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On Improvement of Uniform Convergence of Lagrange Interpolation Polynomials

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Abstract

Due to the Lagrange interpolation polynomials do not converge uniformly to arbitrary continuous functions, in this paper, a new interpolation polynomial is constructed by using the weighted average method to the interpolated functions. It is proved that the interpolation polynomial not only converges uniformly to arbitrary continuous functions, but also has the best approximation order and the highest convergence order.

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1. Introduction

It is known that the Lagrange interpolation polynomials are well used in Natural Science, Engineering Science and so on. However, from the famous theorem of Bernstein[1], we also know that the Lagrange interpolation polynomials do not converge uniformly to arbitrary continuous functions. And thus, for many scholars, one of the interesting works is how to improve the convergence and the convergence order of the interpolation polynomials. Till now, significant works have been made, see the review articles for the relative works by Shen [2] and Xie [3]. Further references may be found in [4∼6]. To improve the convergence of $L_n(f;x)$, a new class of interpolation polynomials, denoted by $F_n(f;x)$, is constructed in this paper by using the weighted average method to the interpolating functions. It is shown that the new interpolation polynomial not only converges uniformly to $f(x) \in C^l_{[-1,1]}, (0 \leq l \leq 4)$, but also has

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the best approximate order, moreover, and that the highest convergence order of $F_n(f;x)$ can not exceed $1/n^5$.

2. Construction of Interpolation polynomial and main results

Let $f(x) \in C^l_{[-1,1]}, (0 \le l \le 4)$, $x = \cos\theta(\theta \in [0,\pi])$, where $C^l_{[-1,1]}$ is a space of continuous functions with continuous derivative of degree l. It is easy to show that

$$x_k = \cos\theta_k = \cos\frac{2k+1}{2n+1}\pi, \quad k = 0,1,2,\cdots,n \tag{1}$$

are zero nodes of the Jacobi orthogonal polynomial

$$(1+x)P_n(x) = (1+\cos\theta)\cos\frac{(2n+1)\theta}{2}\Big/\cos\frac{\theta}{2}. \tag{2}$$

Consequently, the basic Lagrange interpolation functions are given by

$$\mu_k(x) = (-1)^{k+1}\frac{(1+x)P_n(x)\sqrt{2(1-x_k)}}{(2n+1)(x-x_k)}k = 0,1,\cdots,n-1,$$

$$\mu_n(x) = (-1)^n\frac{P_n(x)}{2n+1}. \tag{3}$$

The interpolation polynomial $F_n(f;x)$ is constructed as follows:
Let

$$\triangle^5_h f(x_k) = \frac{1}{2^5}\sum_{i=0}^{5}\begin{pmatrix}5\\ i\end{pmatrix} f(x_{k+i-2}), \tag{4}$$

where $h = \frac{\pi}{n}$, $[a]$ mean the integer part of a, $\begin{pmatrix}5\\ i\end{pmatrix} = \frac{5!}{i!(5-i)!}$, moreover, $\triangle^5_h f(x_k)$ is the central difference of degree 5 of $f(x)$ at the node x_k with the step $h = \frac{\pi}{n}$ and $x_k = \cos\theta_k = \cos\frac{2k+1}{2n+1}\pi$. However, the following information must be presented

(i) From $\cos\theta_j = \cos\theta_{-j-1}$ as $k+i-2 = -j(j=0,1)$, assume that

$$f(x_j) = f(x_{-j-1});$$

(ii) From $\cos\theta_{n+j} = \cos\theta_{n-j}$ as $k+i-2 = n+j(j=0,1)$, assume that

$$f(x_{n+j}) = f(x_{n-j}).$$

Consequently, $F_n(f;x)$ is denoted by

$$F_n(f;x) = \sum_{k=0}^{n} B_k\mu_k(x), \tag{5}$$

where $B_k = f(x_k) + \Delta_h^5 f(x_k)$. In this paper, we obtain

Theorem 1 For $f(x) \in C^j_{[-1,1]}, (0 \le j \le 4)$, the following expression is valid for any $x \in [-1,1]$, i.e.,

$$|F_n(f;x) - f(x)| = O\left(E_n(f) + \frac{1}{n^j}\omega(f^{(j)}, \frac{1}{n}) + \frac{1}{n^{j+1}}\right), \tag{6}$$

where O is independent of $n, x, f, f', \cdots, f^{(4)}$, $E_n(f)$ is the minimum deviation with $f(x)$, $\omega(f^{(j)}, \delta)$ is the modulus of continuity of $f^{(j)}(x)$.

