

Structural Stability, Morse's Lemma and Singular Economies¹

Elvio Accinelli

UASLP, Facultad de Economía
San Luis Potosí-México
elvio.accinelli@eco.uaslp.mx

Abstract

In this paper we consider economies whose consumption spaces are subsets of a Hilbert space with non-empty interior, and we introduce the Negishi approach to characterize the social equilibria of these economies. Using Morse's lemma, we analyze the main characteristics of two-agents-economies, and classify them. We show that the characteristics of "similar economies", in the sense introduced in [Debreu, G. (1970)], can be very different from the structural stability point of view. Finally, we show that the useful mathematical definition of structural stability is not enough to characterize structural stability in economics.

Keywords: Structural stability, Morse functions, singular economies

1 The fundamentals of the economy

In [Debreu, G. (1969)], a mathematical formalization of the intuitive concept of similar economies is given. Intuitively, this concept means that two economies are similar if their endowments and utilities are not very different. We characterize each economy as a set $\mathcal{E} = \{X, u_i, w_i, i \in I\}$ where $X \subset H$ is the consumption subset, and H is a Hilbert space, u_i the utility function of the i -th agent, w_i its endowments, and I the finite set index, one for each agent. To fix ideas let us suppose that the economies \mathcal{E} and \mathcal{E}' have the same utilities and consumption subsets, then they are similar if their respective endowments are close, that is $\|w - w'\| < \epsilon$ where $\epsilon > 0$ and small enough. Economies will be different if they have different endowments so, each economy will be represented by its endowments.

In this paper, we show that for a meager subset of economies, represented by its endowments w and called singular economies, all "similar economy" can

¹This research is supported by CONACYT project No. 46209

have very different characteristics from the original one. This means that the structural characteristics of an economy represented by w , can be very different of the same structural characteristics of all “similar economy,” w' , no matter how close w and w' are. Despite the fact that this set of economies is a meager subset, it plays a crucial role in the theoretical economics. We do not argue that this set of singular economies has the same importance as its topological complement, the set of regular economies. However, it is not true that this topological meager set can be ignored without implications. For instance, great economic changes, and then the existence of the economic crises, imply the existence of singular economies.

Economic crises do not occur when the economies are regular, but they do occur. It is not possibility to explain the existence of economic crises without to have recourse to the set of singular economies. Recall that the subset of the regular economies is **a dense an open subset in the set of the economies**, see [Debreu, G. (1970)]. This means that all perturbation in the fundamentals of a regular economy does not change **its** main structural characteristics, like the cardinality or regularity of the walrasian equilibria. But in a small enough neighborhood of a singular economy all economy is regular. So, small perturbation of the fundamentals of a singular economy implies that this economy stop being singular, and this is not a trivial change. It may be that from a mathematical point of view this set can be ignored, but this is not possible from the theoretical economics points of view. Essential phenomena in nature or in society are frequently connected with singularities or some other kind of meager subset.

To explain the importance of this singular set, we will consider a two-agents pure exchange economy. In our model the commodity space is a Hilbert space symbolized by H . So, we can consider economies with contingent goods in time or in states of the world. In order to use differential techniques we assume that the commodity space has a non empty interior positive cone, (see for instance [Chichilnisky, G. and Zhou, Y. (1988)]). The utility functions $u_i : H_+ \rightarrow R$ are smooth enough that is they belong to the set $C^\infty(X; R)$, are Fréchet differentiable, and strictly quasi-concave functions. We assume that endowments $w_i \in H_{++}$, $i = 1, 2$; where H_+ is the positive cone of H and H_{++} its interior. The differential of $u_i, i = 1, 2$, at x will be denoted by du_x . Recall that this is a function into the Banach space $L(H_+, R)$ of bounded linear transformations from H into R . If u_i is differentiable for all $x \in H_+$ we say that u_i is differentiable. Respectively if $du(x)$ is differentiable we denote $d^2u(x) = d(du(x))$, and $d^2u_x \in L(H_+L(H_+, R))$.

