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Identification of a Spatially Dependent Conductivity in Two Dimensions

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Abstract

From some measured data on the boundary of a domain in two dimensions, we get the unknown conductivity coefficient in the whole domain by applying Backus-Gilbert method on some integral identity. Practical procedures are discussed and some random samples of numerical experiments are displayed.

1-Introduction

It is well known that the conductivity problem:

$$\nabla.(A(\vec{x})\nabla u) = 0 \,,$$

is the mathematical model of many physical problems such as the steady state filtration of ground water in a porous media [14] , and the electrical impedance tomography [4,6] . If u and the boundary flux are given, one can determine the filtration, or the conductivity, $A(\vec{x})$. (The inverse problem).

There are many methods to solve the inverse problem such as the method of characteristics [12], and the method of output least squares [7] .Here we use a new technique based on the combination of Backus-Gilbert method[3] with some integral identity which explicitly relating changes in coefficients to changes in measured data. The advantage of this method is that refining the reconstructed unknown can be accomplished with simple additional calculations.

2- The direct problem and the corresponding integral identity

Consider the situation in the direct problem (2. 1) .Suppose that functions $f \in H^{1/2}(\partial U)$ and $g \in H^{-1/2}(\partial U)$ are given and consider the problem of determining a function pair $\{u = u(\vec{x}; A(\vec{x})), A = A(\vec{x})\}$ such that all of the following conditions are satisfied,

$$\begin{aligned} \nabla.(A(\vec{x})\nabla u(\vec{x})) = 0 \ \text{ in } U := \{(x, y) : 0 \prec x, y \prec 1\} \\ u = f \quad \text{on } \partial U \qquad\qquad (2.\ 1) \\ A(\vec{x}).\nabla u.\vec{n} = g \ \text{ on } \partial U \end{aligned}$$

Here we suppose $A = A(\vec{x})$ denotes a strictly positive and suitable smooth (at least Lipschitz continuous function on$\bar{U}$). This ensures that the boundary value problem consisting of the first two conditions listed above has a solution at least in a weak sense, if the coefficient $A = A(\vec{x})$ were given. Then we can view the third of the listed conditions as an additional constraint, which may be used to determine $A(\vec{x})$ in addition to$u(\vec{x})$.

With $f \in H^{1/2}(\partial U)$ fixed, suppose that the pair $\{u_1, A_1\}$ satisfies the first two conditions listed above and produces the data $u_1|_{\partial U} = g_1 \in H^{-1/2}(\partial U)$ and the pair $\{u_2, A_2\}$ is similarly associated with $u_2|_{\partial U} = g_2 \in H^{-1/2}(\partial U)$. Then

$$\begin{aligned} \nabla.(A_1(\vec{x})\nabla u_1(\vec{x}) - A_2(\vec{x})\nabla u_2(\vec{x})) = 0 \ \text{ in } U \\ u_1 - u_2 = 0 \ \text{ on } \partial U \\ \vec{n}.[A_1(\vec{x})\nabla u_1 - A_2(\vec{x})\nabla u_2(\vec{x})] = g_1 - g_2 \ \text{ on } \partial U \end{aligned}$$

If we rewrite the partial differential equation in the form

$\nabla.[A_1(\vec{x})\{\nabla u_1(\vec{x}) - \nabla u_2(\vec{x})\}] = -\nabla.\{A_1(\vec{x}) - A_2(\vec{x})\}\nabla u_2(\vec{x})$ in U

Then integration by parts and attention to the boundary constraints leads to the integral identity,

$$\int_{\partial U} \Delta g(s)\theta(s)ds = \int_U \Delta A(\vec{x})\nabla u_2(\vec{x}).\nabla\Phi(\vec{x})d\vec{x} \qquad (2.\ 2)$$

Where $\Phi = \Phi(\vec{x})$ denotes the solution of the adjoint problem,

$$\begin{aligned} \nabla.[A_1(\vec{x})\nabla\Phi(\vec{x})] = 0 \ \text{ in } U \\ \qquad\qquad (2.\ 3) \\ \Phi = \theta \ \text{ on } \partial U \end{aligned}$$

Clearly when the coefficient in the boundary value problem is changed from $A = A_1(\vec{x})$ to $A = A_2(\vec{x})$, there is a corresponding change in the measured output, from $g = g_1(\vec{x})$ to$g = g_2(\vec{x})$. For our purpose, we choose$\theta = f \succ 0$.

In the next section (3.1), Backus-Gilbert method is presented. It looks far from the previous discussion, but after we simplify it and apply it on the integral identity in section (3.2), the idea will be very clear.

3.1. Backus – Gilbert method

Consider the problem of approximating an unknown function $\Delta A(x,y) \in H^s((0,1)\times(0,1))$ from a finite set of moments $\mu_i := \langle \Delta A, \sigma_i \rangle_{H^s \times H^{-s}}$, where $\sigma_i \in H^{-s}((0,1)\times(0,1))$ are known (generalized) functions , i = 1,2,…,N.
This problem can be stated as follows: Define

$$\Lambda : H^s\left((0,1)\times(0,1) \right) \to R^N$$

$$\Delta A \to \Lambda\left(\Delta A\right) := \left(\langle \Delta A, \sigma_1\rangle_{H^s\times H^{-s}}, ..., \langle \Delta A, \sigma_N \rangle_{H^s\times H^{-s}}\right)$$

Find $\Delta A \in H^s((0,1)\times(0,1))$ such that $\Lambda(\Delta A) = (\mu_1, ..., \mu_N)$

Now, for $(x_0, y_0) \in (0,1)\times(0,1)$, fixed, assume that

$$\Delta A(x_0, y_0) = \sum_{i=1}^{N} \Phi_i(x_0, y_0)\mu_i \,, \qquad (3.1)$$

, for $\mu_i = \langle \Delta A, \sigma_i \rangle_{H^s\times H^{-s}}$ and $\Phi_i \in H^s((0,1)\times(0,1))$ are unknown functions need to be determined. Then

$$\Delta A(x_0, y_0) = \sum_{i=1}^{N} \Phi_i(x_0, y_0)\langle \Delta A, \sigma_i \rangle_{H^s\times H^{-s}}$$

$$= \left\langle \Delta A, \sum_{i=1}^{N} \Phi_i(x_0, y_0)\sigma_i \right\rangle_{H^s\times H^{-s}}$$

However, this implies

$$\sum_{i=1}^{N} \Phi_i(x_0, y_0)\,\sigma_i(x,y) = \delta((x - x_0),(y - y_0)) \in H^{-s}((0,1)\times(0,1)), s \succ 1 \qquad (3.2)$$

Where $\delta((x - x_0),(y - y_0))$ is the Dirac distribution at (x_0, y_0).[1]

Since, $\langle \Lambda(\Delta A), \vec{\gamma} \rangle_{R^N} = \langle \Delta A, \Lambda^T(\vec{\gamma}) \rangle_{H^s\times H^{-s}}$, where Λ^T denotes the transpose of the operator Λ, and

$$\langle \Lambda(\Delta A), \vec{\gamma} \rangle_{R^N} = (\langle \Delta A, \sigma_1\rangle_{H^s\times H^{-s}}, ..., \langle \Delta A, \sigma_N \rangle_{H^s\times H^{-s}}).\vec{\gamma}$$

$$= \sum_{i=1}^{N} \langle \Delta A, \sigma_i \rangle_{H^s\times H^{-s}}\, \gamma_i$$

$$= \left\langle \Delta A, \sum_{i=1}^{N} \gamma_i\, \sigma_i \right\rangle$$

Then

$$\Lambda^T : R^N \to H^{-s}((0,1)\times(0,1))$$

$$\vec{\gamma} \to \sum_{i=1}^{N} \gamma_i \, \sigma_i$$

Therefore, the equation (3.2) is equivalent to

$$\Lambda^T \, \vec{\Phi}(x_0, y_0) = \delta((x - x_0),(y - y_0)) \qquad (3.3)$$

where $\vec{\Phi}(x_0, y_0) = (\, \Phi_1(x_0, y_0), ..., \Phi_N(x_0, y_0)\,)$

To solve the equation (3.3) for $\vec{\Phi}(x_0, y_0)$, we have to acknowledge that in general there will be no solution. However, the normal equation [7]:

$$(\Lambda^T)^* \, \Lambda^T \, \vec{\Phi}(x_0, y_0) = (\Lambda^T)^* \, \delta((x - x_0),(y - y_0))$$

is always uniquely solvable, where

$(\Lambda^T)^* : H^{-s}((0,1)\times(0,1)) \to R^N$ is the adjoint of Λ^T and is defined by

$$\left\langle \Lambda^T \, \vec{\gamma}, G \right\rangle_{H^{-s}} = \left\langle \vec{\gamma}, (\Lambda^T)^* G \right\rangle_{R^N}$$

