Applied Mathematical Sciences, Vol. 2, 2008, no. 51, 2549 - 2555

Generalized RTS with Discretionary and Nondiscretionary Inputs and Outputs

Sh. Razavyan[a], Gh. Tohidi[b,1] and Kh. Hoseini[b]

[a] Department of Mathematics, Islamic Azad University
Tehran-South Branch, Tehran, Iran

Department of Mathematics, Islamic Azad University
Tehran-Central Branch, Tehran, Iran

Abstract

In some situations, some inputs and outputs are nondiscretionary. Therefore, following Zarepisheh and Soleimani-damaneh [Euro. J. Opera. Res. (2007)], which uses the linear programming problem for determining RTS with discretionary data, in this paper we introduce a procedure for determining RTS with discretionary and nondiscretionary inputs and outputs.

Keywords: Linear programming; Data envelopment analysis; Parametric analysis; Returns to scale; Efficiency; Nondiscretionary data

1 Introduction

Returns to scale (RTS) is an important topic in performance analysis, which helps managers to make decisions about the expansion or contraction of the operation of the Decision Making Unit (DMU) under assessment. RTS can provide useful information on the optimal size of DMUs, or on whether small in size DMUs over- or under-perform larger ones, and vice versa, i.e., it is used to determine whether a technically efficient DMU can improve its productivity by resizing the scale of its operations. In economics, RTS are sometimes defined using the notion of elasticity (Starrett [3]). If elasticity is greater than one, then increasing returns to scale prevail for that particular DMU; if it is equal

[1]Corresponding author, P.O. Box 14515-459, e-mail: ghatohidi@yahoo.com

to one, then constant returns to scale prevail for that particular DMU; and if it is less than one, then decreasing returns to scale prevail for that particular DMU. In fact, the elasticity measure exhibits the rate of proportional variation of outputs with respect to the proportional variation of inputs in a local sense; i.e., in a sufficiently small interval of variations Hadjicostas and Soteriou [2].

Zarepisheh and Soleimani-damaneh [4] propose a procedure for generalized RTS of DMUs with discretionary data, in this paper we introduce a method for determining generalized RTS with discretionary and nondiscretionary inputs and outputs. The rest of this paper organized as follows: in Section 2 we propose a method for determining RTS of DMUS with nondiscretionary inputs and outputs by BCC model. Sections 3 contain a numerical example. Finally Section 3 includes some conclusions.

2 Estimation of the rate of variation

In some situations, some inputs and outputs are nondiscretionary. These data are beyond of the control of a DMU's management, which also need to be considered for efficiency evaluation. The BCC model[1] in input oriented with nondiscretionary is as follows:

$$\begin{array}{lll} \beta_o^* = & \min \ \beta_o & \\ \text{s.t.} & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \beta_o x_{io}^{D}, & i \in D \\ & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \beta_o x_{io}^{ND}, & i \in ND \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, & r = 1, \ldots, s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, & j = 1, \ldots, n. \end{array} \tag{1}$$

where, D and ND represent the associated sets containing discretionary and nondiscretionary inputs, respectively.

Assume that Increasing Returns to Scale (IRS) prevail at $\text{DMU}_o(x_o^D, x_o^{ND}, y_o)$. The Right Hand Side (RHS) vector of model (1) is $(0, x_o^{ND}, y_o, 1)$. Now in the optimal simplex tableau of (1) we perturb the RHS vector in the direction of $(0, 0, y_o, 0)$, and this leads to $(0, x_o^{ND}, y_o, 1)+\delta(0, 0, y_o, 0) = (0, x_o^{ND}, (1+\delta)y_o, 1)$ $(\delta \geq 0)$ as the new RHS vector. Now using the parametric analysis, we can obtain an interval, $\delta \in [0, \delta^1]$, such that the optimal value of the perturbed problem is a linear function with respect to $\delta \in [0, \delta^1]$. Regarding model (1) the optimal value exhibits the minimum proportional increase in the (D) inputs of DMU_o when all outputs of this DMU are multiplied by $(1 + \delta)$(for $\delta \in [0, \delta^1]$). In fact the optimal value of the perturbed problem is $\beta(1 + \delta)$ for $\delta \in [0, \delta^1]$.

3 Numerical example

Cosider 6 DMUs with 2 inputs and 1 output as Table 1, where 2^{th} input is nondiscretionary.

