\delta \times (0, T), \tag{3}$$

$$\rho \theta' - k\Delta\theta - \lambda \Delta w' = 0 \qquad in\ \Omega^\delta \times (0, T), \tag{4}$$

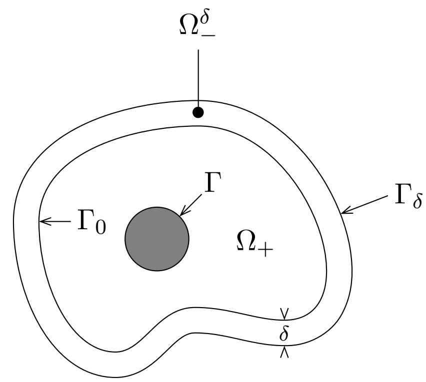

Figure 1: The plate and the thin layer

with Dirichlet conditions on Γ

$$u = 0,\ w = \partial_\nu w = 0,\ \theta = 0,\ \phi = 0 \quad on\ \Gamma \times (0, T), \tag{5}$$

and free boundary conditions on $\Gamma_\delta \times (0, T)$

$$\begin{aligned} C[\varepsilon(u) + f(\nabla w)]\nu &= 0\ ;\ D\left[\Delta w + (1-\mu)B_1 w\right] = 0\ , \\ k\partial_\nu\theta + \lambda\partial_\nu w' &= 0\ ;\ k\partial_\nu\phi - \lambda u'\nu = 0, \end{aligned}$$

$$D\left[\partial_\nu\Delta w + (1-\mu)\partial_s B_2 w\right] - \rho\partial_\nu w'' - C[\varepsilon(u) + f(\nabla w)]\nu.\nabla w + \lambda\partial_\nu\theta = 0. \tag{6}$$

We define also the transmission conditions on $\Gamma_0 \times (0, T)$ by

$$\begin{gathered} [[u]] = [[\phi]] = [[\theta]] = 0,\ \ [[w]] = [[\partial_\nu w]] = 0\ ; \\ [[C[\varepsilon(u) + f(\nabla w)]\nu]] = 0\ ;\ \ [[D[\Delta w + (1-\mu)B_1 w]\ ]] = 0\ ; \\ [[\ k\partial_\nu\theta + \lambda\partial_\nu w']] = 0\ ;\ \ [[k\partial_\nu\phi - \lambda u'\nu]] = 0\ ; \\ \left[\left[D\left[\partial_\nu\Delta w + (1-\mu)\partial_s B_2 w\right] - \rho\partial_\nu w'' - C[\varepsilon(u) + f(\nabla w)]\nu.\nabla w + \lambda\partial_\nu\theta\right]\right] = 0. \end{gathered} \tag{7}$$

With (1) and (4) we associate the initial conditions

$$u(0) = u_0\ ,\ u'(0) = u_1,\ w(0) = w_0\ ,\ w'(0) = w_1\ \theta(0) = \theta_0\ ,\ \phi(0) = \phi_0 \ \text{ in } \Omega^\delta. \tag{8}$$

The variables w and $u = (u_1, u_2)$ represent respectively the vertical and in-plane displacement of the plate, while θ and ϕ describe the temperature affecting the vertical displacement and the horizontal (in-plane) displacement, respectively. By [[ ]] we denote the jump through Γ_0 of a function or distribution defined on Ω^δ that admits in some sens traces on Γ_0. The fourth order tensor C belongs to S the space of 2×2 symmetric matrices, and it is defined by

$$C(\zeta) = \frac{E}{(1-\mu^2)}\left[\mu(tr\zeta)I_S + (1-\mu)\zeta\right],$$

for any ζ in S, where I_S is the identity matrix and $(tr\zeta)$ is the trace of ζ. Moreover, the strain tensor is given by $\epsilon(u) = 1/2\left(\nabla u + \nabla^T u\right)$. The function f is given by $f(s) = (1/2)s \otimes s,\ s \in \mathcal{R}^2$ and the boundary operators are defined by

$$B_1 w \equiv 2\nu_1\nu_2\partial^2_{xy}w - \nu_1^2\partial_y^2 w - \nu_2^2\partial_x^2 w \ \ ,\ B_2 w \equiv (\nu_1^2 - \nu_2^2)\partial^2_{xy}w + \nu_1\nu_2(\partial_y^2 w - \partial_x^2 w).$$