2 The Negishi-approach for a two-agents economy

When the positive cone of the underlying commodity space has a non empty interior, the corresponding price is extremely large, which is the reason that the excess demand function cannot be well defined. To avoid using this function to characterize the walrasian equilibria we introduce the Negishi approach. From this approach we introduce the excess utility function that allow us to characterize the equilibria set without having to consider the excess demand function. On the other hand this approach establishes an immediate relation between changes in the fundamentals of an economy and changes in the social weights of the agents of the economy. In this way, this approach allow us to analyze the social repercussions of the economic changes. This is a complementary motivation to follow this approach.

In this work we consider only two-agents economies where the consumption subset X is the positive cone of a Hilbert space. So, an economy is a set symbolized by:

$$\mathcal{E} = \{X, u_i, w_i, i = 1, 2\}.$$

Although originally the Negishi approach was used in the case of economies whose consumption spaces are subsets of R^n , [Negishi, T. (1960)], we extend this method to the case where the consumption space is a subset of a Hilbert space². Given an economy \mathcal{E} we say that $x = (x_1, x_2) \in X \times X$ is a feasible allocation if $x_1 + x_2 \leq W$, where $W = w_1 + w_2$. We denote the simplex in R^2 by: $\Delta = \{\lambda \in R^2 : \lambda_1 + \lambda_2 = 1, \lambda_1, \lambda_2 \geq 0\}$, and by

$$\mathcal{F} = \{(x_1, x_2) \in X \times X : x_1 + x_2 \leq W\}$$

we denote the set of feasible allocations.

For each $\lambda \in \Delta$, consider the following maximization problem:

$$\begin{aligned} \mathcal{U}_\lambda(x) &= \lambda_1 u_1(x_1) + \lambda_2 u_2(x_2) \\ \text{s.t. } x_1 + x_2 &= W; \end{aligned} \tag{1}$$

Recall that a feasible allocation \bar{x}^* is a Pareto optimal allocation if and only if there exists $\bar{\lambda} \in \Delta$ such that

$$\mathcal{U}_{\bar{\lambda}}(\bar{x}) \geq \mathcal{U}_{\bar{\lambda}}(x) \quad \forall x \in \mathcal{F}. \tag{2}$$

i.e. if and only if x^* is a solution of the problem (1) where $\lambda = \bar{\lambda}$. See [Negishi, T. (1960)].

²This method can be extended to a Banach lattice see [Accinelli, E.; Puchet, M (2005)].

Under the hypothesis of strict concavity of the utility functions for each $\lambda \in \Delta$ there exists one and only one $x^*(\lambda)$ solving the problem given (2). Moreover, the following theorem holds:

Theorem 1 *Under the above hypotheses the function $\lambda \rightarrow x^*(\lambda)$ define a C^1 diffeomorphism between Δ_+ (the interior of Δ) and the set of Pareto optimal allocations.*

Proof: Consider the system of equations defined by the first order condition of problem (1) combined with the total resources constraint, and define de function $F : R^2 \times H^2 \times H^* \rightarrow R^3$ defined by $F(\lambda, x) = (F_1(\lambda, x_1, \gamma), F_2(\lambda, x_2, \gamma), F_3(x))$ where:

$$F_1(\lambda, x_1, \gamma) = \lambda_1 du_1(x_1) - \gamma$$

$$F_2(\lambda, x_2, \gamma) = \lambda_2 du_2(x_2) - \gamma$$

$$F_3(x) = x_1 + x_2 - W.$$

Here $\gamma \in H^*$ is the Lagrange multiplier for the maximization problem: Suppose that the point $(\lambda^*, x^*, \gamma^*) \in \Delta_+ \times H_+^2 \times H^*$ solves the equality $F(\lambda^*, x^*, \gamma) = 0$ holds. It follows that $d_x F(\lambda^*, x^*, \gamma^*) \in L(H^2, R^2)$ is surjective. Since H^2 is a Hilbert space, then the kernel of $d_x F(\lambda^*, x^*)$ splits H^2 and then from the surjective implicit function theorem, see [Zeidler, E. (1993)], it follows that there exist a relative neighborhood $\Lambda \subset \Delta_+$ of λ^* such that $x^*(\cdot) : \Lambda \rightarrow PO$ is a C^{k+2} . Then, it follows that the excess utility function is also a C^{k+2} function in the relative Λ -neighborhood of (λ^*) .[.]

