Applied Mathematical Sciences, Vol. 2, 2008, no. 52, 2577 - 2590

Exact Solution of Time Fractional Partial Differential Equation

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Abstract

In this paper, a time fractional partial differential equation is considered, where the fractional derivative is defined in the Caputo sense. A function transform and Laplace transform method along with an intermediate step of Mellin transform have been applied to achieve an exact solution in terms of the H-function and the complement error function. A number of special cases is also considered.

Mathematics Subject Classification: 26A33, 44A20, 35A22

Keywords: Time fractional partial differential equation, Laplace transform, Mellin transform, H-function

1. Introduction

Time fractional partial differential equations, obtained from the standard partial differential equations by replacing the integer order time derivative by a fractional derivative (of order $0 < \alpha \leq 1$, in Riemann-Liouville or Caputo sence), have been studied and treated in different contexts by several research workers. The fractional diffusion equation, the fractional wave equation, the fractional advection-dispersion equation, the fractional kinetic equation and other fractional partial differential equations have been studied and explicit solutions have been achieved by Mainardi, Pagnini and Saxena[9], Langlands[4], Mainardi, Pagnini and Gorenflo[8],Mainardi and Pagnini[6,7], Yu and Zhang[16],Liu, Anh, Turner and Zhang[5], Saichev and Zaslavsky[11],Saxena, Mathai and Haubold[12], Wyss[15] , Schneider and Wyss[13] and several other research works can be found in the literature. In these works , the techniques of using integral transforms (Laplace, Fourier and Mellin transforms) were used to maintain exact solutions of fractional partial differential equations explicitly.

In this paper, we consider the time fractional partial differential equation

$$\frac{\partial^\alpha C(x,t)}{\partial t^\alpha} = a\frac{\partial^2 C(x,t)}{\partial x^2} - b\frac{\partial C(x,t)}{\partial x} - \lambda C(x,t) \tag{1}$$

for $x > 0, t > 0, m-1 < \alpha \leq m$,where $\frac{\partial^\alpha C(x,t)}{\partial t^\alpha}$ is the fractional derivative with respect to time t ,$a > 0, b \geq 0$ and $\lambda \geq 0$. Subject to initial conditions

$$C^{(k)}(x,0) = C_k(x), \qquad k = 0,1,...,m-1 \tag{2}$$

This time fractional partial differential equation contains several known partial differential equations as special cases. For example:

(i) Setting $\lambda = 0$ and $0 < \alpha \leq 1$, then we get the equation

$$\frac{\partial^\alpha C(x,t)}{\partial t^\alpha} = a\frac{\partial^2 C(x,t)}{\partial x^2} - b\frac{\partial C(x,t)}{\partial x}, \quad C(x,0) = C_0(x) \tag{3}$$

which is the time fractional advection-dispersion equation. Moreover if $\alpha = 1$, then the equation is the standard advection-dispersion equation.

(ii) If $b = 0$ in (3), then we get the equation

$$\frac{\partial^\alpha C(x,t)}{\partial t^\alpha} = a\frac{\partial^2 C(x,t)}{\partial x^2}, \quad C(x,0) = C_0(x) \tag{4}$$

which is the time fractional diffusion equation, and setting $\alpha = 1$ yields the standard diffusion equation.

(iii) Setting $b = 0$, $\lambda = 0$ and $1 < \alpha \leq 2$, then we get

$$\frac{\partial^\alpha C(x,t)}{\partial t^\alpha} = a\frac{\partial^2 C(x,t)}{\partial x^2}, \quad C^{(k)}(x,0) = C_k(x), \quad k = 0,1 \tag{5}$$

which is the fractional wave equation. For $\alpha = 2$, we get the D'Alembert wave equation

$$\frac{\partial^2 C(x,t)}{\partial t^2} = a\frac{\partial^2 C(x,t)}{\partial x^2}, \quad C^{(k)}(x,0) = C_k(x), \quad k = 0,1 \tag{6}$$

In this paper, we handle in details the solution of Eqs.(1) and (2) for the cases when $0 < \alpha \leq 1$ and when $1 < \alpha \leq 2$.