$D = \frac{E}{(1-\mu^2)}$ represents the flexural rigidity of the plate ; E is the young's modulus, μ is the poisson ratio of the material and ρ is its mass density. k is the coefficient of thermal conductivity and $\lambda = \frac{D}{2}(\alpha(1+\mu))$, where α denotes the coefficient of thermal expansion. We assume that $E > 0,\ 0 < \mu < 1/2$ and that the coefficients described above are independent of δ and are piecewise constant: $E = E_+$ in Ω_+ and E_- in Ω_-^δ ; $\mu = \mu_+$ in Ω_+ and μ_- in Ω_-^δ ; $\rho = \rho_+$ in Ω_+ and ρ_- in Ω_-^δ ; $k = k_+$ in Ω_+ and k_- in Ω_-^δ ; $\alpha = \alpha_+$ in Ω_+ and α_- in Ω_-^δ .

Setting

$$\begin{aligned} W(\Omega^\delta) &= \left\{w \in H^2(\Omega^\delta)\ ;\ w_{|\ \Gamma} = \partial_\nu w\ |_\Gamma = 0\ \right\}, \\ V(\Omega^\delta) &= \left\{w \in H^1(\Omega^\delta)\ ;\ w_{|\ \Gamma} = 0\ \right\}, \\ U(\Omega^\delta) &= \left\{u \in \left(H^1(\Omega^\delta)\right)^2\ ;\ u_{|\Gamma} = 0\ \right\}, \end{aligned}$$

and denotting by $\langle , \rangle_D$ the scalar product in $\left[L^2(D)\right]^k,\ k \in N$, the variational formulation of the problem cited above reads :

$$\left\{ \begin{array}{l} \rho\left[\langle u', \varphi\rangle_{\Omega^\delta}\right]' + \langle C[\epsilon(u) + f(\nabla w)], \epsilon(\varphi)\rangle_{\Omega^\delta} + \lambda\langle\nabla\phi, \varphi\rangle_{\Omega^\delta} = 0,\ \forall\varphi \in U(\Omega^\delta) \\ \qquad \rho\left[\langle w', \psi\rangle_{\Omega^\delta} + \langle\nabla w', \nabla\psi\rangle_{\Omega^\delta}\right]' + a(w,\psi) + \\ \qquad \langle C[\epsilon(u) + f(\nabla w)]\nabla w, \nabla\psi\rangle_{\Omega^\delta} - \lambda\langle\nabla\theta, \nabla\psi\rangle_{\Omega^\delta} = 0,\ \forall\psi \in W(\Omega^\delta), \\ \rho\langle\phi, \zeta\rangle'_{\Omega^\delta} + k\langle\nabla\phi, \nabla\zeta\rangle_{\Omega^\delta} - \lambda\langle u', \nabla\zeta\rangle_{\Omega^\delta} = 0 \qquad \forall\zeta \in V(\Omega^\delta), \\ \rho\langle\theta, \eta\rangle'_{\Omega^\delta} + k\langle\nabla\theta, \nabla\eta\rangle_{\Omega^\delta} + \lambda\langle\nabla w', \nabla\eta\rangle_{\Omega^\delta} = 0 \qquad \forall\eta \in V(\Omega^\delta) \end{array} \right. \tag{9}$$

where:

$$a(w,\psi) = D\int_{\Omega^\delta}\left\{\left(\partial_x^2 w + \mu\partial_y^2 w\right)\partial_x^2\psi + 2(1-\mu)\partial^2_{xy}w\partial^2_{xy}\psi + \left(\partial_y^2 w + \mu\partial_x^2 w\right)\partial_y^2\psi\right\}d\Omega^\delta.$$

2.1 The scaling problem

Depending on the thickness δ, the functional setting of our problem is not suited for giving a precise meaning to an asymptotic expansion of the solution. Hence, the first step of the analysis is a scaling inside the thin layer in order to remove the dependence of the space domain on the small parameter δ. So, we perform a dilation in the normal direction of the layer Ω_-^δ (of ratio δ^{-1}) to get a fixed geometry. Let ν be the inner unit normal to Γ_0 and τ be the tangent unit vector field to Γ_0 such that the basis (τ, ν) is direct in each point of Γ_0. Denote by s a curvilinear abscissa (arc length) along Γ_0 oriented according to τ. Thus, the scaling is given by a parametrisation of the thin shell Ω_-^δ by the manifold $\Omega_- = \Gamma_0 \times (0, 1)$ through the mapping

$$\left\{ \begin{array}{l} \Omega_- \longrightarrow \Omega^\delta \\ (s, z) \longrightarrow X = s + \delta z \nu(s) \end{array} \right. \tag{10}$$