Note that the set of Pareto optimal allocations $x^*(\lambda)$ solving (1), does not depend on the distributions of the endowments, but only in the total resources of the economy.

Given the diffeomorphism $x^* : \Delta_+ \rightarrow \mathcal{PO}$, we can define an efficient C^1 path in $\Delta_+ \times \mathcal{F}$

$$\mathcal{C}_N = \{(\lambda, x(\lambda)) \in S_n \times \mathcal{F} : \mathcal{U}_\lambda(x(\lambda)) \geq \mathcal{U}_\lambda(x)\}. \tag{3}$$

Here by the symbol \mathcal{PO} we denote the set of feasible Pareto optimal allocations. This differentiable path will be called the Negishi path represents and efficient path for the economy. If utility remain fixed, this path does not change if the total resources remain constant, i.e. does not depend on the redistribution of initial endowments

Let us now to introduce the excess utility function. Let $x^*(\lambda)$ be a feasible allocation solving the maximization of \mathcal{U}_λ , i.e. $(\lambda x^*(\lambda)) \in \mathcal{C}_N$ then, we define the function $e(\cdot, w, u) : R_+ \rightarrow R^2$ given by: $e(\lambda, w, u) = (e_1(\lambda, w, u), e_2(\lambda, w, u))$ where $e_i(\lambda, w, u) = d[u(x_i^*(\lambda))](x_i^*(\lambda) - w_i)$, $i = 1, 2$.

The following properties of the excess utility function are given in [Accinelli, E.; Puchet, M (2005)]:

1. $e(\alpha\lambda, w, u) = e(\lambda, w, u)$, $\alpha \in R_+, \forall \lambda \in \Delta_+$.
2. $\lambda e(\lambda, w, u) = 0 \forall \lambda \in \Delta_+$.

By means of Δ_+ we symbolize the strictly positive elements of Δ . The existence of the function $x^*(\lambda)$ is an immediate consequence of the first order conditions of the maximization problem and from the implicit mapping theorem for Banach spaces, see [Abraham, R.; Robbin, J. (1967)]. Property (1) allows us to consider the excess utility function as a function whose domain is Δ_+ , that is $e(\cdot, w, u) : \Delta_+ \rightarrow R^2$.

We say that $\lambda \in \Delta_+$ is a *social equilibrium* for the economy \mathcal{E} with utility functions u and initial endowments w if and only if $e(\lambda, w, u) = 0$. This subset of Δ_+ will be symbolized by:

$$\mathcal{SE}_{wu} = \{\lambda \in \Delta_+ : e(\lambda, w, u) = 0\}.$$

As it is well known, there exists a one to one correspondence between the set of social equilibria and the set of Walrasian equilibria, see for instance, ([Accinelli, E. (1994)]).

Theorem 2 *Under the above hypothesis the excess utility function $e(\cdot, w, u) : \Delta_+ \rightarrow R^2$ is a differentiable map.*

Proof: It follows straightforward from the fact that the function $\lambda \rightarrow x^*(\lambda)$ is differentiable, see theorem(1).[.]

