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An Alternating Group Explicit Iterative Method for Solving Four-order Parabolic Equations

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Abstract

A class of alternating group explicit iterative method using the C-N scheme is derived for solving four order parabolic equations, and the analysis for convergency is given. In the end, the numerical result is given, which shows the iterative method is convergent, high precision, and is suitable for parallel computing.

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1 Preface

In Science and engineering computing, parabolic equations are important partial differential equations. many researches on numerical algorithms for low order parabolic equations have been done so far, but schemes for high order parabolic equations which are of high precision and are of constant stability have been scarcely presented.

With the development of parallel computer, researches on parallel computing method is getting more and more popular, In [1], Evans D J presented a class of alternating group method(AGE), which is constantly stable and is suitable for parallel computing. Later, The method is widely cared, and many alternating group schemes for kinds of partial differential equations are presented in [2-6] and [7, 9, 10]. In this thesis, an alternating group explicit iterative method using the C-N scheme is derived for solving four-order parabolic equations. The result of convergence analysis for the iterative method shows that the method is convergent. In the end, results of numerical experiments are given, which shows that the method is of high precision.

In this thesis, a time-dependent periodic boundary problem of four order parabolic equation (1) is considered.

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial^4 u}{\partial x^4} = 0, \quad -\infty < x < \infty, \quad 0 \le t \le T \\ u(x,0) = f(x), \\ u(x+l,t) = u(x,t). \end{cases} \tag{1}$$

Let h and τ be the grids spacings in the x, and t directions respectively. $h = \frac{L}{m}$, then in an period we have $x_i = ih(i = 0,\ 1, \cdots, m)$, $t_n = n\tau(n = 0,\ 1, \cdots, \frac{T}{\tau})$. $(x_i,\ t_n)$ is denoted by $(i,\ n)$, and the numerical solution at the node is u_i^n, while the exact solution is $u(x_i,\ t_n)$. By (1), $u(x_0,\ t) = u(x_m,\ t)$, $r = \frac{\tau}{2h^4}$.

2 The Alternating Group Iterative Method

The purpose of this paper is to get the numerical results of the (n+1)th time layer with the results of the (n)th time layer given. The Crank-Nicolson scheme for solving (1) is presented as below:

$$\begin{aligned} &ru_{i-2}^{n+1} - 4ru_{i-1}^{n+1} + (1+6r)u_i^{n+1} - 4ru_{i+1}^{n+1} + ru_{i+2}^{n+1} \\ &= -ru_{i-2}^{n} + 4ru_{i-1}^{n} + (1-6r)u_i^{n} + 4ru_{i+1}^{n} - ru_{i+2}^{n} \end{aligned} \tag{2}$$

Let $U^n = (u_1^n,\ u_2^n,\ \cdots,\ u_m^n)^T$, then we have $AU^{n+1} = F^n$. here F^n is known.

$$A = \begin{pmatrix} 6 & -4 & 1 & & & & 1 & -4 \\ -4 & 6 & -4 & 1 & & & & 1 \\ 1 & -4 & 6 & -4 & 1 & & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & 1 & -4 & 6 & & -4 & 1 \\ 1 & & & 1 & -4 & & 6 & -4 \\ 1 & -4 & & & 1 & & -4 & 6 \end{pmatrix}_{(m-1)\times(m-1)}$$

Let $m = 4k$, k is an integer, $A = \frac{1}{2}(G_1 + G_2)$

$$G_1 = \begin{pmatrix} G_{2p} & & & & \\ & \cdots & & & \\ & & \cdots & & \\ & & & \cdots & \\ & & & & G_{2p} \end{pmatrix}_{(m-1)\times(m-1)} \qquad G_2 = \begin{pmatrix} G_{p1} & & & & \widetilde{G} \\ & G_{2p} & & & \\ & & \cdots & & \\ & & & G_{2p} & \\ \widetilde{G}^T & & & & G_{p1} \end{pmatrix}_{(m-1)\times(m-1)}$$

$$\widetilde{G} = \begin{pmatrix} 0 & 0 & 2r & -8r \\ 0 & 0 & 0 & 2r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} G_{p1} = \begin{pmatrix} 1+6r & -4r & r & 0 \\ -4r & 1+6r & -4r & r \\ r & -4r & 1+6r & -4r \\ 0 & r & -4r & 1+6r \end{pmatrix}$$

$$
G_{2p}=\begin{pmatrix}
1+6r & -4r & r & & & & & \\
-4r & 1+6r & -4r & r & & & & \\
r & -4r & 1+6r & -4r & 2r & & & \\
 & r & -4r & 1+6r & -8r & 2r & & \\
 & & 2r & -8r & 1+6r & -4r & r & \\
 & & & 2r & -4r & 1+6r & -4r & r \\
 & & & & r & -4r & 1+6r & -4r \\
 & & & & & r & -4r & 1+6r
\end{pmatrix}
$$

Then the alternating group iterative method can be derived as below:

$$
\begin{cases}
(\theta I+G_1)\tilde{u}^{n+1}=(\theta I-G^2)u^n+F_n \\
(\theta I+G_2)u^{n+1}=(\theta I-G^1)\tilde{u}^{n+1}+F_n
\end{cases}
\tag{3}
$$