The fractional derivative ${}_0D_t^\alpha C(x,t) = \frac{\partial^\alpha C(x,t)}{\partial t^\alpha}$ is considered in the Caputo sense and is defined for $m - 1 < \alpha \leq m$ as

$${}_0D_t^\alpha C(x,t) = \frac{\partial^\alpha C(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{\partial^m C(x,\tau)}{\partial t^m} \frac{d\tau}{(t-\tau)^{\alpha-m+1}}$$

$$= J_0^{(\alpha-m)}{}_0D_t^m C(x,t) \tag{7}$$

, where J^α is the Riemann-Liouville fractional integral operator defined as

$$J^\alpha C(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} C(x,\tau) d\tau, \quad \alpha > 0 \tag{8}$$

and

$$J_0^\alpha D_t^\alpha C(x,t) = C(x,t) - \sum_{k=0}^{m-1} \frac{x^k}{k!} C^{(k)}(x,0), \quad m-1 < \alpha \leq m. \tag{9}$$

For more detailed discussion on this fractional derivative, we refer the reader to e.g. [3,10].

The method used to achieve the exact solution of (1) and (2) depends on the Laplace transform defined by

$$\widetilde{f}(p) = L\{f(t);p\} = \int_0^\infty e^{-pt} f(t)dt, \tag{10}$$

and on the Mellin transform defined by

$$f^*(s) = M\{f(t);s\} = \int_0^\infty t^{s-1} f(t)dt. \tag{11}$$

For detailed properties and tables of Laplace and Mellin transforms, one can refer to [2].

2. The Green function

Using Eq.(9), then Eq.(1) can be expressed as the following integral equation

$$C(x,t) = \sum_{k=0}^{m-1} C_k(x)\frac{x^k}{k!} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}$$

$$\left[a\frac{\partial^2 C(x,\tau)}{\partial x^2} - b\frac{\partial C(x,\tau)}{\partial x} - \lambda C(x,\tau)\right] d\tau \tag{12}$$

for $m-1 < \alpha \leq m$. Assume $C(x,t) = u(\xi,t)\exp(\mu\xi)$, $\mu = \dfrac{b}{2\sqrt{a}}$ and $\xi = \dfrac{x}{\sqrt{a}}$, then Eq.(12) becomes

$$u(\xi,t) = \sum_{j=0}^{m-1} g_j(\xi)\frac{t^j}{j!} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[\frac{\partial^2 u(\xi,\tau)}{\partial \xi^2} - \theta^2 u(\xi,\tau)\right] d\tau \tag{13}$$

,where $g_j(\xi) = u^{(j)}(\xi,0)$,$j = 1,2$ and $\theta^2 = \mu^2 + \lambda$.

The Laplace transform of Eq.(13) with respect to t can be written for $p > 0$ as

$$\widetilde{u}(\xi,p)=\sum_{j=0}^{m-1} g_j(\xi)p^{-j-1}+p^{-\alpha}\left[\frac{\partial^2\widetilde{u}(\xi,p)}{\partial\xi^2}-\theta^2\widetilde{u}(\xi,p)\right] \tag{14}$$

or

$$\frac{\partial^2\widetilde{u}(\xi,p)}{\partial\xi^2}-\omega^2\widetilde{u}(\xi,p)=-\sum_{j=0}^{m-1} p^{\alpha-j-1}g_j(\xi) \tag{15}$$

,where $\omega^2=\theta^2+p^\alpha$.
According to Schneider and Wyss [8], Eq.(15) has the solution

$$\widetilde{u}(\xi,p)=\sum_{j=0}^{m-1}\int_0^\infty \widetilde{G}_\theta^{\alpha,j}(\mid \xi-y\mid,p)g_j(y)dy \tag{16}$$

,where the Green function is

$$\widetilde{G}_\theta^{\alpha,j}(r,p)=p^{\alpha-j-1}\left(\frac{r}{2\pi\omega}\right)^{1/2}k_{1/2}(\omega r)$$

$$=p^{\alpha-j-1}\left(\frac{r}{2\pi}\right)^{1/2}(\theta^2+p^\alpha)^{-1/4}k_{1/2}(r\sqrt{\theta^2+p^\alpha})\,,\qquad j=0,1 \tag{17}$$

where $k_{1/2}(z)$ is the modified Bessel function of the third kind [1].