DMUs	DMU_A	DMU_B	DMU_C	DMU_D	DMU_E	DMU_F
input_1	3	4	7	9	13	7
input_2	2	1	2	1	3	3
output	1	3	7	8	9	2

We consider DMU_A as unit under assessment. Model (1) corresponding to this DMU is as follows:

$$\begin{aligned} \beta_A^* = \quad & \min \quad \beta \\ \text{s.t.} \quad & 3\lambda_1 + 4\lambda_2 + 7\lambda_3 + 9\lambda_4 + 13\lambda_5 + 7\lambda_6 \leq 3\beta, \\ & 2\lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_4 + 3\lambda_5 + 3\lambda_6 \leq 2, \\ & \lambda_1 + 3\lambda_2 + 7\lambda_3 + 8\lambda_4 + 9\lambda_5 + 2\lambda_6 \geq 1, \\ & \textstyle\sum_{j=1}^{6} \lambda_j = 1, \lambda_j \geq 0, \quad j = 1, \ldots, 6. \end{aligned} \tag{2}$$

An optimal tableau for this problem is as,

	z	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	β	s_1	s_2	R_3	R_4	RHS
z	1	0	0	-0.33	-0.83	-2	-1.16	0	-0.33	0	0.16	0.83	1
λ_2	0	0	1	3	3.5	4	0.5	0	0	0	0.5	-0.5	0
s_2	0	0	0	3	2.5	5	1.5	0	0	1	0.5	-2.5	0
β	0	0	0	-0.33	-0.83	-2	-1.16	1	-0.33	0	0.16	0.83	1
λ_1	0	1	0	-2	-2.5	-3	-0.5	0	0	0	-0.5	1.5	1

where s_1 and s_2 are the slack variables of the first and the second constraints, respectively. Also R_3 and R_4 are the artificial variables of the third and fourth constraints. Now we do the parametric analysis, when the RHS vector of the problem is perturbed of

$$b' = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \bar{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \bar{b'} = B^{-1}b' = \begin{pmatrix} 0 & 0 & 0.5 & -0.5 \\ 0 & 1 & 0.5 & -2.5 \\ -0.33 & 0 & 0.16 & 0.83 \\ 0 & 0 & -0.5 & 1.5 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.5 \\ -0.5 \\ -0.16 \\ 0.5 \end{pmatrix}.$$

Since $\bar{b'_1} < 0$ and $\bar{b_1} = 0$, then using the algorithm of the parametric analysis in linear programming, λ_2 leaves the basis and since $\forall j \; y_{1j} > 0$ then δ_1^-=0. (perturbed problem is infeasible for any $\delta > 0$). Hence there isn't decreasing in output, we now do the parametric analysis, when the RHS vector of the problem is perturbed in the direction of

$$b' = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \bar{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \bar{b'} = B^{-1}b' = \begin{pmatrix} 0 & 0 & 0.5 & -0.5 \\ 0 & 1 & 0.5 & -2.5 \\ -0.33 & 0 & 0.16 & 0.83 \\ 0 & 0 & -0.5 & 1.5 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.16 \\ -0.5 \end{pmatrix}.$$

$$\delta_1^+ = \frac{1}{-(-0.5)} = 2, \beta(\delta) = c_B\bar{b} + \delta c_B\bar{b}' = 1 + \delta(0,0,1,0)\begin{pmatrix} 0.5 \\ 0.5 \\ 0.16 \\ -0.5 \end{pmatrix} = 1 + 0.16\delta \Longrightarrow m_1^+ = 0.16.$$

Since $\frac{1}{m_1^+}$=6.25> 1, IRS prevail at DMU$_A$. The optimal table for $\delta = \delta_1^+$ is as follows:

	z	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	β	s_1	s_2	R_3	R_4	RHS
z	1	0	0	-0.33	-0.83	-2	-1.16	0	-0.33	0	0.16	0.83	1.32
λ_2	0	0	1	3	3.5	4	0.5	0	0	0	0.5	-0.5	1
s_2	0	0	0	3	2.5	5	1.5	0	0	1	0.5	-2.5	1
β	0	0	0	-0.33	-0.83	-2	-1.16	1	-0.33	0	0.16	0.83	1.32
λ_1	0	1	0	-2	-2.5	-3	-0.5	0	0	0	-0.5	1.5	0

The above table is optimal for $(x_A^D, x_A^{ND}, (1+\delta_1^+)y_A) = (3, 2, 3)$.