3 Convergence Analysis

Kellogg Lemma[8] Let $\theta>0$, and $G+G^T$ is positive, then $(\theta I+G)^{-1}$exists, and

$$
\begin{cases}
\|(\theta I+G)^{-1}\|_2\le\theta^{-1} \\
\|(\theta I-G)(\theta I+G)^{-1}\|_2<1
\end{cases}
\tag{4}
$$

Let

$$
\bar{G}_1=\begin{pmatrix}
\bar{G}_{2p} & & & \\
 & \cdots & & \\
 & & \cdots & \\
 & & & \cdots & \\
 & & & & \bar{G}_{2p}
\end{pmatrix}_{(m-1)\times(m-1)}
,\bar{G}_2=\begin{pmatrix}
\bar{G}_{p1} & & & & \hat{\hat{G}} \\
 & \bar{G}_{2p} & & & \\
 & & \cdots & & \\
 & & & \bar{G}_{2p} & \\
\hat{\hat{G}}^T & & & & \bar{G}_{p1}
\end{pmatrix}_{(m-1)\times(m-1)}
$$

$$
\bar{G}_{2p}=\begin{pmatrix}
6 & -4 & 1 & & & & & \\
-4 & 6 & -4 & 1 & & & & \\
1 & -4 & 6 & -4 & 2 & & & \\
 & 1 & -4 & 6 & -8 & 2 & & \\
 & & 2 & -8 & 6 & -4 & 1 & \\
 & & & 2 & -4 & 6 & -4 & 1 \\
 & & & & 1 & -4 & 6 & -4 \\
 & & & & & 1 & -4 & 6
\end{pmatrix}
,\hat{\hat{G}}=\begin{pmatrix}
0 & 0 & 2 & -8 \\
0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
,\bar{G}_{p1}=\begin{pmatrix}
6 & -4 & 1 & 0 \\
-4 & 6 & -4 & 1 \\
1 & -4 & 6 & -4 \\
0 & 1 & -4 & 6
\end{pmatrix}
$$

then we have $G_1=I+r\bar{G}_1,\ G_2=I+r\bar{G}_2$.

From the construction of the matrixes $\bar{G}_1$ and $\bar{G}_2$, we can see they are nonnegative definite real matrixes. Thus, when $r>0$, $G_1,\ G_2,\ (G_1+G_1^T)$ and $(G_2+G_2^T)$ are all positive matrixes. Then $\|(\theta I-G_1)(\theta I+G_1)^{-1}\|_2<1,\ \|(\theta I-G_2)(\theta I+G_2)^{-1}\|_2<1$.

From (3), we can obtain $U^{n+1}=GU^n,\ G=(\theta I+G_2)^{-1}(\theta I-G_1)(\theta I+G_1)^{-1}(\theta I-G_2)$ is growth matrix.

Let $\overline{G} = (\theta I + G_2)G(\theta I + G_2)^{-1} = (\theta I - G_1)(\theta I + G_1)^{-1}(\theta I - G_2)(\theta I + G_2)^{-1}$, then $\rho(G) = \rho(\overline{G}) \leq \|\overline{G}\|_2 < 1$. Thus, we have the Theorem as below:

Theorem The alternating group iterative method given by (3) is convergent.

4 Numerical experiments

We consider the following time-dependent periodic boundary problem of four order parabolic equation as below:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial^4 u}{\partial x^4} = 0, \quad 0 < x \leq 2\pi, \quad 0 \leq t \leq T \\ u(x, 0) = sinx, \\ u(x + 2\pi, t) = u(x, t). \end{cases} \tag{5}$$

The exact solution for the problem is $u(x,t) = e^{-t} sinx$. Let A.E. denote absolute error. A.E.$=|u_i^n - u(x_i, t_n)|$. We compare the numerical results of (4) with the exact solution in the follow tables. Let "Exact" and "Numer" denote the exact solution and numerical solution respectively.

Table 1: Comparison with exact results at

$$m = 16, h = 2\pi/16, \tau = 10^{-3}, t = 100\tau, \theta = 1$$

i	2	4	6	8	10	12
A.E.	6.401×10^{-4}	9.053×10^{-4}	6.401×10^{-4}	4.851×10^{-11}	6.401×10^{-4}	9.053×10^{-4}
Exact	0.639817	0.904837	0.639817	4.849×10^{-8}	-0.639817	-0.904837
Numer	0.640457	0.905743	0.640457	4.854×10^{-8}	-0.640457	-0.905743

Table 2: Comparison with exact results at

$$m = 16, h = 2\pi/16, \tau = 10^{-4}, t = 100\tau, \theta = 1$$

i	3	5	7	9	11	13
A.E.	9.147×10^{-5}	9.147×10^{-5}	3.789×10^{-5}	9.147×10^{-5}	9.147×10^{-5}	9.147×10^{-5}
Exact	0.914687	0.914687	0.378876	-0.914687	-0.914687	-0.914687
Numer	0.914778	0.914778	0.378914	-0.914778	-0.914778	-0.914778

5 Conclusions

From the results of numerical experiments we can see that the numerical solution for the method marked by (3) is approximate to the exact solution, which accords to the conclusion of convergence analysis, and shows the iterative method is of high precision. On the other hand, the iterative method (3) is suitable for parallel computing obviously.

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