We identify Γ_0 with $\Gamma_0 \times \{0\}$ and we set $\Gamma_- = \Gamma_0 \times \{1\}$ and $\Omega = \Omega_+ \cup \Gamma_0 \cup \Omega_-$. To each function ψ defined on Ω_-^δ is associated a function ψ^δ defined in Ω_- through the variable change (10) by

$$\psi^\delta(s, z) := \psi(X)$$

Denoting by $R = R(s)$ the curvature of Γ_0 at s, we have the Frenet's relations

$$\partial_s \nu = -R\tau \qquad \text{and} \qquad \partial_s \tau = R\nu.$$

Thus, relying on the relation $x = s + \delta z \nu(s)$, we obtain

$$\partial_z = \nu_1 \partial_x + \nu_2 \partial_y \qquad \text{and} \qquad \partial_s = (1 - Rz)\left(\nu_2 \partial_x - \nu_1 \partial_y\right).$$

In Ω_-^δ, a given vector field φ will be decomposed into normal and tangential components : $\varphi = \varphi_\tau \tau + \varphi_\nu \nu$. By the scaling (10), φ is transformed into

$$\varphi^\delta(s, z) = \varphi_\tau(s, \delta z)\tau + \delta \varphi_\nu(s, \delta z)\nu.$$

Likewise, we also express the integrals involved in (9) by :

$$\int_{\Omega_-^\delta} \psi d\Omega_-^\delta = \delta \int_\Sigma \int_0^1 \widehat{\psi}\,(1 - R\delta z) ds dz.$$

The problem (9) is then transformed into an equivalent problem posed over a set independent of δ . The details of this transformation being otherwise too long and very technical, are omitted (the reader is referred to [9], where explicit details are given for the full von karman system without thermal effects).

3 Approximate boundary conditions

The identification of the approximate boundary conditions relies on the application of the formal asymptotic expansion method. The idea consists in approximating the solution by the series given by its asymptotic expansion truncated at a given order. The conditions given by this approximation on Γ_0 give the desired boundary conditions.

The standard asymptotic expansion method leads to write the solution $(u^\delta, w^\delta, \phi^\delta, \theta^\delta)$ of the scaled problem as a formal expansion with respect to δ :

$$\begin{aligned} u^\delta &= u^0 + \delta u^1 + \delta^2 u^2 + \delta^3 u^3 + \dots\dots\dots\dots \\ w^\delta &= w^0 + \delta w^1 + \delta^2 w^2 + \delta^3 w^3 + \dots\dots\dots\dots \\ \phi^\delta &= \phi^0 + \delta \phi^1 + \delta^2 \phi^2 + \delta^3 \phi^3 + \dots\dots\dots\dots \\ \theta^\delta &= \theta^0 + \delta \theta^1 + \delta^2 \theta^2 + \delta^3 \theta^3 + \dots\dots\dots\dots \end{aligned} \tag{11}$$

Hereafter, we will use the index + (resp. $-$) to denote the restriction of the different terms of the asymptotic expansion or the data to Ω_+ (resp.Ω_-). Moreover, we make the following assumption on the data : we suppose that there exists smooth enough functions $w_+^*,\ w_+^{**},\ w_+^{***},\ u_+^*,\ u_+^{**},\ u_+^{***},\ \theta_+^*$ et ϕ_+^* independent of δ, such that :

$$\begin{aligned} w_{0+}^\delta &= w_+^*, w_{0-}^\delta = w_+^* \mid_{\Gamma_0}, w_{1+}^\delta = w_+^{**}, w_{1-}^\delta = w_+^{**} \mid_{\Gamma_0}, \partial_z w_{1-}^\delta = \delta \overset{***}{w}_+, \\ u_{0+}^\delta &= u_+^*, (u_{0\tau}^\delta)_- = (u_\tau^*)_+ \mid_{\Gamma_0}, (u_{0\nu}^\delta)_- = 0, u_{1+}^\delta = u_+^{**}, \left((u_{1\tau}^\delta)_-, \frac{1}{\delta}(u_{1\nu}^\delta)_- \right) = u_+^{***}, \\ \theta_{0+}^\delta &= \theta_+^*,\ \phi_{0+}^\delta = \phi_+^*\ , \theta_{0-}^\delta = \theta_+^* \mid_{\Gamma_0}, \phi_{0-}^\delta = \phi_+^* \mid_{\Gamma_0}\ . \end{aligned}$$