Note that

$$\left\langle \Lambda^T \, \vec{\gamma}, G \right\rangle_{H^{-s}} = \left\langle \sum_{i=1}^{N} \gamma_i \, \sigma_i \, , G \right\rangle_{H^{-s}} = \sum_{i=1}^{N} \gamma_i \left\langle \sigma_i \, , G \right\rangle_{H^{-s}}$$

, which implies that

$$(\Lambda^T)^* G = (\left\langle \sigma_1, G \right\rangle_{H^{-s}}, ..., \left\langle \sigma_N, G \right\rangle_{H^{-s}}) \in R^N$$

Now,

$$(\Lambda^T)^* \, \Lambda^T \, \vec{\Phi}(x_0, y_0) = (\Lambda^T)^* [\sum_{i-1}^{N} \Phi_i(x_0, y_0) \sigma_i \,] = \sum_{i=1}^{N} \Phi_i \, (x_0, y_0)(\Lambda^T)^* \, \sigma_i$$

$$= \sum_{i=1}^{N} \Phi_i(x_0, y_0)(\left\langle \sigma_1, \sigma_i \right\rangle_{H^{-s}}, ..., \left\langle \sigma_N, \sigma_i \right\rangle_{H^{-s}}) = \sum_{i=1}^{N} \Phi_i(x_0, y_0)(\left\langle J\sigma_1, \sigma_i \right\rangle_{H^s \times H^{-s}}, ..., \left\langle J\sigma_N, \sigma_i \right\rangle_{H^s \times H^{-s}})$$

and

$$(\Lambda^T)^* \delta((x - x_0),(y - y_0)) = (\left\langle \sigma_1, \delta_{(x_0, y_0)} \right\rangle_{H^{-s}}, ..., \left\langle \sigma_N, \delta_{(x_0, y_0)} \right\rangle_{H^{-s}})$$

$$= (\left\langle J\sigma_1, \delta_{(x_0, y_0)} \right\rangle_{H^s \times H^{-s}}, ..., \left\langle J\sigma_N, \delta_{(x_0, y_0)} \right\rangle)$$

where $J : H^{-s}((0,1)\times(0,1)) \to H^s((0,1)\times(0,1))$, denote the duality isomorphism [11], defined by

$$J\sigma(x, y) := \sum_{k=1}^{\infty} \lambda_k^{-s} \left\langle \sigma, \omega_k \right\rangle_{H^{-s} \times H^s} \, \omega_k(x, y)$$

where $\{\omega_k\}$ denote an orthonormal basis of eigenfunctions in $H^s((0,1)\times(0,1))$ with eigenvalues $\{\lambda_k^2\}$

In this case,

$$\left\langle J\sigma_i, \delta_{(x_0, y_0)} \right\rangle_{H^s \times H^{-s}} = \sum_{k=1}^{\infty} \lambda_k^{-s} \left\langle \sigma_i, \omega_k \right\rangle \omega_k(x_0, y_0)$$

and

$$\left\langle J\sigma_i, \sigma_j \right\rangle_{H^s \times H^{-s}} = \sum_{k=1}^{\infty} \lambda_k^{-s} \left\langle \sigma_i, \omega_k \right\rangle_{H^{-s} \times H^s} \left\langle \sigma_j, \omega_k \right\rangle_{H^{-s} \times H^s}$$

Finally, to approximate $A(x_0, y_0)$, we generate $M_{N\times N} = \left[\langle \sigma_i, \sigma_j \rangle_{H^{-s}} \right]$ and $\vec{d}_{1\times N} = \left[\langle J\sigma_i, \delta_{(x_0,y_0)} \rangle_{H^s \times H^{-s}} \right]$, and solve $M \left[\vec{\Phi}(x_0, y_0) \right] = \vec{d}$ .Then

$$\Delta K(x_0, y_0) = \left[\vec{\Phi}(x_0, y_0) \right].\vec{\mu}$$
$$= \left[M^{-1}\vec{d} \right].\vec{\mu}$$

3.2. Applying Backus – Gilbert method on the integral identity

Before we go forward, let us summarize Backus –Gilbert method. For simplicity, take $N = 1$.

Then, Backus –Gilbert method says that: To approximate an unknown function $\Delta A(x, y)$ at each point (x_0, y_0) by knowing a value $\mu_1 = \langle \Delta A, \sigma_1 \rangle_{H^s \times H^{-s}} := \int_0^1 \int_0^1 (\Delta A)\sigma_1 \, dx\,dy$, then

$$\Delta A(x_0, y_0) = \frac{\mu_1}{\sum_{k,l=1}^{\infty} \lambda_{k,l}^{-s} \left[\int_0^1 \int_0^1 \sigma_1 \omega_{k,l}^2 \right]^2} \sum_{k,l=1}^{\infty} \lambda_{k,l}^{-s} \left[\int_0^1 \int_0^1 \sigma_1 \omega_{k,l} \right] \omega_{k,l}(x_0, y_0)$$

$$,(x_0, y_0) \in U := (0,1)\times(0,1) \qquad (3.4)$$

Remark (1)

When $\Delta A = 0$ at the boundary of $U, \partial U$, it is reasonable to consider the eigenfunctions $\{\omega_{k,l}(x, y)\} = \{(\sin k\pi x)(\sin l\pi y\}$ and the eigenvalues $\{\lambda_{k,l}\} = \{(k\pi)^2 + (l\pi)^2\}$ $, k, l = 1, 2, ...$ (for the Laplacian operator)

Now, to apply this method on the integral identity (2 .2), put $\mu_1 = \int_{\partial U} (g_2(x, y) - g_1(x, y))\Phi(\partial U, A_1) dx dy$, and $\sigma_1 = \nabla u(x, y, A_1).\nabla\Phi(x, y, A_2)$

Then, the integral identity can be written of the form

$$\mu_1 = \int_0^1 \int_0^1 \nabla A(x, y).\sigma_1(x, y)\, dx\, dy \qquad (3.5)$$

Therefore, we can apply Backus –Gilbert method and use the equation (3. 4) to get $\Delta A(x_0, y_0)$, $(x_0, y_0) \in U$

Lemma (3.1): $\Delta A(x, y)$, defined in (3. 4), satisfies the equation (3. 5)

Proof:

$$\int_0^1\int_0^1 \Delta A(x,y)\sigma_1(x,y)dxdy = \int_0^1\int_0^1 \frac{\mu_1}{\sum_{k,l=1}^{\infty}\lambda_{k,l}^{-s}\left[\int_0^1\int_0^1 \sigma_1\omega_{k,l}\right]^2}\sum_{k,l=1}^{\infty}\lambda_{k,l}^{-s}\left[\int_0^1\int_0^1 \sigma_1\omega_{k,l}\right]\omega_{k,l}(x,y).\sigma_1(x,y)dxdy$$

$$= \frac{\mu_1}{\sum_{k,l=1}^{\infty}\lambda_{k,l}^{-s}\left[\int_0^1\int_0^1 \sigma_1\omega_{k,l}\right]^2}.\sum_{k,l=1}^{\infty}\lambda_{k,l}^{-s}\left[\int_0^1\int_0^1 \sigma_1\omega_{k,l}\right]\left[\int_0^1\int_0^1 \sigma_1\omega_{k,l}\right]$$

$$= \mu_1$$

4- Practical procedures and applications

What we will do to apply (3 . 5) is to pick an initial coefficient , say $A_1(x,y)$, satisfies $A_1 = A$ at ∂U where $A(x,y)$ is the unknown coefficient .So , $\Delta A\big|_{\partial U} := A\big|_{\partial U} - A_1\big|_{\partial U} = 0$. Then the remark (1) in section (3. 2) is satisfied. Since we know $A_1(x,y)$, we get $u(x,y,A_1)$ and $\Phi(x,y;A_1) \equiv u(x,y;A_1)$, by choosing $\Phi(\partial U) \equiv \theta := f$, as a direct problems. Therefore, we calculate both

