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Solving Fuzzy Linear System of Equations by Using Householder Decomposition Method

M. Matinfar[1], S. H. Nasseri[1,2] and M. Sohrabi[1]

[1] Department of Mathematics, University of Mazandaran, Babolsar, Iran

[2] Research Center of Algebraic Hyperstructures and Fuzzy Mathematics, Iran

Corresponding author. Nasseri@umz.ac.ir (Hadi Nasseri)

Abstract

Linear systems have important applications to many branches of science and engineering. In many applications, at least some of the parameters of the system are represented by fuzzy rather than crisp numbers. So it is immensely important to develop numerical procedures that would appropriately treat general fuzzy linear systems and solve them. In this paper, householder decomposition method for solving fuzzy system of linear equations is considered. The method in detail is discussed and illustrated by solving some numerical examples.

Keywords: fuzzy linear systems, householder method, QR-decomposition

1 Introduction

Linear system of equations is important for studying and solving a large proportion of the problems in many topics in applied mathematics and engineering. One of the most applications of fuzzy number arithmetic is treating fuzzy linear systems, several problems in various areas such as economics, engineering and physics boil down to the solution of a linear system of equations. Friedman et al. [6] introduced a general model for solving a fuzzy $n \times n$ linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector. They used the parametric form of fuzzy numbers and replaced the original fuzzy $n \times n$ linear system by a crisp $2n \times 2n$ linear system. After that, in the literature of fuzzy linear system of equations, various methods is proposed to solve these systems [1, 2, 3, 4, 7]. In this study, we focus on solving rectangular fuzzy linear system of equations (FLSE) based on matrix decomposition methods. This paper is organized in 4 sections. In

Section 2, we first state some basic definitions and notations of fuzzy sets and then give the definition of the linear system of equations (FLSE). In Section 3, we deal with Householder process to obtain a QR-decomposition for the coefficient matrix of the extended linear system of the FLSE. Numerical examples are given in Section 4.

2 Preliminaries

We represent an arbitrary fuzzy number by an ordered pair of functions $(\underline{u}(r), \overline{u}(r)), 0 \leq r \leq 1$, which satisfies the following requirements:

1. $\underline{u}(r)$ is a bounded left semicontinuous non-decreasing function over [0, 1],
2. $\overline{u}(r)$ is a bounded left semicontinuous non-increasing function over [0, 1],
3. $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

In particular, if $\underline{u}$, $\overline{u}$ are linear functions we have an important kind of fuzzy numbers entitled triangular fuzzy number. The set of all triangular fuzzy numbers is denoted by E. For arbitrary fuzzy numbers $x = (\underline{x}(r), \overline{x}(r))$,$y = (\underline{y}(r), \overline{y}(r))$ and a real number k, we define the addition and the scalar multiplication of fuzzy numbers by using the extension principle as:

1. $x + y = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$.
2. $kx = \begin{cases} (k\underline{x}, k\overline{x}), & k \geq 0, \\ (k\overline{x}, k\underline{x}), & k < 0. \end{cases}$

Definition 2.1 *Two fuzzy numbers* $x(r) = (\underline{x}(r), \overline{x}(r))$, $y(r) = (\underline{y}(r), \overline{y}(r))$ *are equal,i.e,* $x = y$ *if and only if*

$$\underline{x}(r) = \underline{y}(r), \overline{x}(r) = \overline{y}(r), \quad 0 \leq r \leq 1. \tag{1}$$

Definition 2.2 *Linear system*

$$\begin{cases} a_{11}(\underline{x}_1(r), \overline{x}_1(r)) + \ldots + a_{1n}(\underline{x}_n(r), \overline{x}_n(r)) = (\underline{y}_1(r), \overline{y}_1(r)), \\ \vdots \\ a_{m1}(\underline{x}_1(r), \overline{x}_1(r)) + \ldots + a_{mn}(\underline{x}_n(r), \overline{x}_n(r)) = (\underline{y}_m(r), \overline{y}_m(r)), \end{cases} \tag{2}$$

is called a fuzzy linear system of equations (FLSE)[6], if the coefficient matrix $A = (a_{ij})$, *be a crisp matrix and* $y_i \in E$ *for* $1 \leq i \leq m$ *and* $1 \leq j \leq n$.

