Applied Mathematical Sciences, Vol. 2, 2008, no. 53, 2619 - 2624

On a General Solution of a Certain Linear Differential Equation with Variable Coefficients

Namh A. Abid

Al-Zaytoonah University
P.O.Box 130 Amman 11733 – Jordan
namh_abed@yahoo.com

Azmi K. Al-Madi

Al-Zaytoonah University
P.O.Box 130 Amman 11733 - Jordan

Abstract

In this paper a general solution of a certain nth order linear differential equation variable coefficients of the form

$$\sum_{j=0}^{n}(-1)^{n-j}\frac{f^{(j)}(x)}{a^{j}\,j!}y^{(j)}(x)=0. \qquad (1)$$

being established. Some properties between the coefficients in (1) are constructed.

Mathematics Subject Classification: 65L99

Keywords: Linear differential equations, General solution

1. Introduction

In [1], Bayne and his colleagues have generalized the basic differential equation $y' - ay = 0$ to nth order equation and they have solved it. In our paper we consider the equation $y' - axy = 0$ as a basic equation and generalized it to nth order and solve it. Notice that when x =1, our equation is reduced to the basicequation given in [1], and thus their results are special cases of our results.

It is well known that the differential equation,

$$y' - axy = 0\,, \qquad (2)$$

can be easily solved, see [2]. It follows, from (2), that

$$(y' - axy)' - ax(y' - axy) = 0\,, \qquad (3)$$

or

$$y'' - 2axy' + (a^2x^2 - a)y = 0\,, \qquad (4)$$

and therefore equation (4) can be easily solved.

In this paper, equation (2) will be generalized to nth order equation, and then

we will follow the technique offered by [1] to obtain its general solution. The following theorem is necessary for our main results.

Theorem (1) : Assume that (2) is satisfied. Then

$$y^{(n)} + f_{n-1}y^{(n-1)} + \ldots + f_j y^{(j)} + \ldots + f_1 y' + f_0 y = 0\,. \qquad (5)$$

Proof: Using induction on n, the result follows at once.

From now on, equation (5) will be denoted by $y^{(n)}$-equation and our aim is to find those coefficients f_j, j=0,1,2, … ,n. This will be found in our main results.

2. Main Results

In this section we will prove our main results. It is not so difficult to use the induction on n to prove the following first result.

Theorem (2): If both n and j are even; $n \geq j$, then the coefficient of $y^{(j)}$ in $y^{(n)}$ – equation is equivalent to

$$f_j(x) = (-1)^{n-j}\binom{n}{j}[a^{n-j}x^{n-j} - a^{n-j-1}\binom{n-j}{2}x^{n-j-2}.1 +$$

$$a^{n-j-2}\binom{n-j}{4}1.3x^{n-j-4} + \ldots + (-1)^{\frac{n-j}{2}}a^{\frac{n-j}{2}}1.3.5\ldots(n-j-1)]\,. \qquad (6)$$

Using the same technique, and observe that the constant term equal to zero when j is odd, we have

Theorem (3): If n and j are both odd, then (6) is the coefficient of $y^{(j)}$ in $y^{(n)}$-equation.

Theorem (4): If n is even and j is odd, then the sum of all terms except the constant term in (6) is the coefficient of $y^{(j)}$ in $y^{(n)}$ - equation.

Theorem (5): If n is odd and j is even, then the sum of all terms except the constant term in (6) is the coefficient of $y^{(j)}$ in $y^{(n)}$- equation.

Remark (1): Let $f_j(x)$ be the coefficient of $y^{(j)}$ in $y^{(n)}$ -equation and f_{j-1} be the coefficient of $y^{(j-1)}$in $y^{(n)}$ -equation, then it follows, from (6), that

$$-\frac{f'_{j-1}}{aj} = -\frac{(-1)^{(n-j+1)}}{aj}\binom{n}{j-1}\frac{d}{dx}\Big[a^{n-j+1}x^{(n-j+1)} -$$

$$-a^{n-j}\binom{n-j+1}{2}x^{(n-j-1)}.1 + a^{n-j-1}\binom{n-j+1}{4}1.3.x^{(n-j-3)} - ...\Big]$$

$$= \frac{(-1)^{(n-j)}}{aj}\binom{n}{j-1}\Big[a^{n-j+1}(n-j+1)x^{(n-j)} - a^{n-j}(n-j-1)\binom{n-j+1}{2}x^{(n-j-2)}.1 +$$

$$a^{n-j-1}(n-j-3)\binom{n-j+1}{4}1.3x^{(n-j-4)}\Big] \qquad (7)$$

Using the identity

$$\binom{n-j+1}{2k}(n-j-2k+1) = (n-j+1)\binom{n-j}{2k}; k = 0,1,2,...,\frac{n-j}{2};$$

(n-j) even, we have

$$-\frac{f'_{j-1}}{aj} = \frac{(-1)^{n-j}}{aj}(n-j+1)\binom{n}{j-1}\Big[a^{n-j+1}x^{n-j} - a^{n-j}\binom{n-j}{2}x^{n-j-2}.1 +$$

$$a^{n-j-1}\binom{n-j}{4}1.3.x^{n-j-4} + ...\Big]$$

$$= (-1)^{n-j}\binom{n}{j}\Big[a^{n-j}x^{n-j} - a^{n-j-1}\binom{n-j}{2}x^{n-j-2}.1 + ...\Big] = f_j .$$

Using Remark (1), we have

Corollary (1): If f(x) is the coefficient of y in the nth order differential equation, then

$$(-1)^{n-j}\frac{f^{(j)}(x)}{a^j j!} \qquad (8)$$

is the coefficient of $y^{(j)}$ in $y^{(n)}$ - equation.