(i) $\tilde{\sigma}_1 := \nabla u(x,y;A_1).\nabla\Phi(x,y;A_1) = \nabla u(x,y;A_1).\nabla u(x,y,A_1)$

, and

(ii) $g_1(x,y) := A_1(x,y)\nabla u(x,y,A_1).\vec{n}$ at ∂U

Since we can measure the flux,

$$g(x,y) := A(x,y)\nabla u(x,y;A).\vec{n} \text{ at } \partial U$$

we can calculate

$$\mu_1 := \int_{\partial U}(g - g_1)\Phi(\partial U)$$

Now, apply (3. 4) to get the next iterate,

$$A_2(x_0,y_0) := A_1(x_0,y_0) + \mu_1 \frac{\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}[(k\pi)^2+(l\pi)^2]^{(-s)}\int_0^1\int_0^1 \tilde{\sigma}_1(\sin l\pi x)(\sin k\pi y)dxdy}{\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}[(k\pi)^2+(l\pi)^2]^{(-s)}[\int_0^1\int_0^1 \tilde{\sigma}_1(\sin l\pi x)(\sin k\pi y)dxdy]^2}\sin(k\pi x_0)\sin(l\pi y_0)$$

$(x_0,y_0) \in U$ (3.6)

From (3. 6), we know $A_2(x,y), (x,y) \in U$, so we can get $u(x,y;A_2)$ and $\Phi(x,y,A_2)$, as a direct problems. As before, we can know

$\tilde{\sigma}_2 := \nabla u(x,y;A_2).\nabla\Phi(x,y,A_2) = \nabla u(x,y;A_2).\nabla u(x,y;A_2)$,

$g_2(x,y) := A_2(x,y)\nabla u(x,y;A_2).\vec{n}$ at ∂U , and

$$\mu_2 := \int_{\partial U}(g - g_2)\Phi(\partial U)$$

Apply (3. 4) to get the third iterate,

$$A_3(x_0,y_0) := A_2(x_0,y_0) + \mu_2 \frac{\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}[(k\pi)^2+(l\pi)^2]^{(-s)}\int_0^1\int_0^1 \tilde{\sigma}_2(\sin l\pi x)(\sin k\pi y)dxdy}{\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}[(k\pi)^2+(l\pi)^2]^{(-s)}[\int_0^1\int_0^1 \tilde{\sigma}_2(\sin l\pi x)(\sin k\pi y)dxdy]^2} \sin(k\pi x_0)\sin(l\pi y_0)$$

$(x_0, y_0) \in U$ (3.7)

Continue as before, we get a monotone sequence $\{A_n(x,y)\}$ converges to the unknown coefficient $A(x,y)$, whenever $\{\mu_n\}$ converges to zero, according to the following lemma

Lemma (4.1): According to the initial coefficient $A_1(x,y)$,

(i) $g_1 \succ g$ imply $\{A_n(x,y)\}$ is a decreasing sequence

(ii) $g_1 \prec g$ implies $\{A_n(x,y)\}$ is an increasing sequence

Proof:

If $g_1(x,y) \succ g(x,y)$ then $\mu_1 \prec 0$, since $\Phi(\partial U) \equiv \theta \succ 0$. So from (3. 6), $A_2(x,y) \prec A_1(x,y)$, since $\sigma_1 \equiv \nabla u(x,y,A_1).\nabla u(x,y,A_1) \equiv (\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 \succ 0$. Then $g_2(x,y) \prec g_1(x,y)$ and $\mu_2 \prec 0$. Therefore, from (3.7), $A_3(x,y) \prec A_2(x,y)$ and so on .Then $\{A_n(x,y)\}$ is a decreasing sequence. While if $g_1(x,y) \prec g(x,y)$, then $\mu_1 \succ 0$ so, from (3. 6), $A_2(x,y) \succ A_1(x,y)$, then $g_2(x,y) \succ g_1(x,y)$ and $\mu_2 \succ 0$. So, from (3.7), $A_3(x,y) \succ A_2(x,y)$ and so on. Then $\{A_n(x,y)\}$ is an increasing sequence.

Applications

In the following examples, we consider:

$f(x,0) = \ln(x+2)$; $f(0,y) = \ln(2-y)$; $f(1,y) = \ln(3-y)$; $f(x,1) = \ln(x+1)$, $0 \prec x, y \prec 1$

In example (I), assume the actual (unknown) coefficient $A(x,y) = x - y + 2$

Example (I) is divided into two examples (I .1) and (I . 2) , according to the choice of the initial coefficient . In example (I.1), the initial coefficient is:

$$A_1(x,y) = a_1(x) + b_1(y) + a_2(x).b_2(y) , 0 \le x, y \le 1$$

Where:

$a_1(x) = 5x^2 - 4x + 2$; $b_1(y) = 5y^2 - 4y + 2$; $a_2(x) = -\frac{1}{2}(x - a_1(x))$; $b_2(y) = (y - b_1(y))$, $0 \le x, y \le 1$

In example (I .2), the initial coefficient is:

$$A_1(x,y) = a_{11}(x) + b_{11}(y) + a_{22}(x).b_{22}(y) , 0 \le x, y \le 1$$

Where:

$a_{11}(x)=3x^2-2x+2; b_{11}(y)=3y^2-2y+2; a_{22}(x)=-\frac{1}{2}(x-a_{11}(x)); b_{22}(y)=(y-b_{11}(y))$

The only reason for these choices is to guarantee that $A_1\big|_{\partial U}=A\big|_{\partial U}$.

In example (II), assume the actual (unknown) coefficient:

$A(x,y)=e^{2x(1-x)}+e^{2y(1-y)}-e^{2x(1-x)+2y(1-y)}$

(So, $A(0,y)=A(x,0)=A(x,1)=A(1,y)=1 \quad ,0\le x,y\le 1$)

It is reasonable to choose the initial coefficient $A_1(x,y)=1, 0\le x,y\le 1$

In example (III), assume the actual (unknown) coefficient:

$A(x,y)=c(x)-d(y)+c(x).d(y), 0\le x,y\le 1$

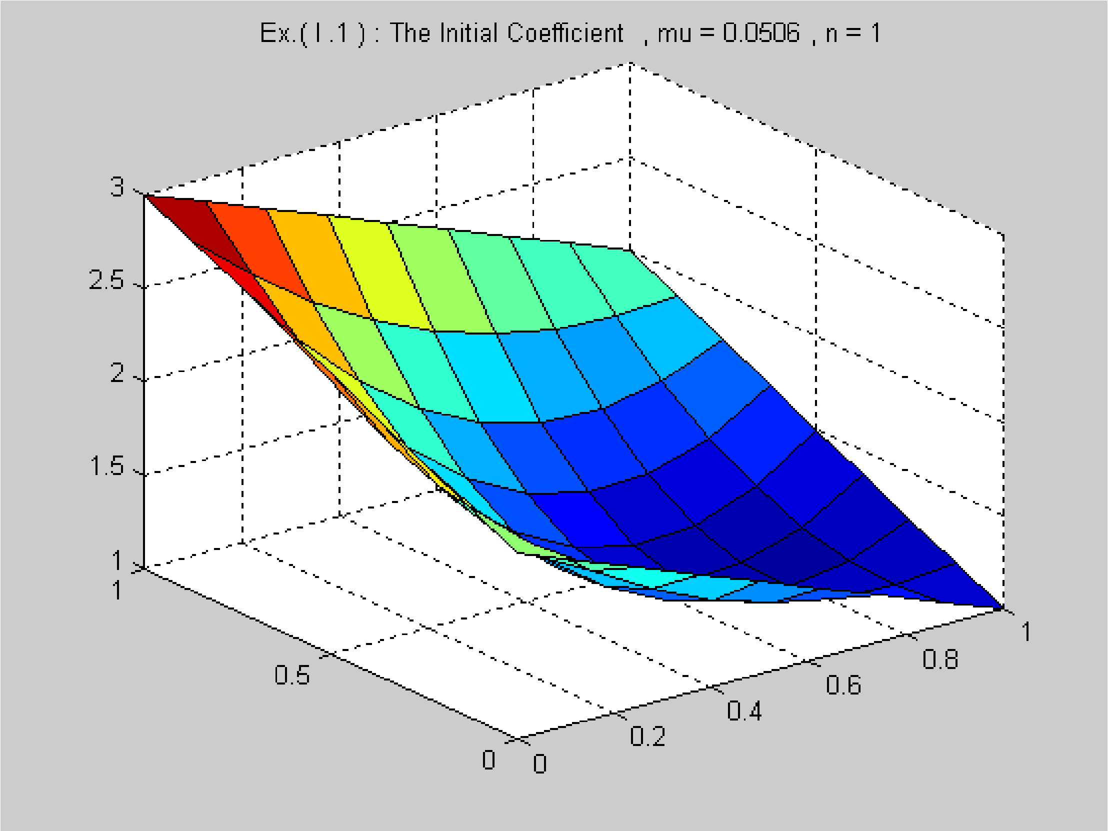
Where:

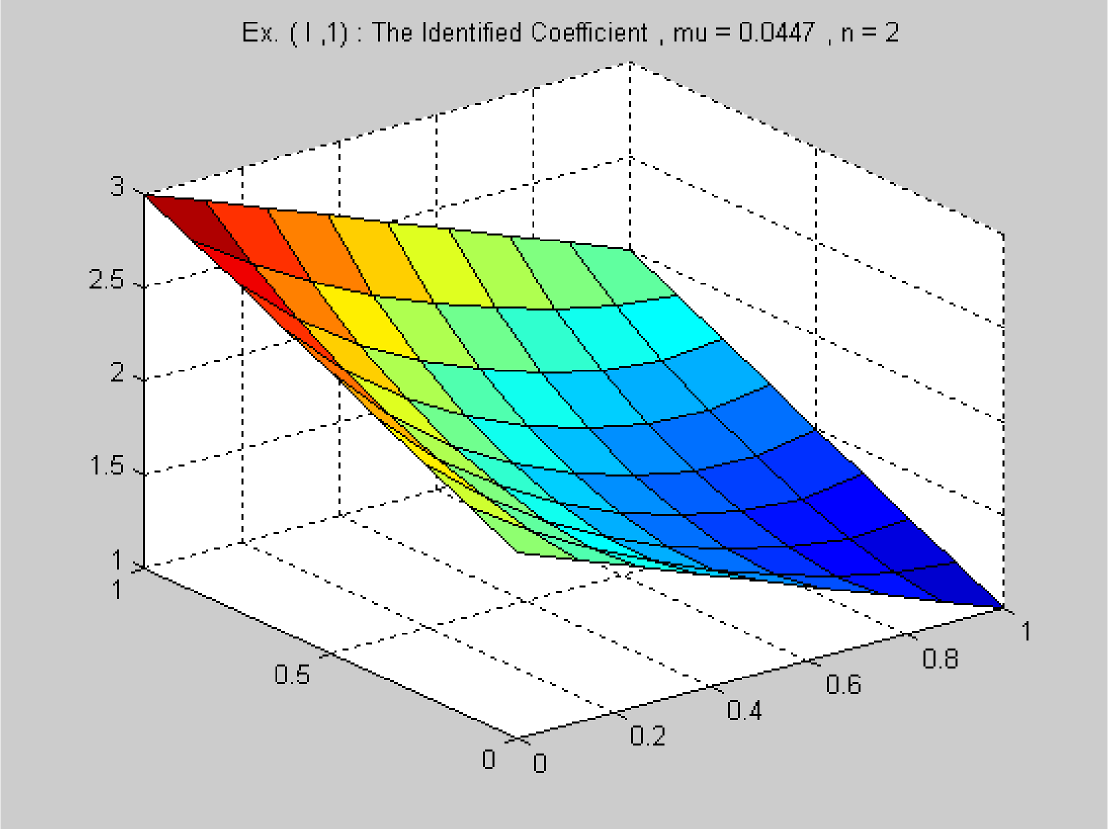
$c(x)=10x^2-10x+1, d(y)=10y^2-10y-1$

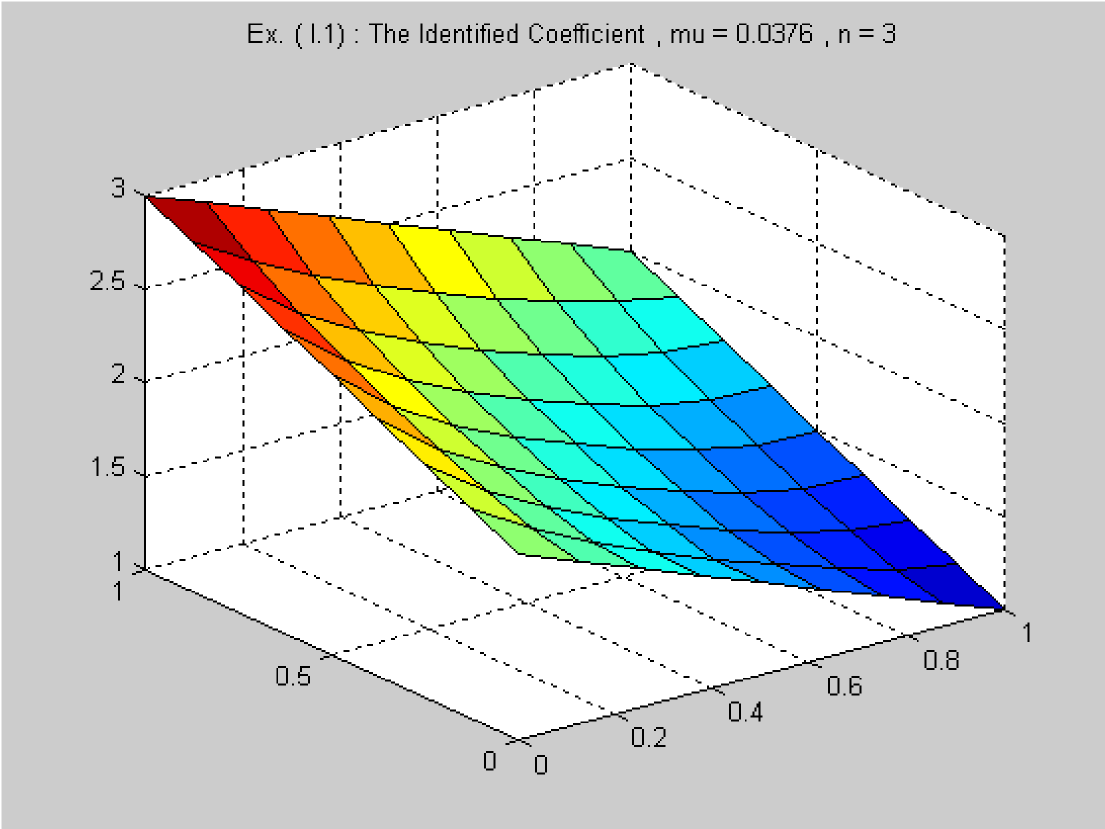
(So, $A(0,y)=A(x,0)=A(x,1)=A(1,y)=1 \quad ,0\le x,y\le 1$)