A direct transition to the time domain (i.e. inverting the Laplace transform) does not seem to be feasible. This difficulty is circumvented by passing through the intermediate step of the Mellin transform (11) which is connected to the Laplace transform by the relation

$$f^*(s)=\frac{1}{\Gamma(1-s)}\int_0^\infty p^{-s}\widetilde{f}(p)dp \tag{18}$$

Then we get

$$G_\theta^{*\alpha,j}(r,s)=\frac{1}{\Gamma(1-s)}\int_0^\infty p^{-s}\widetilde{G}_\theta^{\alpha,j}(r,p)dp\ ,\qquad j=0,1 \tag{19}$$

or

$$G_\theta^{*\alpha,j}(r,s) = \left(\frac{r}{2\pi}\right)^{1/2} \frac{1}{\Gamma(1-s)}$$

$$\int_0^\infty p^{-s+\alpha-j-1}(\theta^2+p^\alpha)^{-1/4} k_{1/2}\left(r\sqrt{\theta^2+p^\alpha}\right) dp \ , \qquad j=0,1 \tag{20}$$

Let $q = p^{\alpha/2}$, then for $j = 0, 1$ Eq.(18) yields

$$G_\theta^{*\alpha,j}(r,s) = \frac{2}{\alpha}\left(\frac{r}{2\pi}\right)^{1/2} \frac{1}{\Gamma(1-s)}$$

$$\int_0^\infty q^{\left(2-\frac{2s}{\alpha}-\frac{2j}{\alpha}\right)-1} \left(\theta^2+q^2\right)^{-1/4} k_{1/2}\left(r\sqrt{\theta^2+q^2}\right) dq \tag{21}$$

Now, using the formula [2]

$$M\left\{\left(t^2+b^2\right)^{-1/2\nu} k_\nu\left[a\left(t^2+b^2\right)^{1/2}\right]; s\right\} = a^{\frac{-s}{2}} 2^{\frac{s}{2}-1} b^{\frac{s}{2}-\nu} \Gamma(s/2) k_{\nu-\frac{s}{2}}(ab) \tag{22}$$

$min\left\{Re(a), Re(b), Re(s)\right\} > 0$, we get

$$G_\theta^{*\alpha,j}(r,s) = \frac{1}{\alpha\sqrt{\pi}}\left(\frac{2\theta}{r}\right)^{1/2-s/\alpha-j/\alpha} \frac{\Gamma(1-s/\alpha-j/\alpha)}{\Gamma(1-s)} k_{\frac{s}{\alpha}+\frac{j}{\alpha}-\frac{1}{2}}(r\theta) \tag{23}$$

Now from the H-function representation (A2), we have

$$\frac{\Gamma(1-j/\alpha-s/\alpha)}{\Gamma(1-s)} = H^{*1,0}_{\ 1,1}\left(z \left|^{(1,1)}_{(1-j/\alpha,1/\alpha)}\right.\right)(-s) \tag{24}$$

Let $\varphi(p) = \frac{1}{2}\exp\left\{\frac{-1}{2}\theta r(p^\alpha + p^{-\alpha})\right\}$ and $\phi(p) = p^{-\alpha}\varphi(p)$
Then, from the properties of the Mellin transform [2]

$$\varphi^*(s) = \frac{1}{\alpha} k_{s/\alpha}(\theta r), \phi^*(s) = \frac{1}{\alpha} k_{\frac{s}{\alpha}+\frac{j}{\alpha}-\frac{1}{2}}(\theta r) \tag{25}$$

Thus the Mellin transform $G_\theta^{*\alpha}(r,s)$ can be written as

$$G_{\theta}^{*\alpha,j}(r,s) = \frac{1}{\sqrt{\pi}}\left(\frac{2\theta}{r}\right)^{1/2-j/\alpha}\left(\frac{2\theta}{r}\right)^{-s/\alpha}\phi_j^*(s){H^*}_{1,1}^{1,0}(-s) \tag{26}$$