3. General Form

In this section we will use equation (2) to obtain a general form that can be easily solved. For this purpose, write

$$y^{(1)}-axy = f_1y^{(1)}+f_0y = A_1(x,y)= 0, \text{ then}$$

$$(A_1(x,y))' -ax\, A_1(x,y) = A_2(x,y),\text{say},=0,$$

that means

$$A_2(x,y) = y'' -2axy' + (a^2x^2-a) = f_2y^{(2)}+ f_1y^{(1)} +f_0y =0.$$

Thus, if

$$A_{n-1}(x,y) = f_{n-1}y^{(n-1)}+ f_{n-2}y^{(n-2)} +\ldots + f_2y^{(2)}+ f_1y^{(1)} +f_0y =0,$$

then

$$(A_{n-1}(x,y))' - ax\, A_{n-1}(x,y) = A_n(x,y),\text{say},=0$$

or

$$f_n(x)y^{(n)}+ f_{n-1}(x)y^{(n-1)} +\ldots + f_j(x)y^{(j)}+ \ldots + f_1(x)y^{(1)} +f_0(x)y = \sum_{j=0}^{n} f_{n-j}(x)y^{(n-j)}=0. \qquad (9)$$

Therefore, if we write f(x) instead of $f_0(x)$ (the coefficient of y in (9)), and using Remark (1), one can be using simple induction on n to see that

$$f_{n-j}(x)= (-1)^j \frac{f^{(n-j)}(x)}{a^{n-j}(n-j)!}\ , \qquad (10)$$

which by (9) implies the following result

Theorem (6): If $y' - axy = 0$, then for any integer $n \geq 1$, we have

$$\sum_{j=0}^{n}(-1)^{n-j} \frac{f^{(j)}(x)}{a^j\, j!} y^{(j)}(x) = 0 \qquad (11)$$

where f(x) is the coefficient of y in (9).
The following result gives the general solution of equation (1).

Theorem (7): let

$$\sum_{j=0}^{n}(-1)^{(j)} \frac{f^{(n-j)}(x)}{a^{n-j}(n-j)!} y^{(n-j)} = 0, \qquad (12)$$

then the general solution of (12) is given by :

$$y(x) = e^{\left(\frac{1}{2}ax^2\right)}\sum_{j=1}^{n}\frac{1}{(n-j)!}c_j x^{(n-j)}. \qquad (13)$$

Proof: We shall use induction on (n) to show that (13) is satisfied.

For n=2,equation (12) will reduce to:

$$y'' - 2axy' + \left(a^2x^2 - a\right)y = 0. \tag{14}$$

or

$$\left(y' - axy\right)' - ax\left(y' - axy\right) = 0.$$

Hence equation (14) has,

$$y = e^{\left(\frac{1}{2}ax^2\right)}\left(c_1 x + c_2\right),$$

as a general solution, and the result is true for the case n=2. Let the result be true for the integer (n), and that

$$\sum_{j=0}^{n+1}(-1)^j \frac{f^{(n-j+1)}(x)}{a^{n-j+1}(n-j+1)!} y^{(n-j+1)} = 0\,. \tag{15}$$

This equation can be written as

$$\left(\sum_{j=0}^{n}(-1)^j \frac{f^{(n-j)}(x)}{a^{n-j}(n-j)!} y^{(n-j)}\right)' - ax\left(\sum_{j=0}^{n}(-1)^j \frac{f^{(n-j)}(x)}{a^{n-j}(n-j)!} y^{(n-j)}\right) = 0, \tag{16}$$

it follows, from the case when n=2, that

$$\sum_{j=0}^{n}(-1)^{(j)} \frac{f^{(n-j)}(x)}{a^{n-j}(n-j)!} y^{(n-j)} = 0, \tag{17}$$

as a general solution of equation (16) . We use induction hypothesis to have

$$y = e^{\left(\frac{1}{2}ax^2\right)}\left[\frac{1}{n!}x^n + \frac{1}{(n-i)!}x^{(n-1)} + \ldots\ldots + c_n x + c_{n+1}\right] \tag{18}$$

as a general solution of (17). This completes the proof.

Using the same technique as in Theorem (7), we have:

<u>Theorem (8):</u> The general solution to

$$\sum_{j=0}^{n}(-1)^j \frac{f^{(n-j)}(x)}{a^{n-j}(n-j)!} y^{(n-j)} = g(x) \tag{19}$$

where g is a continuous function, is given by

$$y = e^{\left(\frac{1}{2}ax^2\right)} \iint \ldots \int g(x) e^{\left(\frac{-1}{2}ax^2\right)} dx \ldots\ldots dx \,. \qquad (20)$$

References

[1] R.E. Bayne, J.E. Joseph, M.H. Kwack, and T.H. Lawson, A Note on Linear Differential Equations with Constant Coefficients, Missouri Journal Of Mathematical Articles, 10, Issue #1 (1-5), Winter 1998.

[2] W.E. Boyce and R.C. DiPrima, Elementary Differential Equations, Wiley, New York, 2000.

Received: March 24, 2008