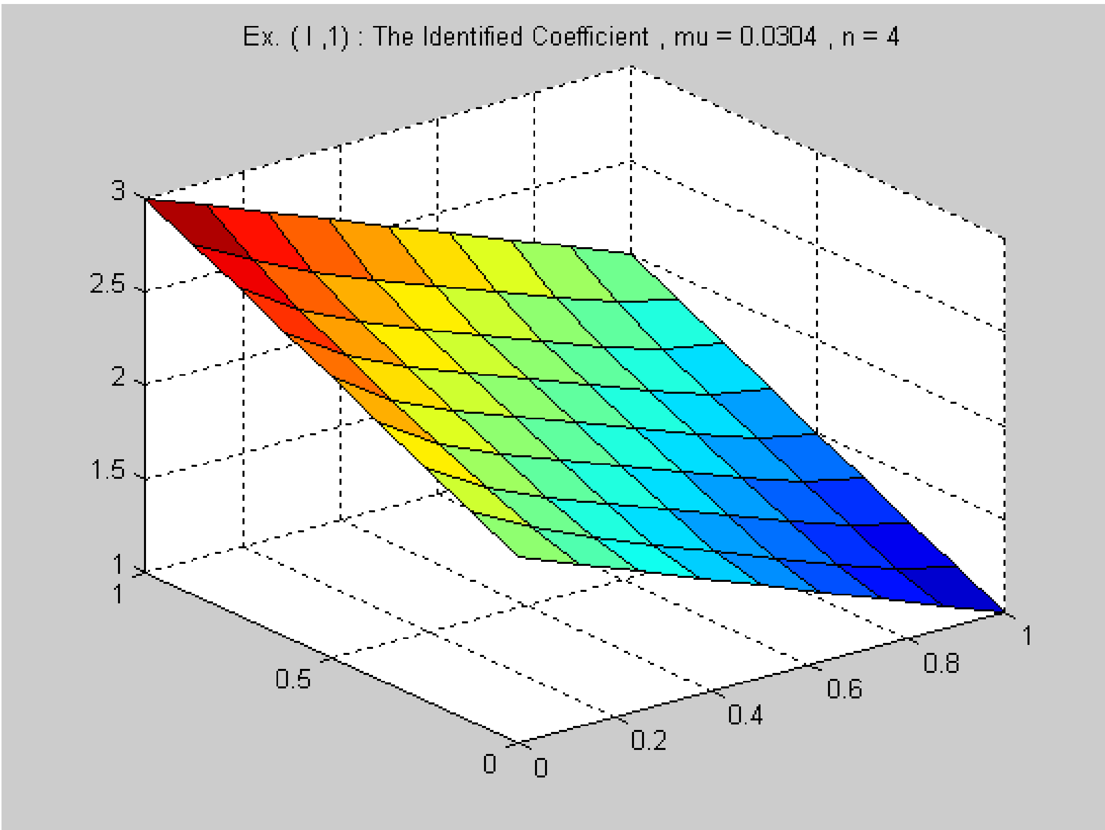
It is reasonable to pick the initial coefficient $A_1(x,y)=1, 0\le x,y\le 1$.

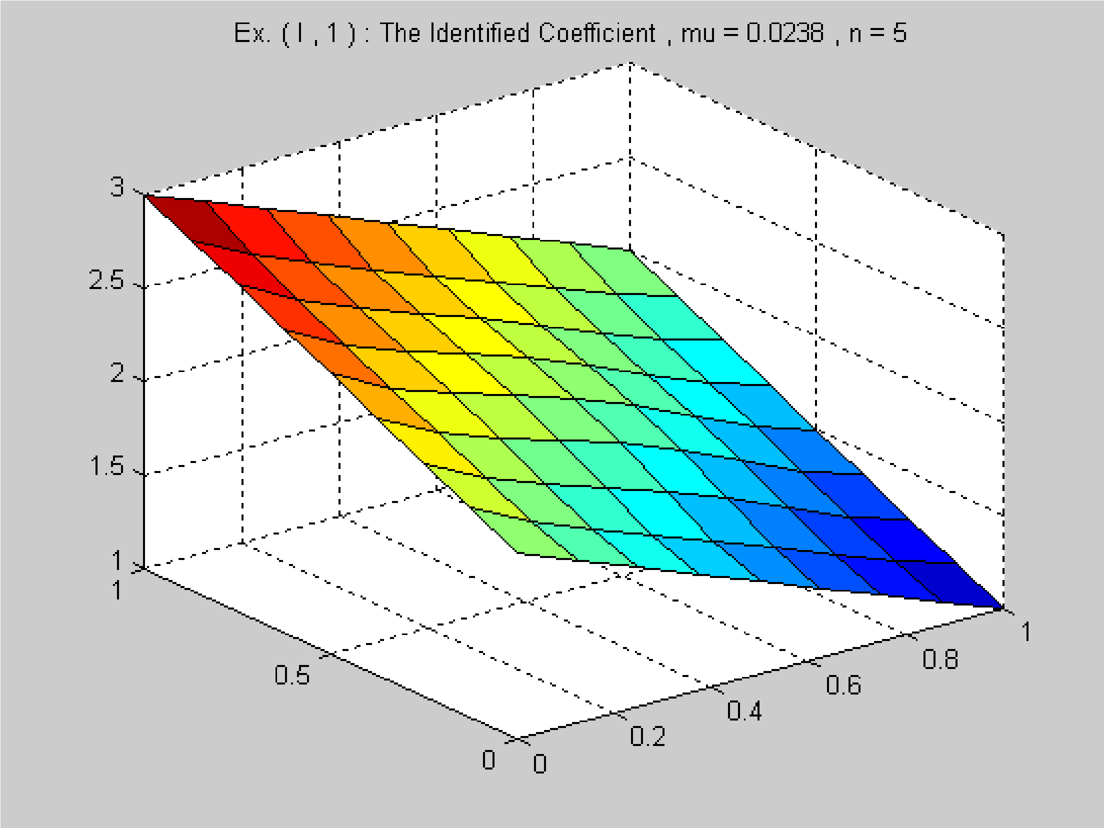
By applying the previous procedures on a Mat Lab program, we get the following figures, which indicate that this method may work:

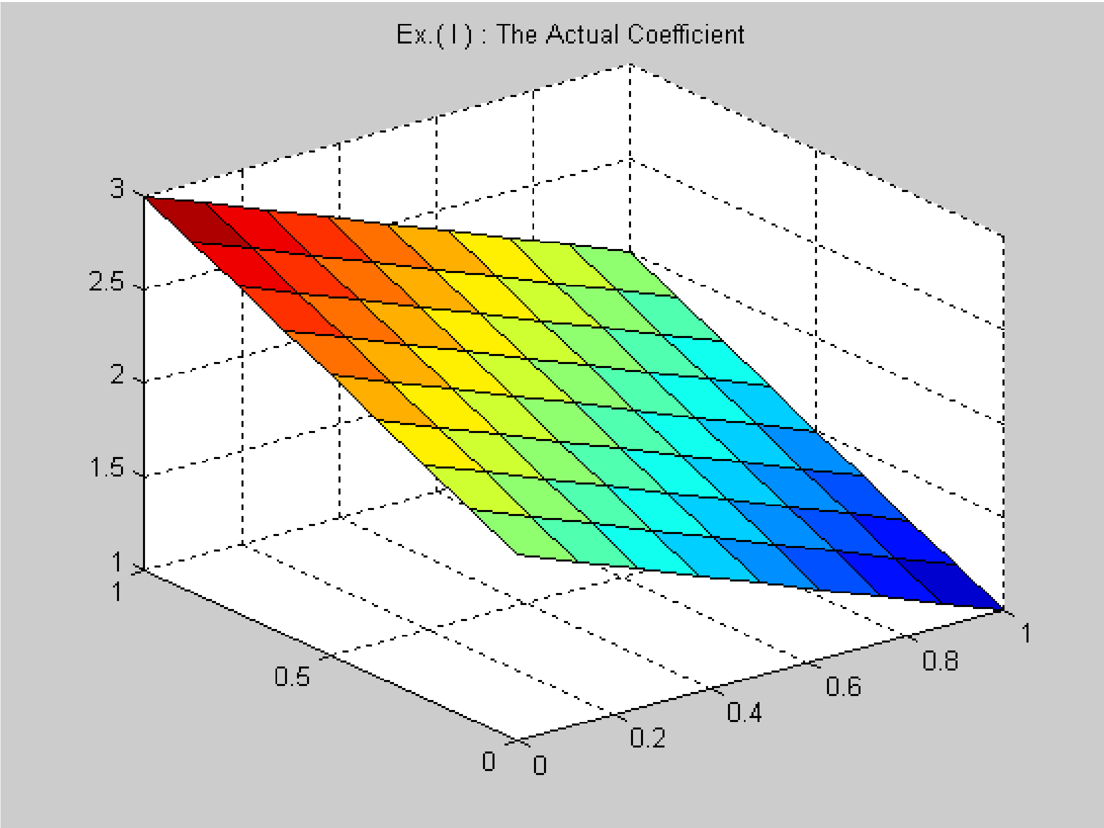

Ex.(I .1) : The Initial Coefficient , mu = 0.0506 , n = 1
3
2.5
2
1.5
1
1
0.5
0
0
0.2
0.4
0.6
0.8
1