According to the basic Ansatz of the method of formal asymptotic expansions, we expand the forms involved in the variational scaled problem in powers of δ and insert the expansions (11) into the equations obtained. We then identify the successive terms $(u^p, w^p, \phi^p, \theta^p),\ p \geqslant 0$ by equating to zero the factors of the successive powers of δ. In so doing, we obtain a hierarchy of variational equations. This leads to the identification of the problem solved by the first term of the asymptotic expansion $(u^0, w^0, \phi^0, \theta^0)$, that is :

$$\left\{ \begin{array}{l} \rho_+ \left[\left\langle (u_+^0)', \varphi \right\rangle_{\Omega+} \right]' + \rho_+ \left[\left\langle (w_+^0)', \psi \right\rangle_{\Omega+} \right]' + \rho_+ \left[\left\langle (\nabla w_+^0)', \nabla \psi \right\rangle_{\Omega+} \right]' + \\ a_+ \left(w_+^0, \psi \right) + \lambda_+ \left\langle \nabla \phi_+^0, \varphi \right\rangle_{\Omega^\delta} + N_+(u_+^0, w_+^0, \varphi, \psi) - \lambda_+ \left\langle \nabla \theta_+^0, \nabla \psi \right\rangle_{\Omega^\delta} \\ +\rho_+ \left\langle \phi_+^0, \zeta \right\rangle_{\Omega^\delta}' + k_+ \left\langle \nabla \phi_+^0, \nabla \zeta \right\rangle_{\Omega^\delta} - \lambda_+ \left\langle (u_+^0)', \nabla \zeta \right\rangle_{\Omega^\delta} + \rho_+ \left\langle \theta_+^0, \eta \right\rangle_{\Omega^\delta}' \\ +k_+ \left\langle \nabla \theta_+^0, \nabla \eta \right\rangle_{\Omega^\delta} + \lambda_+ \left\langle (\nabla w_+^0)', \nabla \eta \right\rangle_{\Omega^\delta} = 0, \end{array} \right. \tag{12}$$

$\forall (\varphi, \psi, \zeta, \eta) \in U^\delta(\Omega_+) \times W^\delta(\Omega_+) \times V^\delta(\Omega_+) \times V^\delta(\Omega_+)$, with initial conditions

$$u_+^0(0) = u_+^*, (u_+^0)'(0) = u_+^{**}, w_+^0(0) = w_+^*, (w_+^0)'(0) = w_+^{**},\quad \phi_+^0(0) = \phi_+^*,\ \theta_+^0(0) = \theta_+^* \quad in\ \Omega_+$$

where

$$N_+(u_+^0, w_+^0, \varphi, \psi) = \int_{\Omega_+} \left\{ C[\epsilon(u_+^0) + f(\nabla w_+^0)].\epsilon(\varphi) + C[\epsilon(u_+^0) + f(\nabla w_+^0)]\nabla w_+^0.\nabla\psi \right\} d\Omega_+.$$

Remark. The problem (12) solved by the first term $(u^0, w^0, \phi^0, \theta^0)$ which corresponds to the approximate problem of order 0, is nothing but the variational form of the full von Karman thermoelastic system posed over the set Ω_+. This model is simply obtained by omitting the thin layer. In other words, the effect of the thin layer is not seen at order 0. Since our aim is to obtain an approximate problem that incorporates this effect, we must go further in the asymptotic expansion and derive the conditions of order 1. To this end, we keep the first two terms of the expansions and define $w^{[1]}$, $u^{[1]}$, $\theta^{[1]}$, $\phi^{[1]}$, as $w^{[1]} = w^0 + \delta w^1$, $u^{[1]} = u^0 + \delta u^1, \theta^{[1]} = \theta^0 + \delta\theta^1,\ \phi^{[1]} = \phi^0 + \delta\phi^1$.

Setting

$$\begin{aligned}
\tilde{U}(\Omega_+) &= \left\{ u \in \left(H^1(\Omega_+)\right)^2 \ ;\ u_{|\Gamma} = 0,\ u_{\tau|\Gamma_0} \in H^1(\Gamma_0) \right\}, \\
\tilde{V}(\Omega_+) &= \left\{ w \in H^1(\Omega_+) \ ;\ w_{|\Gamma} = 0 \ ;\ w_{|\Gamma_0} \in H^1(\Gamma_0) \right\}, \\
\tilde{W}(\Omega_+) &= \left\{ w \in H^2(\Omega_+) \ ; w_{|\Gamma} = \partial_\nu w \mid_\Gamma = 0 \ ;\ w_{|\Gamma_0} \in H^2(\Gamma_0),\ \partial_\nu w \mid_{\Gamma_0} \in H^1(\Gamma_0) \right\}
\end{aligned}$$