Definition 2.3 *A fuzzy vector* $X = (x_1, \ldots, x_n)^t$ *given by*

$$x_i = (\underline{x}_i(r), \overline{x}_i(r)), \quad i = 1, ..., n, \ 0 \leq r \leq 1, \tag{3}$$

is called a solution of the fuzzy linear system (3) , if

$$\underline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \underline{a_{ij}x_j} = \underline{y_i}, \quad \overline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \overline{a_{ij}x_j} = \overline{y_i}. \tag{4}$$

Consequently, in order to solve the system given by Eq.(2) one must solve a $2m \times 2n$ crisp linear system where the right-hand side column is the function vector $(\underline{y}_1(r), \ldots, \underline{y}_m(r), \overline{y_1}(r), \ldots, \overline{y}_m(r))^t$. We get the $2m \times 2n$ linear system

$$\begin{cases} s_{11}\underline{x}_1 + \ldots + s_{1n}\underline{x}_n + s_{1,n+1}\overline{x}_1 + \ldots + s_{1,2n}\overline{x}_n = \underline{y}_1, \\ \vdots \\ s_{m,1}\underline{x}_1 + \ldots + s_{m,n}\underline{x}_n + s_{m,n+1}\overline{x}_1 + \ldots + s_{m,2n}\overline{x}_m = \underline{y}_m, \\ s_{m+1,1}\underline{x}_1 + \ldots + s_{m+1,n}\underline{x}_n + s_{m+1,n+1}\overline{x}_1 + \ldots + s_{m+1,2n}\overline{x}_n = \overline{y}_1, \\ \vdots \\ s_{2m,1}\underline{x}_1 + \ldots + s_{2m,n}\underline{x}_n + s_{2m,n+1}\overline{x}_1 + \ldots + s_{2m,2n}\overline{x}_n = \overline{y}_m, \end{cases} \tag{5}$$

where s_{ij} are determined as follows:
if $a_{ij} \geq 0$ then $s_{ij} = s_{i+m,j+n} = a_{ij}$ and if $a_{ij} \leq 0$, then $s_{i+m,j} = s_{i,j+n} = a_{ij}$, for $(1 \leq i \leq m \ , 1 \leq j \leq n)$.
We can see $A = S_1 + S_2$, such that S_1 contain all of entries non negatives and S_2 is contain all non positives entries of the matrix A. Using matrix notation we get

$$SX = Y, \tag{6}$$

where $S = (s_{ij})$, $1 \leq i \leq 2m$, $1 \leq j \leq 2n$, $X = (\underline{x}_1, \cdots, \underline{x}_n, \overline{x}_1, \cdots, \overline{x}_n)^t$ and $Y = (\underline{y}_1, \cdots, \underline{y}_m, \overline{y}_1, \cdots, \overline{y}_m)^t$.

Therefore, from Eq.(6) we have:

$$\begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix} \begin{pmatrix} \underline{X}_1 \\ \overline{X}_2 \end{pmatrix} = \begin{pmatrix} \underline{Y}_1 \\ \overline{Y}_2 \end{pmatrix}. \tag{7}$$

In [3, 6], we can find the following theorems and here omit their proofs.