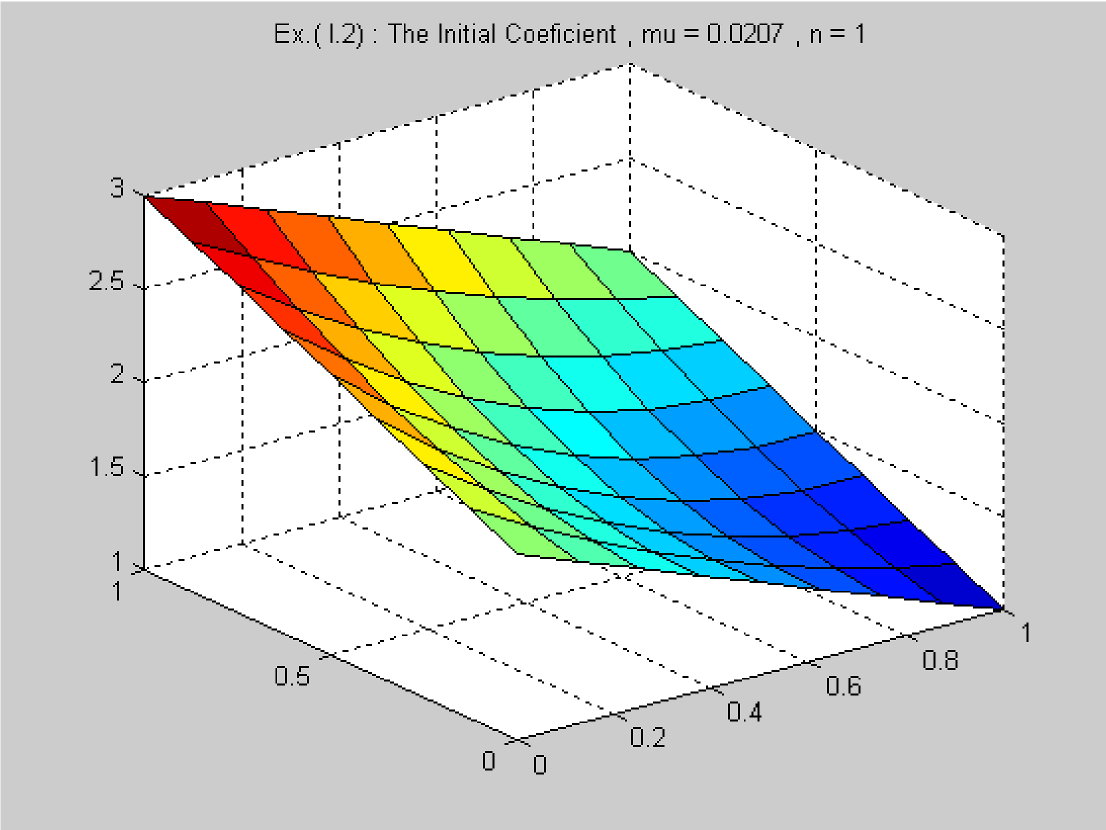

Ex. (I ,1) : The Identified Coefficient , mu = 0.0447 , n = 2
3
2.5
2
1.5
1
1
0.5
0
0
0.2
0.4
0.6
0.8
1

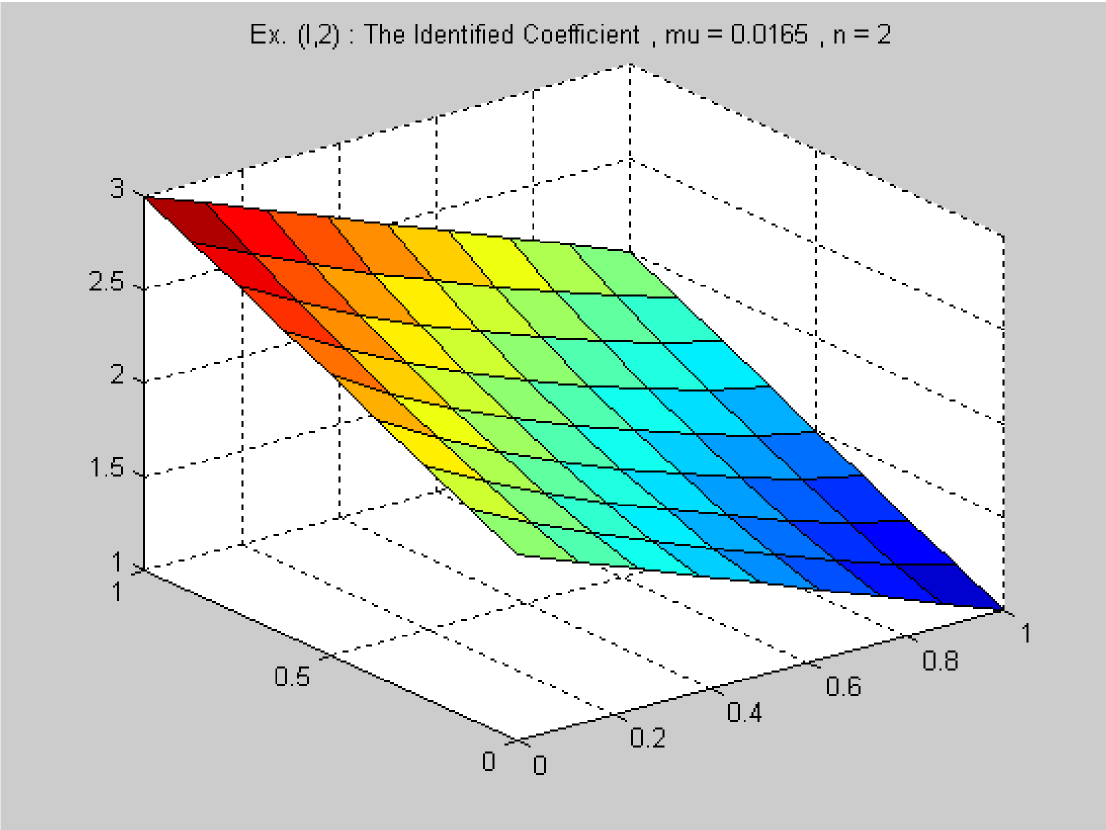

Ex. (I.1) : The Identified Coefficient , mu = 0.0376 , n = 3
3
2.5
2
1.5
1
1
0.5
0
0
0.2
0.4
0.6
0.8
1

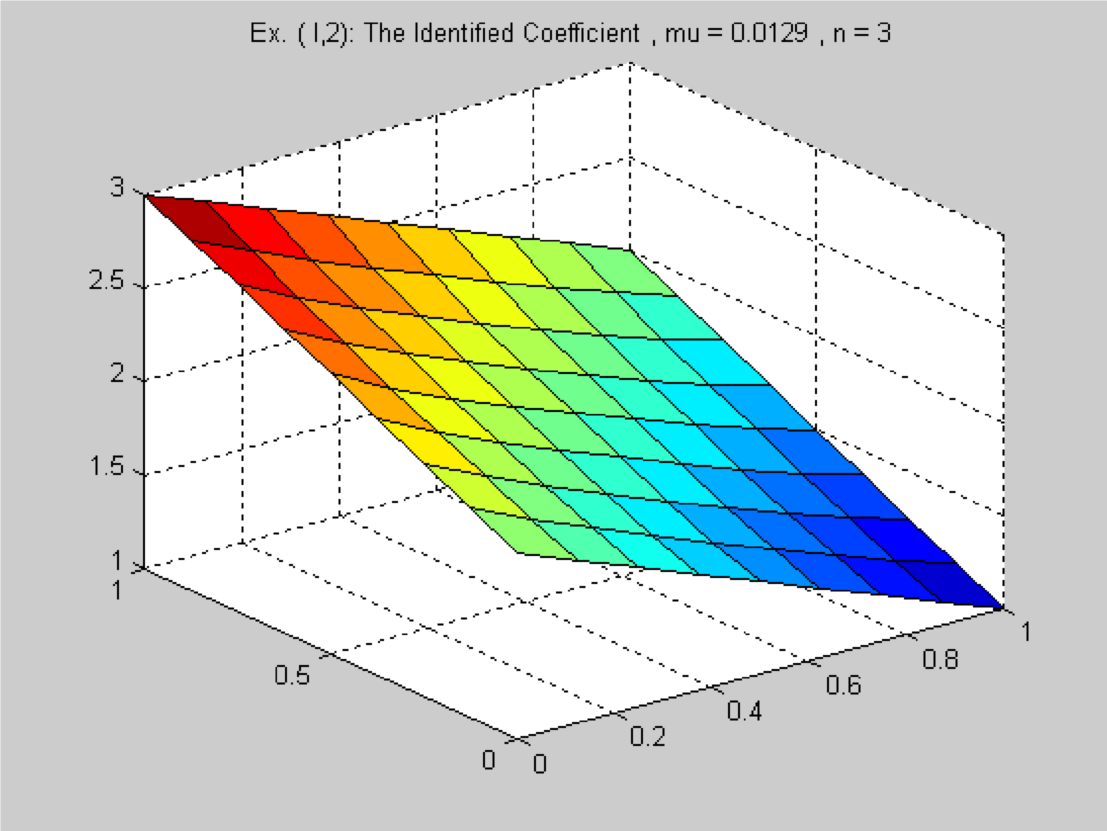

Ex. (I ,1) : The Identified Coefficient , mu = 0.0304 , n = 4
3
2.5
2
1.5
1
1
0.5
0
0
0.2
0.4
0.6
0.8
1