Now $M\{\varphi(at);s\} = a^{-s}\varphi^*(s)$ and $M\{\varphi(t^m);s\} = \frac{1}{|m|}\varphi^*(s/m), m \neq 0$, and by the Mellin convolution theorem $M\left\{\int_0^\infty h\left(\frac{t}{z}\right)\varphi(z)\frac{dz}{z};s\right\} = h^*(s)\varphi^*(s)$,then we get for $j = 0, 1$

$$M\left\{\int_0^\infty \phi_j\left((2\theta/r)^{\frac{1}{\alpha}}\frac{t}{z}\right)H_{1,1}^{1,0}(z^{-1})\frac{dz}{z};s\right\} = \left[\left(\frac{2\theta}{r}\right)^{\frac{1}{\alpha}}\right]^{-s}\phi_j^*(s){H^*}_{1,1}^{1,0}(-s) \tag{27}$$

Let $\eta = \dfrac{1}{z}$,then

$$\int_0^\infty \phi_j\left(\left(\frac{2\theta}{r}\right)^{1/\alpha}\frac{t}{z}\right)H_{1,1}^{1,0}(z^{-1})\frac{dz}{z} = \int_0^\infty \phi_j\left(\left(\frac{2\theta}{r}\right)^{1/\alpha}\eta t\right)H_{1,1}^{1,0}(\eta)\frac{d\eta}{\eta} \tag{28}$$

so the inverse Mellin transform leads to

$$\begin{aligned}
G_{\theta}^{\alpha,j}(r,t) &= \frac{1}{\sqrt{\pi}}(\frac{2\theta}{r})^{\frac{1}{2}-\frac{j}{\alpha}}\int_0^\infty \phi_j\left(\left(\frac{2\theta}{r}\right)^{1/\alpha}\eta t\right)H_{1,1}^{1,0}(\eta)\frac{d\eta}{\eta} \\
&= \frac{1}{\sqrt{\pi}}(\frac{2\theta}{r})^{\frac{1}{2}-\frac{j}{\alpha}}\int_0^\infty \left(\left(\frac{2\theta}{r}\right)^{1/\alpha}\eta t\right)^{j-\frac{\alpha}{2}}\varphi\left(\left(\frac{2\theta}{r}\right)^{1/\alpha}\eta t\right)H_{1,1}^{1,0}(\eta)\frac{d\eta}{\eta} \\
&= \frac{t^{j-\frac{\alpha}{2}}}{2\sqrt{\pi}}\int_0^\infty \eta^{j-\frac{\alpha}{2}}\exp\left[-\theta^2 t^\alpha\eta^\alpha - \frac{r^2}{4}t^{-\alpha}\eta^{-\alpha}\right]H_{1,1}^{1,0}(\eta)\frac{d\eta}{\eta} \qquad (29)
\end{aligned}$$

Let $\sigma = \eta^\alpha$ and using the formulas (A9) and (A10) of the H-functions , then

$$G_{\theta}^{\alpha,j}(r,t) = \frac{t^{j-\frac{\alpha}{2}}}{2\sqrt{\pi}}\int_0^\infty \exp\left[-\theta^2 t^\alpha\sigma - \frac{r^2}{4}t^{-\alpha}\sigma^{-1}\right]H_{1,1}^{1,0}\left[\sigma\ \Big|_{(-1/2,1)}^{(1+j-3\alpha/2,\alpha)}\right]d\sigma \tag{30}$$

So the inverse Laplace transform of Eq.(16) yields

$$u(\xi,t)=\sum_{j=0}^{m-1}\int_0^\infty G_\theta^{\alpha,j}\left(|\xi-y|\,,t\right)g_j(y)dy \quad , \quad j=0,1 \tag{31}$$

3. The complete solution and special cases:

In this section, we find the exact solution of the initial problem given in (1) and (2), Eq. (31) implies that

$$C(x,t)=e^{\frac{\mu}{\sqrt{a}}x}\sum_{j=0}^{m-1}\int_0^\infty G_\theta^{\alpha,j}\left(\left|\frac{x}{\sqrt{a}}-y\right|,t\right)g_j(y)dy \quad , \quad j=0,1 \tag{32}$$