and denoting $\boldsymbol{\gamma}_T(\psi) = \partial_s^2\psi - R(s)\partial_\nu\psi$, $\boldsymbol{\gamma}_S(\psi) = -\partial_s\partial_\nu\psi - R(s)\partial_s\psi$, $N_T(\varphi, \psi) = \partial_s\varphi_\tau - R(s)\varphi_\nu + \frac{1}{2}(\partial_s\psi)^2$, we obtain that $(u_+^{[1]}, w_+^{[1]}, \phi_+^{[1]}, \theta_+^{[1]})$ solves the following problem

$$\left\{\begin{array}{l}
\rho_+ \left[\left\langle (u_+^{[1]})', \varphi \right\rangle_{\Omega+} + \left\langle (w_+^{[1]})'_+, \psi \right\rangle_{\Omega+} + b_+ \left((w_+^{[1]})', \psi \right) + \left\langle \phi_+^{[1]}, \zeta \right\rangle_{\Omega+} + \left\langle \theta_+^{[1]}, \eta \right\rangle_{\Omega+} \right]' + \\
N_+ \left(u_+^{[1]}, w_+^{[1]}, \varphi, \psi \right) + a_+ \left(w_+^{[1]}, \psi \right) + k_+ \left\langle \nabla\phi_+^{[1]}, \nabla\zeta \right\rangle_{\Omega+} + k_+ \left\langle \nabla\theta_+^{[1]}, \nabla\eta \right\rangle_{\Omega+} + \\
\lambda_+ \left[\left\langle \nabla\phi_+^{[1]}, \varphi \right\rangle_{\Omega+} + \left\langle \nabla\theta_+^{[1]}, \nabla\psi \right\rangle_{\Omega+} - \left\langle (u_+^{[1]})', \nabla\zeta \right\rangle_{\Omega+} + \left\langle (\nabla w_+^{[1]})', \nabla\eta \right\rangle_{\Omega+} \right] + \\
\boldsymbol{\delta} \left\{ \rho_- \left[\left\langle (u_+^{[1]})', \varphi \right\rangle_{\Gamma_0} + \left\langle (w_+^{[1]})'_+, \psi \right\rangle_{\Gamma_0} + b_{\Gamma_0} \left((w_+^{[1]})', \psi \right) + \left\langle \phi_+^{[1]}, \zeta \right\rangle_{\Gamma_0} + \left\langle \theta_+^{[1]}, \eta \right\rangle_{\Gamma_0} \right]' + \right. \\
a_{\Gamma_0}(w_+^{[1]}, \psi) + N_{\Gamma_0}(u_+^{[1]}, w_+^{[1]}, \varphi, \psi) - \lambda_- \left\langle \partial_s\phi_+^{[1]}, \varphi_\tau \right\rangle_{\Gamma_0} + \frac{\lambda_-^2}{k_-} \left\langle \left(u_{+\nu}^{[1]} \right)', \varphi_\nu \right\rangle_{\Gamma_0} \\
-\lambda_- \left\langle \partial_s\theta_+^{[1]}, \partial_s\psi \right\rangle_{\Gamma_0} + k_- \left\langle \partial_s\phi_+^{[1]}, \partial_s\zeta \right\rangle_{\Gamma_0} + \frac{\lambda_-^2}{k_-} \left\langle \left(\partial_\nu w_+^{[1]} \right)', \partial_\nu\psi \right\rangle_{\Gamma_0} + \\
\left. k_- \left\langle \partial_s\theta_+^{[1]}, \partial_s\eta \right\rangle_{\Gamma_0} + \lambda_- \left\langle \partial_s \left(w_+^{[1]} \right)', \partial_s\eta \right\rangle_{\Gamma_0} + \lambda_- \left\langle \left(u_{+\tau}^{[1]} \right)', \partial_s\zeta \right\rangle_{\Gamma_0} \right\} = O(\delta^2),
\end{array}\right. \tag{13}$$

$\forall(\varphi,\psi,\zeta,\eta)\in\tilde{U}(\Omega_+)\times\tilde{W}(\Omega_+)\times\tilde{V}(\Omega_+)\times\tilde{V}(\Omega_+)$, where

$$\begin{aligned}
a_{\Gamma_0}(w,\psi) &= \int_{\Gamma_0} E_-\left(\boldsymbol{\gamma}_T(w)\boldsymbol{\gamma}_T(\psi)+\frac{1}{(1+\mu_-)}\boldsymbol{\gamma}_S(w)\boldsymbol{\gamma}_S(\psi)\right)ds,\\
b_{\Gamma_0}(w,\psi) &= \int_{\Gamma_0}(\partial_s w\partial_s\psi+\partial_\nu w\partial_\nu\psi)\,ds,\\
N_{\Gamma_0}(u,w,\varphi,\psi) &= E_-\int_{\Gamma_0} N_T(u,w)(\partial_s\varphi_\tau-R(s)\varphi_\nu+\partial_s w\partial_s\psi)ds.
\end{aligned}$$