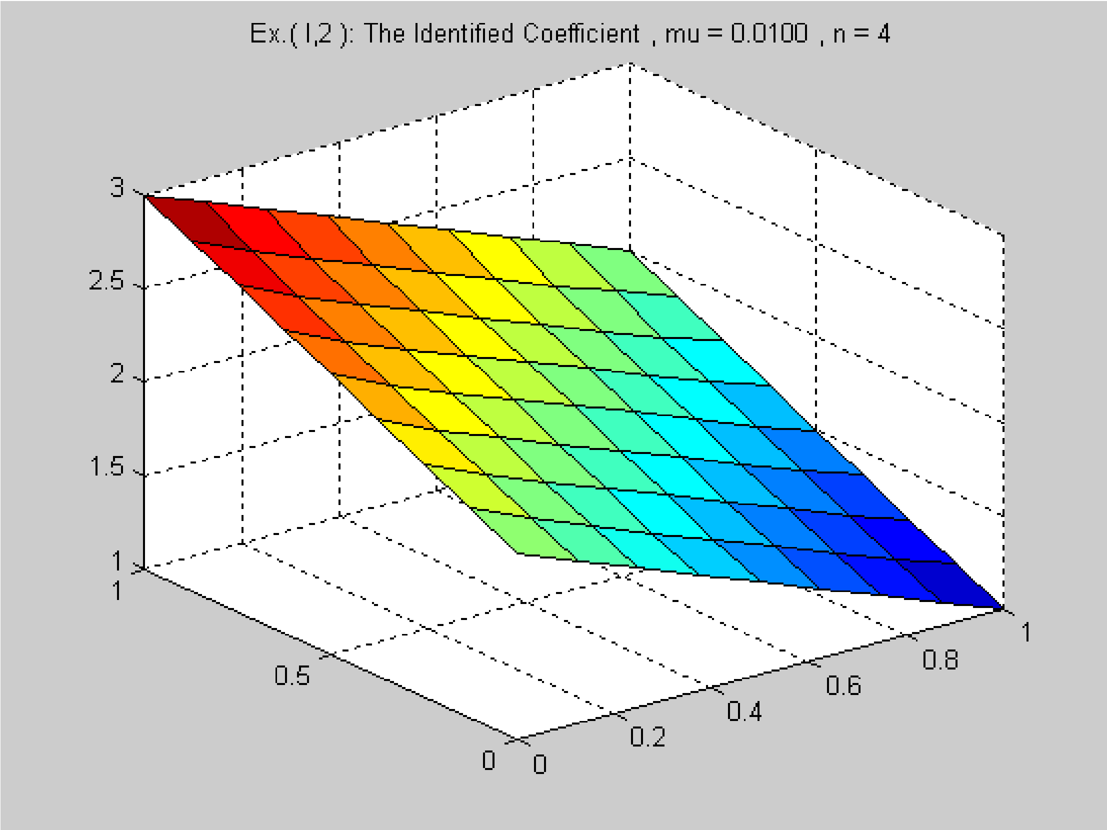

Ex. (I , 1) : The Identified Coefficient , mu = 0.0238 , n = 5
3
2.5
2
1.5
1
1
0.5
0
0
0.2
0.4
0.6
0.8
1