Definition 1 (Similar economies) *We say that two economies \mathcal{E} and $\bar{\mathcal{E}}$ are ϵ -similar if*

$$(1) \sup_{i=1,2} \{\|u_i - \bar{u}_i\| + \|w_i - \bar{w}_i\|\} \leq \epsilon,$$

where $\|u_i - \bar{u}_i\| = \sup_{x \in H_+} |u_i(x) - \bar{u}_i(x)| + \|du_i(x) - d\bar{u}_i(x)\| + \|d^2u_i(x) - d^2\bar{u}_i(x)\|$ and, $\|w_i - \bar{w}_i\|$ is the inner product $(\langle w_i - \bar{w}_i, w_i - \bar{w}_i \rangle)^{\frac{1}{2}}$.

$$(2) \text{ The total resources are fixed, i.e. } w_1 + w_2 = \bar{w}_1 + \bar{w}_2 = W.$$

For fixed utility functions, the Negishi path of two similar economies is the same. Then, if w and \bar{w} are the endowments of two ϵ -similar economies, with utility functions denoted by u , it follows that $\|e(\lambda, w, u) - e(\lambda, \bar{w}, u)\| \leq M\|w - \bar{w}\| < \epsilon'$, where $\epsilon' = M\epsilon$. Here $M = \max_i M_i$, satisfying $M_i = \max_{x_i(\lambda)} \|\frac{\partial u_i}{\partial x_i}(x_i(\lambda)); x_i \in \mathcal{F}_i\|$ and

$$\mathcal{F}_i = \{(\lambda, x(\lambda)) \in \mathcal{C}_N : u_i(w_i) \leq u_i(x_i(\lambda)) \leq u_i(W)\}.$$

3 Singular economies

Consider an economy parameterized by its endowments, suppose that utilities and consumption spaces are given. The social equilibria set is given by $\mathcal{SE}(\bar{u}) = \{(\lambda, w) \in \Delta_+ \times H_{++}^2 : e(\lambda, w, \bar{u}) = 0\}$ the set of social weights and endowments, such that $e_i(\lambda, w, \bar{u}) = d\bar{u}_i(x_i(\lambda))(x_i(\lambda) - w_i) = 0$, $i = 1, 2$. We do not consider economies where some $\lambda_i = 0$, $i = 1, 2$, because this suppose that the i -th consumer is out of the market³. From item (1) in the previous section it follows that the linear transformation $d_\lambda e(\lambda, w, \bar{u})$ of the excess utility function has rank equal to 1 at most, i.e, $rank [d_\lambda e(\lambda, w, \bar{u})] \leq 1$. So we say that an economy \mathcal{E} is singular if and only if $rank [d_\lambda e(\lambda, w, \bar{u})] = 0$ for some $\lambda \in \Delta_+$ such that, $e(\lambda, w, \bar{u}) = 0$. Using the two properties of the excess utility function given in the previous section, we can characterize a singular economy from a reduced excess utility function $\bar{e}(\cdot, \cdot, \bar{u}) : (0, 1) \times H_{++}^2 \rightarrow R$ given by $\bar{e} = \bar{e}(\lambda, w, \bar{u})$.

Characterizing the economies from their excess utility maps, we classified the economies as: regular, singular no-degenerated or singular degenerated, such that the zero is a regular, singular no-degenerated or degenerated values of the reduced excess utility function. So, an economy of two agents is singular if the jacobian matrix of the excess utility function, in some $\lambda \in \Delta : e(\lambda, w, u) = 0$, is a singular matrix. And it will be singular degenerate or no-degenerate if the hessian matrix at some of these points is a singular matrix with rank 1 or 0 respectively.

4 Morse's lemma and the singular economies

The Morse theory investigates the local and global behavior of functions $f : M \rightarrow R$ where M is a manifold in R^n . with critical points. The quadratics terms in Taylor expansions play a crucial role in local investigations. Morse's lemma characterizes the real and differentiable maps in three types: regular maps, singular no-degenerate maps and singular degenerate maps. **We recall** that a map is regular if has no singular points, and is singular or singular degenerate if has singular (no-degenerate) or singular degenerate points, see **for instance** [Golubistki, M. and Guillemin, V.(1973)]. As we characterize the economies by means of their excess utility functions, we classify them according to the characteristic, of this maps.