The problem solved by $\left(w^{[1]},u^{[1]},\theta^{[1]},\phi^{[1]}\right)$ suggests to approximate the interior part of the solution of our initial problem by seeking a solution $\left(\widetilde{u}_+,\widetilde{w}_+,\tilde{\phi}_+,\tilde{\theta}_+\right)$ to the problem obtained by taking the right-hand side of the problem (13) equal to 0. Hence, neglecting the $O(\delta^2)$ term in (13) and using the assumptions on the initial conditions, we are led to the approximate problem of order 1 :

$$\left\{\begin{array}{l}
\rho_+\left[\langle(\widetilde{u}_+)',\varphi\rangle_{\Omega+}+\left\langle(\widetilde{w}_+)'_+,\psi\right\rangle_{\Omega+}+b_+\left((\widetilde{w}_+)',\psi\right)+\left\langle\tilde{\phi}_+,\zeta\right\rangle_{\Omega+}+\left\langle\tilde{\theta}_+,\eta\right\rangle_{\Omega+}\right]'+\\
N_+\left(\widetilde{u}_+,\widetilde{w}_+,\varphi,\psi\right)+a_+\left(\widetilde{w}_+,\psi\right)+k_+\left\langle\nabla\tilde{\phi}_+,\nabla\zeta\right\rangle_{\Omega+}+k_+\left\langle\nabla\tilde{\theta}_+,\nabla\eta\right\rangle_{\Omega+}+\\
\lambda_+\left[\left\langle\nabla\tilde{\phi}_+,\varphi\right\rangle_{\Omega+}+\left\langle\nabla\tilde{\theta}_+,\nabla\psi\right\rangle_{\Omega+}-\left\langle\widetilde{u}'_+,\nabla\zeta\right\rangle_{\Omega+}+\left\langle\nabla\widetilde{w}'_+,\nabla\eta\right\rangle_{\Omega+}\right]+\\
\boldsymbol{\delta}\left\{\rho_-\left[\langle(\widetilde{u}_+)',\varphi\rangle_{\Gamma_0}+\left\langle(\widetilde{w}_+)'_+,\psi\right\rangle_{\Gamma_0}+b_{\Gamma_0}\left((\widetilde{w}_+)',\psi\right)+\left\langle\tilde{\phi}_+,\zeta\right\rangle_{\Gamma_0}+\left\langle\tilde{\theta}_+,\eta\right\rangle_{\Gamma_0}\right]'+\right.\\
a_{\Gamma_0}(\widetilde{w}_+,\psi)+N_{\Gamma_0}(\widetilde{u}_+,\widetilde{w}_+,\varphi,\psi)-\lambda_-\left\langle\partial_s\tilde{\phi}_+,\varphi_\tau\right\rangle_{\Gamma_0}+\frac{\lambda_-^2}{k_-}\left\langle(\widetilde{u}_{+\nu})',\varphi_\nu\right\rangle_{\Gamma_0}\\
-\lambda_-\left\langle\partial_s\tilde{\theta}_+,\partial_s\psi\right\rangle_{\Gamma_0}+k_-\left\langle\partial_s\tilde{\phi}_+,\partial_s\zeta\right\rangle_{\Gamma_0}+\frac{\lambda_-^2}{k_-}\left\langle(\partial_\nu\widetilde{w}_+)',\partial_\nu\psi\right\rangle_{\Gamma_0}+\\
\left.k_-\left\langle\partial_s\tilde{\theta}_+,\partial_s\eta\right\rangle_{\Gamma_0}+\lambda_-\left\langle\partial_s\left(\widetilde{w}_+\right)',\partial_s\eta\right\rangle_{\Gamma_0}+\lambda_-\left\langle(\widetilde{u}_{+\tau})',\partial_s\zeta\right\rangle_{\Gamma_0}\right\}=0,
\end{array}\right. \tag{14}$$