1. The case $0<\alpha\leq 1$ *:*

In this case $g(\xi)=u(\xi,0)=C_0\exp(-\mu\xi)$, so the the solution of the fractional partial differential equation is written as

$$C(x,t)=\frac{C_0}{2\sqrt{\pi t^\alpha}}\int_0^\infty \exp(-4\lambda t^{2\alpha}\sigma^2)H_{1,1}^{1,0}\left[\sigma\left|_{(-1/2,1)}^{(1-3\alpha/2,\alpha)}\right.\right]$$

$$\left(\int_0^\infty \exp\left(-\left[\frac{\frac{x}{\sqrt{a}}-y-2\mu t^\alpha\sigma}{2\sqrt{t^\alpha\sigma}}\right]\right)dy\right)d\sigma \tag{33}$$

Now, let $\beta=\dfrac{2\mu t^\alpha\sigma-\frac{x}{\sqrt{a}}+y}{2\sqrt{t^\alpha\sigma}}$, then Eq.(33) yields

$$C(x,t)=\frac{C_0}{\sqrt{\pi}}\int_0^\infty \exp(-4\lambda t^{2\alpha}\sigma^2)\sigma^{1/2}H_{1,1}^{1,0}\left[\sigma\left|_{(-1/2,1)}^{(1-3\alpha/2,\alpha)}\right.\right]\left(\int_{\beta_1}^\infty e^{-\beta^2}d\beta\right)d\sigma \tag{34}$$

where $\beta_1=\dfrac{2\mu t^\alpha\sigma-\frac{x}{\sqrt{a}}}{2\sqrt{t^\alpha\sigma}}$.

And using the property of the H-function (A9), we get

$$C(x,t) = \frac{C_0}{2}\int\limits_0^\infty \exp(-4\lambda t^{2\alpha}\sigma^2) H_{1,1}^{1,0}\left[\sigma \left|_{(0,1)}^{(1-\alpha,\alpha)}\right.\right](erfc\,(\beta_1))\,d\sigma \tag{35}$$

, where $erfc(\beta_1) = \frac{2}{\sqrt{\pi}}\int\limits_{\beta_1}^{\infty}\exp\left(-\beta^2\right)d\beta$

2. The case $1 < \alpha \le 2$:

Let $C_0(x) = C_0$ and $C_1(x) = C_1$, then $g_1(y) = C_0\exp(-\mu y)$ and $g_2(y) = C_1\exp(-\mu y)$ then the solution is

$$\begin{aligned} C(x,t) &= \frac{C_0}{2}\int\limits_0^\infty \exp\left(-4\lambda t^{2\alpha}\sigma^2\right) H_{1,1}^{1,0}\left[\sigma \left|_{(0,1)}^{(1-\alpha,\alpha)}\right.\right](erfc(\beta_1))d\sigma + \\ &\quad \frac{C_1 t}{2}\int\limits_0^\infty \exp\left(-4\lambda t^{2\alpha}\sigma^2\right) H_{1,1}^{1,0}\left[\sigma \left|_{(0,1)}^{(2-\alpha,\alpha)}\right.\right](erfc(\beta_1))d\sigma \end{aligned} \tag{36}$$

3. Let $\lambda = 0$ and $0 < \alpha \le 1$, and assuming $C_0(x) = C_0$, then the solution of Eq.(3) is given by

$$C(x,t) = \frac{C_0}{2}\int\limits_0^\infty H_{1,1}^{1,0}\left[\sigma \left|_{(0,1)}^{(1-\alpha,\alpha)}\right.\right](erfc(\beta_1))\,d\sigma \tag{37}$$

which is the result obtained in [5]

4. Moreover let $b = 0$, and $C_0(x) = C_0$, then the solution of the time fractional diffusion Eq.(4) is

$$C\,(x,t) = \frac{C_0}{2}\int\limits_0^\infty H_{1,1}^{1,0}\left[\sigma \left|_{(0,1)}^{(1-\alpha,\alpha)}\right.\right]\left(erfc\left(\frac{-x}{2\sqrt{t^\alpha \sigma}}\right)\right)d\sigma \tag{38}$$