Definition 2 *Let M be a smooth manifold of dimension n . A function $f : M \rightarrow R$ is called a **Morse function** if every singular point is a no-degenerate singular point.*

³Suppose that $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 = 0$, then if $(\lambda, x(\lambda)) \in C_N$ then $x_1(\lambda) = 0$.

As it is well known, all real map is a Morse map or is close to one of these maps, i.e. the set of Morse functions is a residual set in the set of differentiable real functions $C(M, R)$ with the Whitney topology. The following theorems are classical in mathematics, our main reference **for this point is the text of [Golubistki, M. and Guillemin, V. (1973)]**.

Theorem 3 (Generalized Morse Lemma) *Let M be a smooth manifold of dimension n , and let $f : U(x_0) \subset M \rightarrow R$ be a smooth function, $x_0 \in X$ is a no-degenerate singular point of f Then there exists a local diffeomorphism ψ (in a neighborhood $U(x_0)$ of x_0 such that:*

$$f(\psi(y)) = f(x_0) + d^2 f(x_0) y^2 / 2 \quad (4)$$

is satisfied for all $y \in U(p)$, where $p = \psi(x_0)$, and $U(p) = \psi(U(x_0))$.

The significance of Morse's Lemma is in reducing the family of all smooth real functions vanishing at $p \in M$ ($f(p) = 0$) in R^n with zero as a no-degenerate singular value, to just $n + 1$ simple stereotypes.

Consider now a no-degenerate singular economy $\bar{\mathcal{E}}$ and the corresponding reduced excess utility function $\bar{e}(\cdot, \bar{w}, \bar{u}) : (0, 1) \rightarrow R$. Applying the Morse theorem to this function, where the manifold M is the interval $(0, 1)$ it follows that, in a neighborhood $U_{\bar{\lambda}}$ of a social equilibrium $\bar{\lambda} \in \Delta_+$ of a no-degenerate singular economy \bar{w} , the reduced excess utility functions $\bar{e}(\cdot, \bar{w}, \bar{u})$ by a smooth coordinate transformation, can be reduced just to a one of the 2 simple stereotypes, namely:

$$\bar{e}(\psi(\nu), \bar{w}, \bar{u}) = \pm \nu^2 \quad (5)$$

So, it follows that only two kinds of no-degenerate singular economies exist, and they are characterized by (5).

We will use the following three theorems, to show some of the main characteristics of the no-degenerate singular economies.

Theorem 4 *Let M be a smooth manifold. Let $f : M \rightarrow R$ be a smooth function with a no-degenerate singular point p . Then there exists a neighborhood V of p in M such that no other singular point of f are in V , i. e., no-degenerate singular points are isolates.*

So, no degenerate singular points are isolates. Moreover, if we consider the economies parameterized by their endowments (utilities are fixed), generically in Ω , there exists only one λ such that $e(\lambda, w, \bar{u}) = 0$ is a critical no-degenerate social equilibrium. This follows as a conclusion of the next theorem:

Theorem 5 *Let M be a smooth manifold. The set of Morse functions whose singular values are distinct (i.e., if p and q are distinct singular points of f in M , then $f(p) \neq f(q)$) form a residual set in $C^\infty(X, R)$.*

This means that if the economy $\mathcal{E} = \{u_i, w_i, i \in 1, 2\}$ is singular no-degenerate, then there exists only one critical equilibrium⁴ $\lambda \in \mathcal{E}q(w, u)$.