Regarding the algorithm cost row and the β-row (except for the β-column) are multiplied by $\frac{1}{\beta(\delta_1^+)} = \frac{1}{1.32}$. The following tableau is obtained:

	z	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	β	s_1	s_2	R_3	R_4	RHS
z	1	0	0	0.25	0.62	1.51	0.87	0	0.25	0	-0.12	-0.62	1
λ_2	0	0	1	3	3.5	4	0.5	0	0	0	0.5	-0.5	1
s_2	0	0	0	3	2.5	5	1.5	0	0	1	0.5	-2.5	1
β	0	0	-0.25	-0.62	-1.51	-0.87	1	-0.25	0	0.12	0.62	1	
λ_1	0	1	0	-2	-2.5	-3	-0.5	0	0	0	-0.5	1.5	0

The above table is optimal for $(\beta(\delta_1^+)x_A^D, x_A^{ND}, (1+\delta_1^+)y_A)$=(4,2,3). Now, we do the parametric analysis in the direction of

$$b' = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \bar{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \bar{b}' = B^{-1}b' = \begin{pmatrix} 0 & 0 & 0.5 & -0.5 \\ 0 & 1 & 0.5 & -2.5 \\ -0.25 & 0 & 0.12 & 0.62 \\ 0 & 0 & -0.5 & 1.5 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 1.5 \\ 0.36 \\ -1.5 \end{pmatrix}.$$

Since $\bar{b_3'} < 0$ and $\bar{b_3}$= 0 , then using the algorithm of the parametric analysis in linear programming, λ_1 leaves the basis and λ_3 enters the basis by a dual-simplex iteration. And hence the tableau changes to:

	z	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	β	s_1	s_2	R_3	R_4	RHS
z	1	-0.12	0	0	-0.3	-1.13	-0.8	0	-0.25	0	0.18	0.43	1
λ_2	0	-3	1	0	-0.25	-0.5	-0.25	0	0	0	-0.25	1.75	1
s_2	0	-3	0	0	-1.25	0.5	0.75	0	0	1	-0.25	-0.25	1
β	0	-0.12	0	0	-0.3	-1.13	-0.8	1	-0.25	0	0.18	0.43	1
λ_3	0	-0.5	0	1	1.25	1.5	0.25	0	0	0	0.25	-0.75	0

$$\bar{b'} = B^{-1}b' = \begin{pmatrix} 0 & 0 & -0.25 & 1.75 \\ 0 & 1 & -0.25 & -0.25 \\ -0.25 & 0 & 0.18 & 0.43 \\ 0 & 0 & 0.25 & -0.75 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.75 \\ -0.75 \\ 0.54 \\ 0.75 \end{pmatrix}.$$

Hence,

$$\delta_2^+ = \frac{1}{-(-0.75)} = 1.33, \beta(\delta) = c_B\bar{b} + \delta c_B\bar{b'} = 1 + \delta(0,0,1,0) \begin{pmatrix} -0.75 \\ -0.75 \\ 0.54 \\ 0.75 \end{pmatrix} = 1 + 0.54\delta.$$

Therefor, $m_2^+ = 0.54$ and $\frac{1}{m_2^+} = 1.85$. The optimal table for $\delta = \delta_2^+$ is as follows:

	z	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	β	s_1	s_2	R_3	R_4	RHS
z	1	-0.12	0	0	-0.3	-1.13	-0.8	0	-0.25	0	0.18	0.43	1.72
λ_2	0	-1.5	1	0	-0.25	-0.5	-0.25	0	0	0	-0.25	1.75	0
s_2	0	-1.5	0	0	-1.25	0.5	0.75	0	0	1	-0.25	-0.25	0
β	0	-0.12	0	0	-0.3	-1.13	-0.8	1	-0.25	0	0.18	0.43	1.72
λ_3	0	-0.5	0	1	1.25	1.5	0.25	0	0	0	0.25	-0.75	1

$(\beta(\delta_1^+)x_A^D, x_A^{ND}, (1+\delta_1^+)(1+\delta_2^+)y_A)$=(4,2,7). Regarding the algorithm cost row and the β-row (except for the β-column)are multiplied by $\frac{1}{\beta(\delta_2^+)} = \frac{1}{1.72}$. The following tableau is obtained:

	z	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	β	s_1	s_2	R_3	R_4	RHS
z	1	-0.07	0	0	-0.17	-0.64	-0.5	0	-0.14	0	0.1	0.25	1
λ_2	0	-1.5	1	0	-0.25	-0.5	-0.25	0	0	0	-0.25	1.75	0
s_2	0	-1.5	0	0	-1.25	0.5	0.75	0	0	1	-0.25	-0.25	0
β	0	-0.07	0	0	-0.17	-0.64	-0.5	1	-0.14	0	0.1	0.25	1
λ_3	0	-0.5	0	1	1.25	1.5	0.25	0	0	0	0.25	-0.75	1