5. If $\lambda = b = 0, a = 1, 1 < \alpha \le 2$, we get the time fractional wave equation subject to the initial conditions $C(x,0) = C_0(x)$ and $\frac{\partial C(x,t)}{\partial t} = C_1(x)$, then iff $C_0(x) = C_0$ and $C_1(x) = C_1$, we get the solution of Eq.(5) as

$$\begin{aligned} C(x,t) &= \frac{C_0}{2}\int_0^{\infty} H_{1,1}^{1,0}\left[\sigma \left|_{(0,1)}^{(1-\alpha,\alpha)}\right.\right]\left(erfc\left(\frac{-x}{2\sqrt{t^{\alpha}\sigma}}\right)\right)d\sigma \\ &\quad + \frac{C_1 t}{2}\int_0^{\infty} H_{1,1}^{1,0}\left[\sigma \left|_{(0,1)}^{(2-\alpha,\alpha)}\right.\right]\left(erfc\left(\frac{-x}{2\sqrt{t^{\alpha}\sigma}}\right)\right)d\sigma \end{aligned} \quad (39)$$

4. Conclusion

The time fractional partial differential equation is obtained from the classical partial differential equation by replacing the first order time deravative by a fractional derivative of order $(m-1<\alpha\leq m)$. Using variable transformation, the intermediate steps of Mellin and Laplace transforms, we derive the complete solution of this time fractional partial differential equation. Also special cases have been obtained for $0<\alpha\leq 1$ and $1<\alpha\leq 2$.

Appendix: The H-function

An H-function is defined in terms of a Mellin-Bernes type integral as follows [14]:

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z\left|_{(b_1,\beta_1)\ldots(b_q,\beta_q)}^{(a_1,\alpha_1)\ldots(a_p,\alpha_p)}\right.\right] = \frac{1}{2\pi i}\int_L H_{p,q}^{*m,n}(s)z^{-s}ds \quad (A1)$$

wherem,n,p and q are nonnegative integeres such that $0\leq n\leq p, 1\leq m\leq q$ and empty products are interpreted as unity. The parameters $\alpha_1,...,\alpha_p$ and $\beta_1,...,\beta_q$ are positive real numbers, whereas $a_1,...,a_p$ and $b_1,...,b_q$ are complex numbers.The Fox or H-function is characterized by its Mellin transform:

$$H^{*m,n}_{\ p,q}(s) = H^{*m,n}_{\ p,q}\left[z\left|_{(b_i,\beta j)i=1,\ldots,q}^{(a_j,\alpha_j)j=1,\ldots,p}\right.\right](s) = \frac{A(s)B(s)}{C(s)D(s)} \quad (A2)$$

where

$$A(s) = \prod_{j=1}^{m}\Gamma\left(b_j+\beta_j s\right), B(s) = \prod_{j=1}^{m}\Gamma\left(1-a_j-\alpha_j s\right),$$

$$C(s) = \prod_{j=m+1}^{q}\Gamma\left(1-b_j-\beta_j s\right), D(s) = \prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s).$$

In (A1) $z^{-s} = \exp\{-s\log|z| - i\arg(z)\}$ and $\arg(z)$ is not necessarily the principal value. The parameters are restricted by the condition $P(A) \cap P(B) = \varnothing$,where

$$P(A) = \{poles\ of\ \ \Gamma(1 - a_i + \alpha_i s)\} = \left\{\tfrac{1-a_i+k}{\alpha i} \in C; i = 1, ..., n,\ k \in N_0\right\},$$

$$P(B) = \{poles\ \ of\ \ \Gamma(b_i + \beta_i s)\} = \left\{\tfrac{-b_i-k}{\beta_i} \in C; i = 1, ..., m,\ k \in N_0\right\}, N_0 = \{0, 1, ...\}$$

The integral (A1) converges if one of the following conditions holds

$$L = L(c - i\infty, c + i\infty; P(A), P(B)), |\arg z| < \frac{\omega\pi}{2}, \omega > 0;$$

$$L = L(c - i\infty, c + i\infty; P(A), P(B), |\arg z| < \frac{\omega\pi}{2}, \omega \geq 0, cR < -Re(\gamma);$$

where

$$\omega = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=1}^{q} \beta_j,\ R = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j,$$

$$\gamma = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2} + 1.$$

The H-function are analytic for $z \neq 0$ and multi-valued (single-valued on the Riemann surface of $\log z$).The H-functions may be represented as

$$H_{p,q}^{m,n}\left[z\,\Big|_{(b_1,\beta_1)...(b_q,\beta_q)}^{(a_1,\alpha_1)...(a_p,,\alpha_p)}\right] = \sum_{i=1}^{m}\sum_{k=0}^{\infty} c_{ik}\frac{(-1)^k}{k!\beta_i} z^{\frac{b_i+k}{\beta_i}} \qquad (A3)$$