Theorem 2.4 *The matrix S is a non-singular if and only if the matrix $A = S_1 + S_2$ and $S_1 - S_2$ are both non-singular.*

Theorem 2.5 *If S^{-1} exists it must have the same structure as S, i.e.*

$$S^{-1} = \begin{pmatrix} D & C \\ C & D \end{pmatrix}. \tag{8}$$

Theorem 2.6 *The unique solution X is a fuzzy vector for arbitrary Y if S^{-1} is nonnegative,*

$$(S^{-1})_{ij} \geq 0, 1 \leq i, j \leq 2n. \tag{9}$$

Theorem 2.7 *The inverse of a nonnegative matrix A is nonnegative if and only if A is a generalized permutation matrix.*

Definition 2.8 *Let $X = \{(\underline{x}_i(r), \overline{x}_i(r)), 1 \leq i \leq n\}$ denote the unique solution of $SX = Y$. The fuzzy number $U = \{(\underline{u}_i(r), \overline{u}_i(r)), 1 \leq i \leq n\}$ is defined by*

$$\begin{aligned} \underline{u}_i(r) &= \min\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1), \overline{x}_i(1)\}, \\ \overline{u}_i(r) &= \max\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1), \overline{x}_i(1)\}, \end{aligned} \tag{10}$$

is called the fuzzy solution of $SX = Y$. If $(\underline{x}_i(r), \overline{x}_i(r))$, are fuzzy numbers for every i, $1 \leq i \leq n$ then U is called a strong fuzzy solution. Otherwise, U is a weak fuzzy solution.

3 $QR-$ decomposition method for solving FLSE

Here we first state some important results concerning to the matrix computing which is valid in the crisp environments and then we apply it for solving FLSE.

Theorem 3.1 *If A is an $m \times n$ matrix with full column rank, then A can be factored as $A = QR$, where Q is an $m \times n$ matrix whose column vectors form an orthonormal basis for the column space of A and R is a $n \times n$ invertible upper triangular matrix .*

In above theorem that discussed in [5], an orthonormal basis is constructed easily from an orthogonal basis $v_1, \ldots, v_p$: Simply normalize (i.e., "scale") all the v_k. When working problems by hand, this is easier than normalizing each v_k as soon as it is found (because it avoids unnecessary writing of square roots). This theorem guarantees that every matrix A with full column rank has a $QR-$decomposition, in particular, if A is invertible. The fact that Q has orthonormal column implies that $Q^T Q = I$, so multiplying both side of $A = QR$ by on the left side

$$R = Q^T A. \tag{11}$$

Thus, one method for finding a QR-decomposition of a matrix A with full rank is to apply the Householder process to the column vectors of A, then form the matrix Q from the resulting orthonormal basis vectors, and then find R from (11). We can find Householder method for $QR-$ decomposition (in detail) and the following theorem in [5].

Theorem 3.2 *If A is a matrix with full column rank, and if $A = QR$ is a QR-decomposition of A, then the normal system for $Ax = b$ can be expressed as*

$$Rx = Q^T b, \tag{12}$$

and the solution can be expressed as

$$\hat{x} = R^{-1} Q^T b. \tag{13}$$

Now again consider fuzzy linear system of equations (2) and the its extended parametric system as defined in (6). We are going to apply the above idea for solving $2m \times 2n$ system. So, if S be a non-singular matrix we have

$$Sx = y, \tag{14}$$

then by using a QR-decomposition of S, as

$$S = QR, \tag{15}$$

we have

$$QRx = y, \tag{16}$$

thus

$$Rx = Q^T y, \tag{17}$$

therefore, the solution can be express as

$$\hat{x} = R^{-1} Q^T y. \tag{18}$$

4 Numerical examples

Example 4.1. Use QR-decomposition to obtain a solution of the fuzzy system:

$$\begin{cases} x_1 - x_2 = (-7 + 2r, -3 - 2r), \\ x_1 + 3x_2 = (19 + 4r, 27 - 4r). \end{cases}$$

The extended matrix is $S = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & 3 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}$ and then we obtain:

$$Q = \begin{pmatrix} -0.7071 & 0.6396 & 0.2023 & 0.2236 \\ -0.7071 & -0.6396 & -0.2023 & -0.2236 \\ 0 & 0.4264 & -0.6068 & -0.6708 \\ 0 & 0 & -0.7416 & 0.6708 \end{pmatrix}, R = \begin{pmatrix} -1.4142 & -2.1213 & 0 & 0.7071 \\ 0 & -2.3452 & 0.4264 & -0.6396 \\ 0 & 0. & -0.3484 & -2.4271 \\ 0 & 0 & 0 & 1.7889 \end{pmatrix}.$$

Now by using (18), we have: $\hat{x} = R^{-1}Q^T b = (1+r, 6+r, 3-r, 8-r)$.