3. Proof of Theorem

From the classical interpolation theory, we know that, for any $f(x) \in C^j_{[-1,1]}, (0 \le j \le 4)$, there exists a algebraic polynomial $p(x) \in P_m$ such that

$$|p(x) - f(x)| \le E_n(f), \tag{7}$$

where P_m is a set of algebraic polynomials with degree m and $m \le n$. On the other hand, from one of the properties of the Lagrange interpolation polynomial, namely, a polynomial $p(x) \in P_m$, $(m \le n)$ coincides with its own interpolation polynomial, so we have

$$\sum_{k=0}^{n} \left(p(x_k) + \Delta_h^5 p(x_k)\right) \cdot \mu_k(x) = p(x) + \Delta_h^5 p(x). \tag{8}$$

From Eq.(8), we obtain

$$\begin{aligned} F_n(f;x) - f(x) &= \sum_{k=0}^{n} (f(x_k) + \Delta_h^5 f(x_k)) \cdot \mu_k(x) - f(x) \\ &= \sum_{k=0}^{n} \left(f(x_k) - p(x_k) + \Delta_h^5 f(x_k) - \Delta_h^5 p(x_k)\right) \cdot \mu_k(x) \\ &\quad + \sum_{k=0}^{n} (p(x_k) + \Delta_h^r p(x_k)) \cdot \mu_k(x) - f(x) \\ &= W_1 + W_2. \end{aligned} \tag{9}$$

For W_1, it is not difficult to show that

$$W_1 = \sum_{k=0}^{n} (f(x_k) - p(x_k)) \left(\frac{1}{2^5} \sum_{i=0}^{5} \begin{pmatrix} 5 \\ i \end{pmatrix} \left(\mu_k(x) + (-1)^{3+i} \mu_{k+i-2}(x)\right)\right). \tag{10}$$

Moreover, W_1 is split into three expressions, as follows,

$$W_1 = \left\{ \sum_{k=0}^{s-2} + \sum_{k=s-1}^{n-r-2+s} + \sum_{k=n-r-1+s}^{n} \right\} \clubsuit$$

$$= W_{11} + W_{12} + W_{13}, \tag{11}$$

where $\clubsuit$ denotes

$$(f(x_k) - p(x_k)) \left(\frac{1}{2^5} \sum_{i=0}^{5} \left(\begin{array}{c} 5 \\ i \end{array} \right) \left(\mu_k(x) + (-1)^{3+i} \mu_{k+i-2}(x) \right) \right).$$

Using the method in [5], for any $x \in [-1, 1]$, the following expression is valid, i.e.,

$$\sum_{k=3}^{n-4} \left(\frac{1}{2^5} \sum_{i=0}^{5} \left(\begin{array}{c} 5 \\ i \end{array} \right) \left| \mu_k(x) + (-1)^{3+i} \mu_{k+i-2}(x) \right| \right) = O(1), \tag{12}$$

and thus

$$W_{12} = O\left(E_n(f)\right). \tag{13}$$

Since $\mu_k(x) = O(1)^{[4]}, (k = 0, 1, 2, \cdots, n)$, from Eq.(8), we have

$$W_{11} = O\left(E_n(f)\right), W_{13} = O\left(E_n(f)\right). \tag{14}$$

Combining Eqs.(13) and (14), we obtain

$$W_1 = O\left(E_n(f)\right). \tag{15}$$

For W_2, we have

$$W_2 = \{(p_n(x) - f(x)) + (\Delta_h^r p_n(x) - \Delta_h^r f(x))\} + \Delta_h^r f(x)$$

$$= W_{21} + W_{22}. \tag{16}$$

From Eq.(7), it leads to

$$W_{21} = O\left(E_n(f)\right). \tag{17}$$

To obtain an exact estimation of W_{22}, let $H(\theta) = f(\cos\theta)$. It is not difficult to show that the following expression is valid

$$H^{(j)}(\theta) = \frac{d^j H(\theta)}{d\theta^j} = \sum_{b=1}^{j} \gamma_b f^{(b)} \cdot \cos^{b_1}\theta \sin^{b_2}\theta \triangleq \sum_{b=1}^{j} \gamma_b \delta_b(\theta), \tag{18}$$

where $b = b_1 + b_2$, and γ_b is a constant. Using the similar method in [5] (Eq.(14)), so we have

$$W_{22} = O\left(\frac{1}{n^{j+1}} + \frac{1}{n^j}\omega(f^{(j)}, \frac{1}{n})\right). \tag{19}$$

In sum Theorem 1 is valid. Furthermore, it is not difficult to show that the following results are also valid by using Theorem 1

Corollary 1 For any $f(x) \in C_{[-1,1]}$ and for any $x \in [-1,1]$, the following expression is valid uniformly, i.e.,

$$\lim_{n\to\infty} F_n(f;x) = f(x).$$

Corollary 2 For arbitrary functions with any derivatives, the highest convergence order of $F_n(f;x)$ can not exceed $1/n^5$.

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