Theorem 6 *Let M be a smooth manifold. Then the set of smooth proper mappings $f : M \rightarrow R$ is open and dense in $C^\infty(M, R)$ the space of real smooth functions.*

Consider the subset $\mathcal{E}_u(\cdot) = \{X, u_i, \cdot, I\}$, $I = \{1, 2\}$ of economies with fixed utility functions $u = (u_1, u_2)$ parameterized by the endowments, i.e. for each $w = (w_1, w_2) \in H_{++}^2$ we obtain an economy $\mathcal{E}_u(w) = \{X, u_i, w_i, I\}$, $I = \{1, 2\}$ with its corresponding excess utility function $e(\cdot, w, u)$, parameterized also by $w \in H_{++}$. This can be symbolized by the next diagram:

$$w \rightarrow \mathcal{E}_u(w) \rightarrow e(\cdot, w, u).$$

So, the intuition behind the theorem (6) is the following: fixed utility functions, there exist an open and dense set \mathcal{W}_0 such that for all $w \in \mathcal{W}_0$ the excess utility functions is a Morse function then, generically economies are regular or singular no-degenerate. This means that for fixed utility functions, for all $w \in \mathcal{W}_0$ the economies $\mathcal{E}_u(w)$ are regular or singular no-degenerate, i.e. for fixed utility functions all economy is regular or singular no-degenerate, or is close to one of them.

Let us now introduce the mathematical concept of structural stability. A function $f : M \rightarrow N$ where M and N are m and n dimensional manifolds, is said to be *structurally stable* if and only if there exists a neighborhood $U(f)$ of f in the set of smooth functions, such that if g in $U(f)$ then g is equivalent⁵ to f . The subset of Morse functions is a structural stable set. Then the subset of excess utility function that are Morse's functions conform a structurally stable subset in $C^\infty(M, R)$. We will see that this mathematical fact, it is not enough to characterize as structural stable (in the economical sense) the set of no-degenerate singular economies.

5 Structural stability in economics

The previous theorems (4), (5), (6) say that the subset of the Morse functions is an open and dense subset of the smooth real functions and that for a given Morse function $f : M \rightarrow R$:

⁴Then, only if the economy is singular degenerate it is possible to obtain a continuous set of equilibria, in this cases the excess utility function it is not a Morse function, and so it belongs to a subset nowhere dense in the smooth functions set with the Whitney topology.

⁵Two smooth mappings f and $g : R^n \rightarrow R^m$ are said to be equivalent if and only if there exists mapping ϕ and ψ such that $f = \phi(g(\psi))$.

1. All its singular values are distinct, and finite.
2. All singular values of $g \in N_f(\epsilon)$ are distinct.
3. There exists a neighborhood $N_f(\epsilon)$ of f and radius ϵ (in the Whitney topology) such that all $g : M \rightarrow R \in N_f$ is a Morse function.

This means that the set of Morse functions is a structurally stable subset. From an economical point of view this means that the set of the economies $\mathcal{E} = \{X, w_i, \hat{u}_i, I\}$, $I = \{1, 2\}$, whose excess utility function $e(\cdot, w, \hat{u})$ are Morse functions are structurally stable. Because there exists an ϵ -neighborhood $N_w(\epsilon)$ of w such that all excess utility function $e(\cdot, w', \hat{u})$ with $w' \in N_w(\epsilon)$ is a Morse function too. Then, if the economy represented by w has a Morse excess utility function $e(\cdot, w, \hat{u})$, then all similar economy w' in an ϵ -neighborhood of w has a Morse excess utility function, $e(\cdot, w', \hat{u})$ associate. This means that this set of economies is structurally stable. This is satisfactory from a strictly mathematical point of view, however this concept of structural stability is no satisfactory from the economics point of view, because two economies whose excess utility functions are Morse functions belonging to the same neighborhood $N(\epsilon)$ of radius ϵ (ϵ -similar), can give place to very different equilibrium sets. Suppose that for a Morse excess utility function $e : \Delta_+ \rightarrow R$, the point $\bar{\lambda}$ verify that $e(\bar{\lambda}) = 0$, and suppose also, that this is a singular point for the excess utility function, from the fact that we are assuming that this is a Morse function, we know that there is the only one point with this property. For all other similar economies, the point $\bar{\lambda}$ will not be an equilibrium, because the singular equilibrium disappears or gives place to two new regular equilibria. So, small changes in fundamentals of a singular economy give place to a regular and strongly modified economy that shows a very different behavior from the structural economics point of view, the singularity disappears after perturbations and similar economies can have only regular equilibria. This means that the set of no-degenerate singular economies is not a stable subset from the theoretical economics point of view. This is the main reason why the structural stability criterium useful in mathematics is not enough to characterize structural stability in economics.