The above table is optimal for $\beta(\delta_1^+)\beta(\delta_2^+)x_A^{ND}, x_A^D, (1+\delta_1^+)(1+\delta_2^+)y_A)$=(7,2,7). Now, we do the parametric analysis in the direction of

$$b' = \begin{pmatrix} 0 \\ 0 \\ 7 \\ 0 \end{pmatrix}, \bar{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \bar{b'} = B^{-1}b' = \begin{pmatrix} 0 & 0 & -0.25 & 1.75 \\ 0 & 1 & -0.25 & -0.25 \\ -0.14 & 0 & 0.1 & 0.25 \\ 0 & 0 & -0.25 & -0.75 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 7 \\ 0 \end{pmatrix} = \begin{pmatrix} -1.75 \\ -1.75 \\ 0.7 \\ 1.75 \end{pmatrix}.$$

Since $\bar{b'_1} < 0$ and $\bar{b_1} = 0$, then using the algorithm of the parametric analysis in linear programming, λ_2 leaves the basis and λ_4 enters the basis by a dual-simplex iteration. And hence the tableau changes to:

	z	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	β	s_1	s_2	R_2	R_3	RHS
z	1	-1.09	-0.68	0	0	-0.3	-1.35	0	-0.14	0	0.27	0.94	1
λ_4	0	-6	-4	0	1	2	-5	0	0	0	1	-7	0
s_2	0	-6	-5	0	0	3	-4	0	0	1	1	-9	0
β	0	-1.09	-0.68	0	0	-0.3	-1.35	1	-0.14	0	0.27	0.94	1
λ_3	0	7	5	1	0	-1	6	0	0	0	-1.5	8	1

$$\bar{b}' = B^{-1}b' = \begin{pmatrix} 0 & 0 & 1 & -7 \\ 0 & 1 & 1 & -9 \\ -0.14 & 0 & 0.27 & 0.94 \\ 0 & 0 & -1.5 & 8 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 7 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \\ 1.89 \\ -10.57 \end{pmatrix}.$$

Hence $\delta_3^+ = \frac{1}{-(-10.5)}, \beta(\delta) = c_B\bar{b} + \delta c_B\bar{b}' = 1 + \delta(0,0,1,0)\begin{pmatrix} 7 \\ 7 \\ 1.89 \\ -10.5 \end{pmatrix} =$ $1 + 1.89\delta \Longrightarrow m_3^+ = 1.89$ and $\frac{1}{m_2^+} = 0.53$.

Since $\frac{1}{m_3^+} < 1$, then the algorithm terminates and the results can be interpreted as follows:

1. If the outputs increase from 1 to $(1+\delta_1^+)(1)$=3, then the increase rate is equal to $\frac{1}{m_1^+}$=6.25.

2. If the outputs increase from 3 to $(1+\delta_1^+)(1+\delta_2^+)(1)$=7, then the increase rate is equal to $\frac{1}{m_2^+} = 1.85$.

3. $\beta(\delta_1^+)\beta(\delta_2^+)x_A^D, x_A^{ND}, (1+\delta_1^+)(1+\delta_2^+)y_A)$=(7,2,7) is an most productive scale size and increasing its outputs is not beneficial.

4 Conclusion

In this paper, we introduced RTS of DMUs with nondiscretionary data. The proposed model uses the parametric analysis for determining RTS of DMUs with nondiscretionary data and determines the global variation of outputs with respect to the variation of inputs.

References

[1] Banker, R.D., A. Charnes and W.W. Cooper, (1984), Some models for estimating technical and scale inefficiencies in data envelopment analysis, *Management Science*, 30 1078-1092.

[2] Hadjicostas, P. and Soteriou, A.C., (2006), One sided elasticities and tecnical efficiency in multi-output production: Atheorical framework, *European Jouranl of Operationl Research*, 168 (2), 425-449.

[3] Starrett, D.A., (1977), Measuring returns to scale in the aggregate, and the scale effect of ublic goods, *Econometrica*, 45, 1439-1355.

[4] Zarepisheh, M. and M. Soleimani-damaneh, (2007), Global variation of outputs with respect to the variation of inputs in performance analysis; generalized RTS, *European Jouranl of Operationl Research*.

Received: April 1, 2008