where

$$c_{ik} = \frac{\prod\limits_{j=1, j\neq i}^{m} \Gamma(b_j - (b_i + k)\frac{\beta_j}{\beta_i}) \prod\limits_{j=1}^{n} \Gamma(1 - a_j + (b_i + k)\frac{\alpha_j}{\beta_i})}{\prod\limits_{j=m+1}^{q} \Gamma(1 - b_j + (b_i + k)\frac{\beta_j}{\beta_i}) \prod\limits_{j=n+1}^{p} \Gamma(a_j - (b_i + k)\frac{\alpha_j}{\beta_i})} \qquad (A4)$$

whenever $R \geq 0$ and the poles in $P(A)$ are simple. Similarly,

$$H_{p,q}^{m,n}\left[z\,\Big|_{(b_1,\beta_1)...(b_q,\beta_q)}^{(a_1,\alpha_1)...(a_p,\alpha_p)}\right] = \sum_{i=1}^{m}\sum_{k=0}^{\infty} c_{ik}\frac{(-1)^k}{k!\alpha_i} z^{-\frac{1+\alpha_i+k}{\alpha_i}}, \qquad (A5)$$

where

$$c_{ik}=\frac{\prod\limits_{j=1}^{m}\Gamma\left(b_j+(1-a_i+k)\frac{\beta_j}{\alpha_i}\right)\prod\limits_{j=1,j\neq i}^{n}\Gamma\left(1-a_j+(1-a_i+k)\frac{\alpha_j}{\alpha_i}\right)}{\prod\limits_{j=m+1}^{q}\Gamma\left(1-b_j+(1-a_i+k)\frac{\beta_j}{\alpha_i}\right)\prod\limits_{j=n+1}^{p}\Gamma\left(a_j+(1-a_i+k)\frac{\alpha_j}{\alpha_i}\right)} \qquad (A6)$$

whenever $R\leq 0$ and the poles in $P(A)$ are simple.

In particular, if $R>0$, we obtain from (A3) that

$$H_{p,q}^{1,n}(z)=H_{p,q}^{1,n}\left[z\left|^{(a_1,\alpha_1)\dots(a_p,\alpha_p)}_{(b_1,\beta_1)\dots(b_q,\beta_q)}\right.\right]=\frac{1}{\beta_1}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{B(s_k)}{C(s_k)D(s_k)}z^{-s_k}, \qquad (A7)$$

where $s_k=-b_1+k/\beta_1$.In the case $R<0$, we obtain from (A3) that

$$H_{p,q}^{m,1}(z)=H_{p,q}^{m,1}\left[z\left|^{(a_1,\alpha_1)\dots(a_p,\alpha_p)}_{(b_1,\beta_1)\dots(b_q,\beta_q)}\right.\right]=\frac{1}{\alpha_1}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{A(s_k)}{C(s_k)D(s_k)}z^{-s_k}, \qquad (A8)$$

where $s_k=(k+1-\alpha_1)/\alpha_1$.

The following identities for the H-function are well-known:

$$z^k H_{p,q}^{m,n}\left[z\left|^{(a_1,\alpha_1)\dots(a_p,\alpha_p)}_{(b_1,\beta_1)\dots(b_q,\beta_q)}\right.\right]=H_{p,q}^{m,n}\left[z\left|^{(a_1+k\alpha_1,\alpha_1)\dots(a_p+k\alpha_p,\alpha_p)}_{(b_1+k\beta_1,\beta_1)\dots(b_q+k\beta_q,\beta_q)}\right.\right], \qquad (A9)$$

$$H_{p,q}^{m,n}\left[z\left|^{(a_1,\alpha_1)\dots(a_p,\alpha_p)}_{(b_1,\beta_1)\dots(b_q,\beta_q)}\right.\right]=kH_{p,q}^{m,n}\left[z^k\left|^{(a_1,k\alpha_1)\dots(a_p,k\alpha_p)}_{(b_1,k\beta_1)\dots(b_q,k\beta_q)}\right.\right]. \qquad (A10)$$

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Received: March 20, 2008