Note that big changes in social characteristics, after perturbations in fundamentals of the economy can be observed if and only if the original economy is singular. Perturbations in fundamentals of regular economies do not change the number of equilibria nor the topological properties of the equilibrium sets, i.e. the topological properties of the $\lambda \in \Delta_+ : e(\lambda) = 0$ remain unchanged. This means that small changes in the main characteristics of an economy, can be reversed by small changes in the opposite direction if and only if the economy is regular. In both cases, regular and singular economies small changes in fundamentals (perturbations) will give place to regular economies. However

the cardinality and main characteristics of the equilibrium set of the new (after the perturbations occurs) “similar economies” change if the original (before the the perturbations occurs) economy was a singular one. Perturbations in fundamentals transforms a regular economy in a regular one, and big changes in the equilibrium sets does not occurs. But this is not the case if the original economy is a singular one, where big changes in the structure of the singular economies occurs after perturbations, and this process can not be reversed by means of small modifications of the fundamentals in the opposite direction. On the other hand, the characteristics of the new economies are not completely predictable by the actual theory. We know that this new economies will be regular, but we can not know if the new economy will have one equilibrium minus or two news regular equilibria appear (the singularity disappears). So, in these case similar economies can show very different characteristics in the structure of the equilibrium set.

Because the excess utility functions characterizing the economies are generically Morse functions, and thus stable maps, from a mathematical point of view regular and no-degenerate singular economies are structurally stable. This concept of structural stability does not consider the characteristics of the equilibria set and the possibilities of big changes in this set, as a result of small changes in the fundamentals of the economies. A definition of structural stability with economical significance must necessarily look for the characteristics of the equilibrium set of the similar economies. To give a definition of structural stability with deeper economic meaning, let us consider the following notation. Let $\mathcal{E} = \{X, u_i, w_i, I\}$ be an economy, where the consumption set X is for each agent $i \in I$ a subset of a non empty positive cone of a Hilbert space H . Let $u_i \in C^\infty(X; R), \forall i \in I$ be a smooth utility function, and let $w_i \in H_{++}, \forall i \in I$ be the endowments of the i -th agent. Consider $C^\infty(X, R)$ with the Whitney topology. Let \mathcal{U} be a neighborhood of the utility profile $u = (u_1, \dots, u_n)$ in the product space $(C^\infty(X, R))^n$ with the Whitney product topology, and let \mathcal{V} be a neighborhood of $w = (w_1, \dots, w_n)$ in H^n .

Definition 3 (Structurally stable economies) *We say that an economy $\mathcal{E} = \{X, u_i, w_i, I\}$ is structurally stable if*

1. *There is a relative neighborhood $\mathcal{V}_\mathcal{E} \subset \mathcal{U} \times \mathcal{V}$ where \mathcal{U} is a neighborhood of u and \mathcal{V} a neighborhood of v such that, for all economy $\bar{\mathcal{E}}$ in $\mathcal{V}_\mathcal{E}$ the respective excess utility functions $e_i(\cdot, u, w)$ and $e_i(\cdot, \bar{u}, \bar{w})$ are equivalent, and if*
2. *there exists a bijective correspondence $N_q : \mathcal{SE}_{uw} \rightarrow \mathcal{SE}_{\bar{u}\bar{w}}, \forall (\bar{u}, \bar{w}) \in \mathcal{U} \times \mathcal{V}$.*