$\forall(\varphi,\psi,\zeta,\eta)\in\tilde{U}(\Omega_+)\times\tilde{W}(\Omega_+)\times\tilde{V}(\Omega_+)\times\tilde{V}(\Omega_+)$, with the initial conditions :

$$\begin{array}{c}
\widetilde{u}_+(0)=u_+^*,(\widetilde{u}_+)'(0)=u_+^{**},\widetilde{w}_+(0)=w_+^*,(\widetilde{w}_+)'(0)=w_+^{**},\ \tilde{\phi}_+(0)=\phi_+^*,\ \tilde{\theta}_+(0)=\theta_+^*,\quad in\ \Omega_+\\
\widetilde{w}_+(0)=w^*_{+|\Gamma_0},\quad(\widetilde{w}_+)'(0)=\overset{**}{w}_{+|\Gamma_0},\quad\widetilde{u}_+(0)=u^*_{+|\Gamma_0},(\widetilde{u}_+)'(0)=\overset{***}{u}_+,\quad on\ \Gamma_0\quad(15)\\
(\partial_\nu\widetilde{w}_+)'(0)=\overset{***}{w}_+,\ \tilde{\phi}_+(0)=\phi^*_{+|\Gamma_0},\ \tilde{\theta}_+(0)=\theta^*_{+|\Gamma_0}\quad on\ \Gamma_0
\end{array}$$

We can show, using the Faedo-Galerkin method that this problem have at least one solution $\left(\widetilde{u}_+,\widetilde{w}_+,\tilde{\phi}_+,\tilde{\theta}_+\right)$ such that:

$$\begin{array}{c}
\widetilde{w}_+\in L^\infty\left(0,T;\tilde{W}(\Omega_+)\right),\ (\widetilde{w}_+)'\in L^\infty\left(0,T;\tilde{V}(\Omega_+)\right),(\partial_\nu\widetilde{w}_+)'\in L^\infty\left(0,T;L^2(\Gamma_0)\right),\\
\widetilde{u}_+\in L^\infty\left(0,T;\tilde{U}(\Omega_+)\right),(\widetilde{u}_+)'\in L^\infty\left(0,T;\left[L^2(\Omega_+)\right]^2\right),(\widetilde{u}_+)'_{|\Gamma_0}\in L^\infty\left(0,T;\left[L^2(\Gamma_0)\right]^2\right),\\
\tilde{\phi}_+,\ \ \tilde{\theta}_+\in L^\infty\left(0,T;L^2(\Omega_+)\right)\cap L^2\left(0,T;\tilde{V}(\Omega_+)\right).
\end{array}$$

Remark. The approximate problem written above is the variational form of the following boundary value problem :

$$\rho_+(\widetilde{u}_+)" - div\{C[\epsilon(\widetilde{u}_+) + f(\nabla\widetilde{w}_+)]\} + \lambda_+\nabla\tilde{\phi}_+ = 0 \quad in \ \Omega_+ \times (0,T), \tag{16}$$

$$\rho_+[I-\Delta]\widetilde{w}_+" + D_+\Delta^2\widetilde{w}_+ - div\{C[\epsilon(\widetilde{u}_+) + f(\nabla\widetilde{w}_+)]\nabla\widetilde{w}_+\} + \lambda_+\Delta\tilde{\theta}_+ = 0 \quad in \ \Omega_+ \times (0,T), \tag{17}$$

$$\rho_+\tilde{\phi}_+' - k_+\Delta\tilde{\phi}_+ + \lambda_+ div\tilde{u}_+' = 0 \qquad in\ \Omega_+ \times (0,T), \tag{18}$$

$$\rho_+\tilde{\theta}_+' - k_+\Delta\tilde{\theta}_+ - \lambda_+\Delta\tilde{w}_+' = 0 \qquad in\ \Omega_+ \times (0,T), \tag{19}$$

with Dirichlet conditions on $\Gamma \times (0,T)$

$$\widetilde{u}_+ = 0,\ \widetilde{w}_+ = \partial_\nu\tilde{w}_+ = 0,\ \tilde{\theta}_+ = 0,\ \tilde{\phi}_+ = 0 \qquad on\ \Gamma \times (0,T), \tag{20}$$

and the approximate boundary conditions on $\Gamma_0 \times (0,T)$:

$$^t\boldsymbol{\tau}\left(C[\epsilon(\widetilde{u}_+) + f(\nabla\widetilde{w}_+)]\right)\boldsymbol{\nu} = \delta\left(-\rho_-(\widetilde{u}_\tau)_+'' + E_-\partial_s\left[N_T(\widetilde{u}_+,\widetilde{w}_+)\right] + \lambda_-\partial_s\tilde{\phi}_+\right), \tag{21}$$