Then, the solution can be obtain as follows:

$$x_1 = (\underline{x}_1(r), \overline{x}_1(r)) = (1+r, 3-r),$$
$$x_2 = (\underline{x}_2(r), \overline{x}_2(r)) = (6+r, 8-r).$$

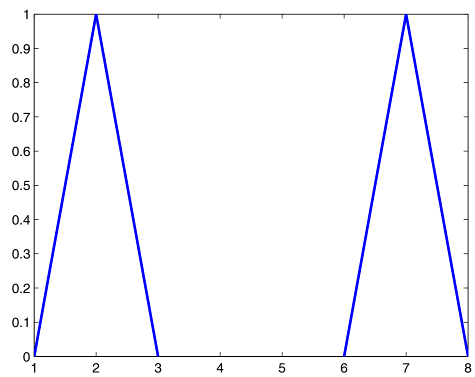

Fig. 4.1. The solution for Example 4.1

As we see, the solution by our method is same with the solution is obtained by LU decomposition method that is proposed in [1].

Example 4.2. Apply QR-decomposition to find a solution of the fuzzy system:

$$\begin{cases} 4x_1 + 2x_2 = (4+4r, 10-2r), \\ -x_1 + 3x_2 = (1+r, 3-r), \\ 4x_1 + 2x_2 = (4+4r, 10-2r). \end{cases}$$

The extended matrix is S= $\begin{pmatrix} 4 & 2 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ -1 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 \end{pmatrix}$ and we obtain QR-decomposition:

$$Q = \begin{pmatrix} -0.8728 & -0.0313 & -0.0069 & 0.1925 \\ 0 & -0.9870 & 0.0057 & -0.1604 \\ -0.4364 & -0.0156 & -0.0034 & 0.0962 \\ 0 & 0 & -0.8938 & -0.0320 \\ 0.2182 & -0.1566 & -0.0345 & 0.9626 \\ 0 & 0 & -0.4469 & -0.0160 \end{pmatrix}, R = \begin{pmatrix} -4.2825 & -2.1821 & 0 & 0.6546 \\ 0 & -3.0394 & 0.9870 & -0.4700 \\ 0 & 0 & -4.4750 & -2.3382 \\ 0 & 0 & 0 & 2.8076 \end{pmatrix}.$$

Now using our method, we obtain:
$\hat{x} = R^{-1}Q^T b = (0.514 + 0.914r, 0.971 + 0.171r, 1.914 - 0.485r, 1.171 - 0.028r)$.
So,

$$x_1 = (\underline{x}_1(r), \overline{x}_1(r)) = (0.514 + 0.914r, 1.914 - 0.485r),$$
$$x_2 = (\underline{x}_2(r), \overline{x}_2(r)) = (0.971 + 0.171r, 1.171 - 0.028r).$$

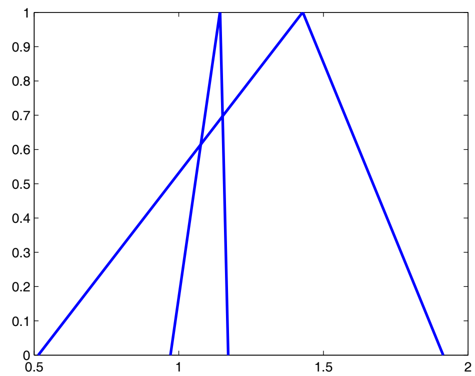

Fig. 4.2. The solution for Example 4.2

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