This definition, which is motivated by economical considerations, is stronger than the standard concept of structural stability in differential topology. This

means that if an economy is structurally stable from this definition, it is also structurally stable considering the previous definition of structural stability. We introduce in this definition a bijective correspondence N_q between the equilibria set of similar economies, because in economics, structural stability must consider the equilibrium sets of similar economies and to say that an economy is structurally stable, the equilibrium sets of the perturbed economies must remain similar to the original one. If this is not the case the main characteristics of the economies will be very different and there would no sense in talking about stability from the economical point of view. From our definition it follows that the only structurally stable subset of economies is the subset of the regular economies, because only for this subset of economies the cardinality of the equilibrium sets of similar economies remain constant. This is a more realistic and useful definition of structural stability in economics. Note that in our work, we consider a restriction of this definition to economies where utilities are fixed $u = (u_1, u_2)$ and $I = \{1, 2\}$. In definition (3) we consider stability under perturbation on tastes and endowments, thus it is possible to extend our work considering changes not only in endowments but also in tastes represented by utility functions, as considered in ([Accinelli, E.; Puchet, M; Piria A: (2003)]).

6 Conclusions

Our main conclusion is the following: “similar economies” i.e. economies with close endowments and the same utilities have similar excess utility functions, however the structural characteristics of similar economies in a neighborhood of a singular economy are very different. From a mathematical point of view regular and no-degenerate singular economies are structurally stable. But this stability makes no sense from the economics point of view, because if we analyze the characteristics of the equilibria sets (one of the main characteristics of an economy) of the similar economies to a singular no-degenerate economy, we will see that they have important differences, and these differences have a deeper economics sense. This make know that perturbations in its fundamentals transform a singular economy in a very different economy. All similar economy to a given regular economy is a regular one, they have the same number of equilibria, and the equilibria of the original and the perturbed economy are close, but this is not the situation if we consider a singular economy. All similar economy to a singular economy is a regular economy, thus small changes in the fundamentals of a singular economy imply big changes in the characteristics and in the social behavior of the original economy. The equilibria set of a perturbed economy from a singular one, is very different from the original one and the respective equilibria are no necessarily close. Moreover, some of the main characteristics of the perturbed economy are structurally unforeseeable before the perturbation. Significant changes in economics (like crises)

can occur only for a singular economy, and these changes can occur from the action of a central planner looking for gradual changes in the economy, and it will not be possible to recover the initial situation with small changes in the opposite direction. So similar economies can be separated by big crises.

References

- [Abraham, R.; Robbin, J. (1967)] “Transversal mappings and flows.” *Ed. Benjamin, NY*.
- [Accinelli, E. (1994)] “Existence and Uniqueness of the Equilibrium for Infinite Dimensional Economies.” *Estudios de Economía* 21, pp. 315-326.
- [Accinelli, E.; Puchet, M (2005)] “Could catastrophe theory become a new tool in understanding singular economies?” *New tools of Economic Dynamics, Leskow, J., Puchet, M, and Punzo, L. eds. Serie: Lecture Notes in Economics and Mathematical Systems* 551, Chapter: 8. *Springer Verlag*.
- [Accinelli, E.; Puchet, M; Piria A: (2003)] “Taste and Singular Economies” *Revista Mexicana de Economía y Finanzas* 2: 1, pp 35-48.
- [Chichilnisky, G. and Zhou, Y. (1988)] “Smooth Infinte Economies.” *Journal of Mathematica Economies* 29, pp. 27-42.
- [Debreu, G. (1969)] “Neighboring economic agents.” *La Dècision. Colloques Internacionaux du CNRS* 171, Paris.
- [Debreu, G. (1970)] “Economies with a finite Set of Equilibria.” *Econometrica* 38, pp 387 - 392.
- [Golubistki, M. and Guillemin,V.(1973)] “Stable Mappings and Their Singularities.” *Springer Verlag*.
- [Negishi, T. (1960)] “Welfare Economics and Existence of an Equilibrium for a Competitive Economy.” *Metroeconomica* 12, pp. 92- 97.
- [Zeidler, E. (1993)] “Non Linear Functional Analysis and its Applications” *Springer Verlag*.

Received: January 21, 2008