$$^t\boldsymbol{\nu}\left(C[\epsilon(\widetilde{u}_+) + f(\nabla\widetilde{w}_+)]\right)\boldsymbol{\nu} = \delta\left(-\rho_-(\widetilde{u}_\nu)_+'' + E_-R(s)N_T(\widetilde{u}_+,\widetilde{w}_+) - \frac{\lambda_-^2}{k_-}(\widetilde{u}_\nu)_+'\right), \tag{22}$$

$$D_+\left[\Delta\widetilde{w}_+ + (1-\mu_+)B_1\widetilde{w}_+\right] = -\delta\left(Q(\widetilde{w}_+) + \rho_-\partial_\nu\widetilde{w}_+'' + \frac{\lambda_-^2}{k_-}(\partial_\nu\tilde{w}_+)'\right), \tag{23}$$

$$\begin{aligned} &D_+\left[\partial_\nu\Delta\widetilde{w}_+ + (1-\mu_+)\partial_s B_2\widetilde{w}_+\right] - \rho_+\partial_\nu\widetilde{w}_+'' - C[\epsilon(\widetilde{u}_+) + f(\nabla\widetilde{w}_+)]\nu.\nabla\widetilde{w}_+ + \lambda_+\partial_\nu\tilde{\theta}_+ = \\ &\delta\ \left(\rho_-\left[\widetilde{w}_+ - \partial_s^2\widetilde{w}_+\right]'' +\ P(\widetilde{w}_+) - E_-\partial_s\left[N_T(\widetilde{u}_+,\widetilde{w}_+)\ \partial_s\widetilde{w}_+\right] + \lambda_-\partial_s^2\tilde{\theta}_+\right), \end{aligned} \tag{24}$$

$$k_+\partial_\nu\tilde{\theta}_+ + \lambda_+\partial_\nu\tilde{w}' = \delta\left(-\rho_-\tilde{\theta}_+'\ + k_-\partial_s^2\tilde{\theta}_+ + \lambda_-\partial_s^2\tilde{w}_+'\right), \tag{25}$$

$$k_+\partial_\nu\tilde{\phi}_+ - \lambda_+\tilde{u}_+'\nu = \delta\left(-\rho_-\tilde{\phi}_+'\ + k_-\partial_s^2\tilde{\phi}_+ + \lambda_-\partial_s(\tilde{u}_\tau)_+'\right), \tag{26}$$

where $^t\boldsymbol{\nu}$ (resp. $^t\boldsymbol{\tau}$) is the transposed vector (resp. matrix) of $\boldsymbol{\nu}$ (resp. $\boldsymbol{\tau}$). With the system above, we associate the initial conditions (15). The operators P and Q are defined by :

$$\begin{aligned} P(\widetilde{w}) &= E_-\left[\partial_s^2\boldsymbol{\gamma}_T(\widetilde{w}) + \frac{2}{1+\mu_-}\partial_s\left(R(s)\boldsymbol{\gamma}_S(\widetilde{w})\right)\right], \\ Q(\widetilde{w}) &= E_-\left[\frac{2}{1+\mu_-}\partial_s\boldsymbol{\gamma}_S(\widetilde{w}) - R(s)\boldsymbol{\gamma}_T(\widetilde{w})\right]. \end{aligned}$$

As it can be seen, the approximate problem of order 1 differs from that of order 0 by the appearance of new additive terms in the right-hand sides of the boundary conditions (21)-(26) posed on Γ_0: the effect of the layer is now seen at order1.

We have, thus, obtained a coupled nonlinear problem posed only over the set Ω_+ , which is nothing but the domain occupied by the plate. However, the effect of the thin layer is taken into account and is completely embodied by the additive terms that are involved in the right hand sides of the boundary conditions imposed along the boundary Γ_0, which is the common portion of the boundaries of the plate and the layer. Indeed, one observes that these terms depend solely on the thickness of the layer and on its elastic and thermal characteristics E_- , μ_-, ρ_-, k_- and λ_-. Moreover, the effect of the thin body is also expressed by means of the new initial conditions imposed on Γ_0.

It is worth noticing that the approximate boundary conditions obtained in this paper are not standard since they involve tangential and time derivatives of order equal to that of the interior differential operator. This type of boundary conditions is called in the Russian literature a Ventcel's condition and the related boundary-value problem is called Ventcel's problem. Here, they model the presence of the layer and express the influence of this latter on the oscillations and the propagation of heat inside the plate.

Finally, let us mention that it is a question of interest to give an estimate for the error between the solution of the approximate problem and the one of the original problem. Because of the nonlinearity and the complexity of the two problems, this question is very delicate and remains to be seen.

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