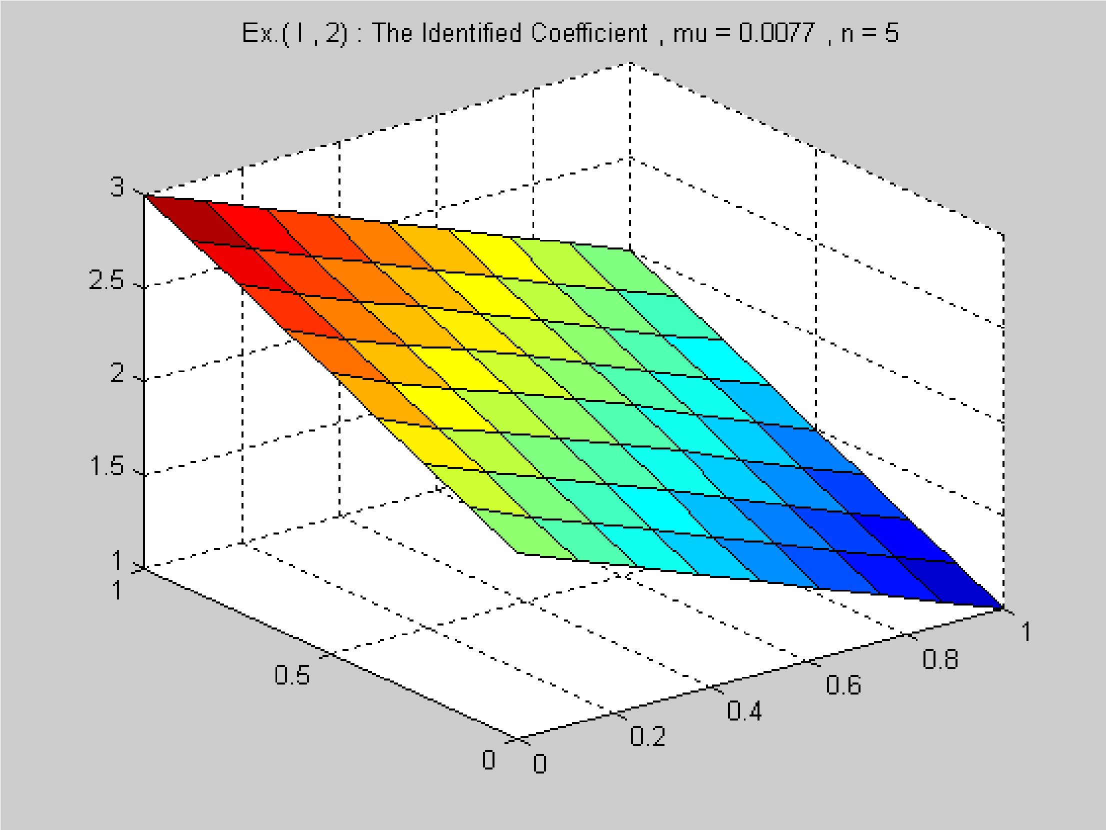

Ex.(I) : The Actual Coefficient
3
2.5
2
1.5
1
1
0.5
0
0
0.2
0.4
0.6
0.8
1

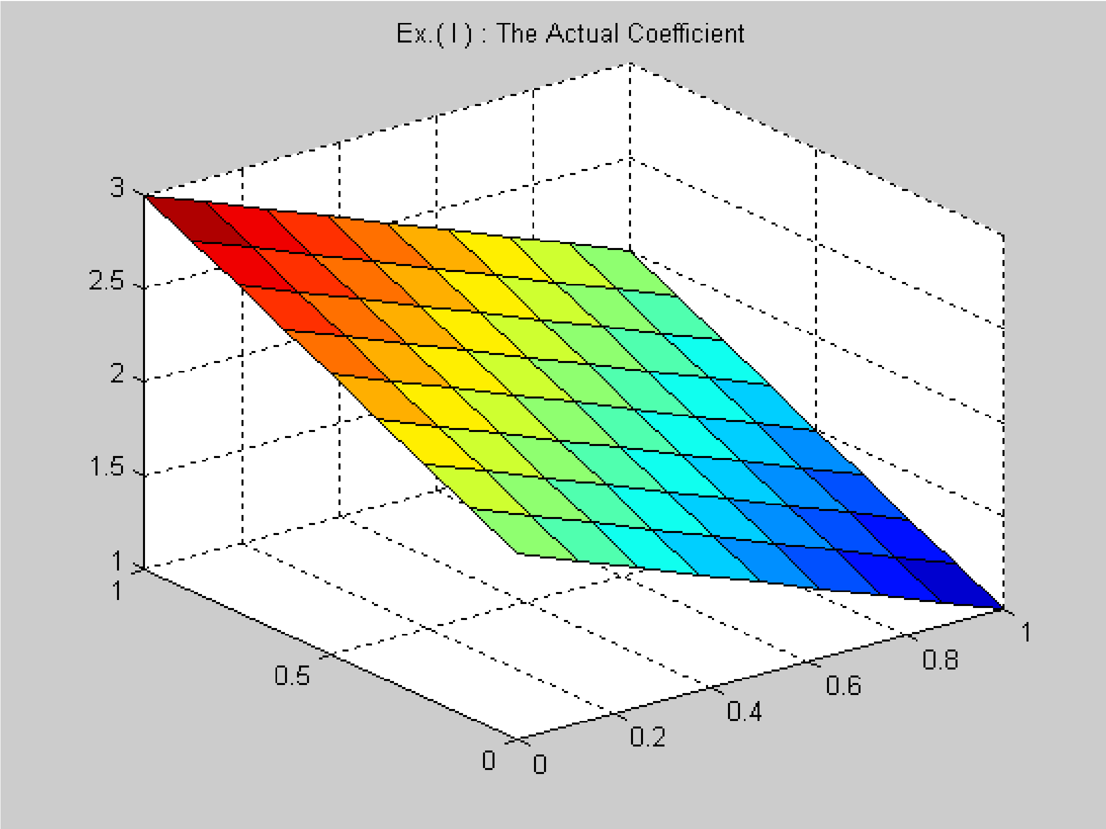

Ex.(I.2) : The Initial Coeficient , mu = 0.0207 , n = 1
3
2.5
2
1.5
1
1
0.5
0
0
0.2
0.4
0.6
0.8
1

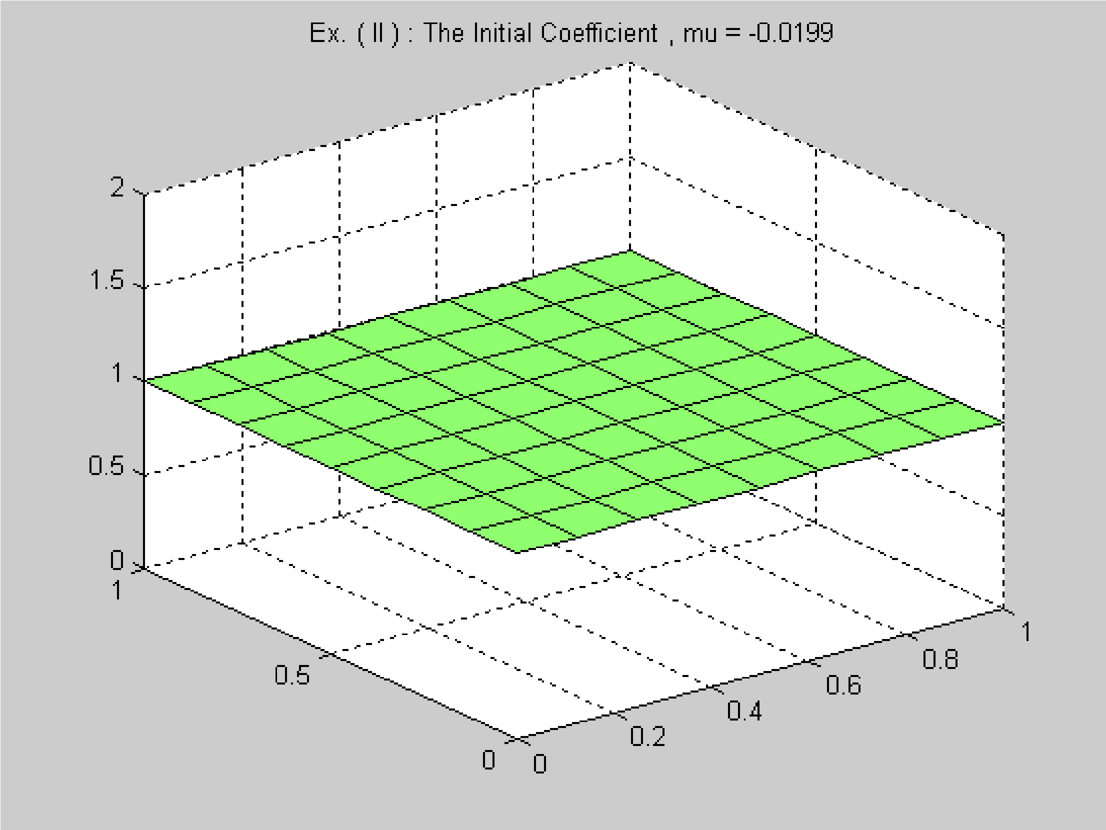

Ex. (I,2) : The Identified Coefficient , mu = 0.0165 , n = 2
3
2.5
2
1.5
1
1
0.5
0
0
0.2
0.4
0.6
0.8
1

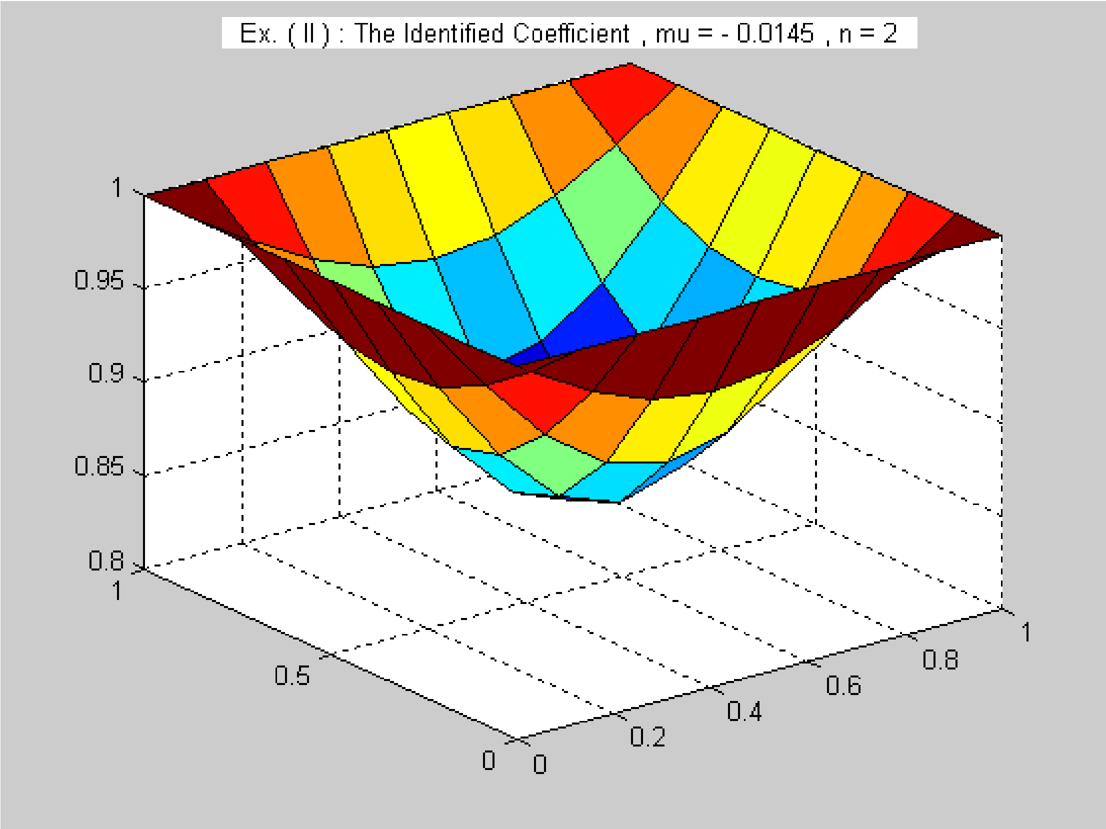

Ex. (I,2): The Identified Coefficient , mu = 0.0129 , n = 3
3
2.5
2
1.5
1
1
0.5
0
0
0.2
0.4
0.6
0.8
1

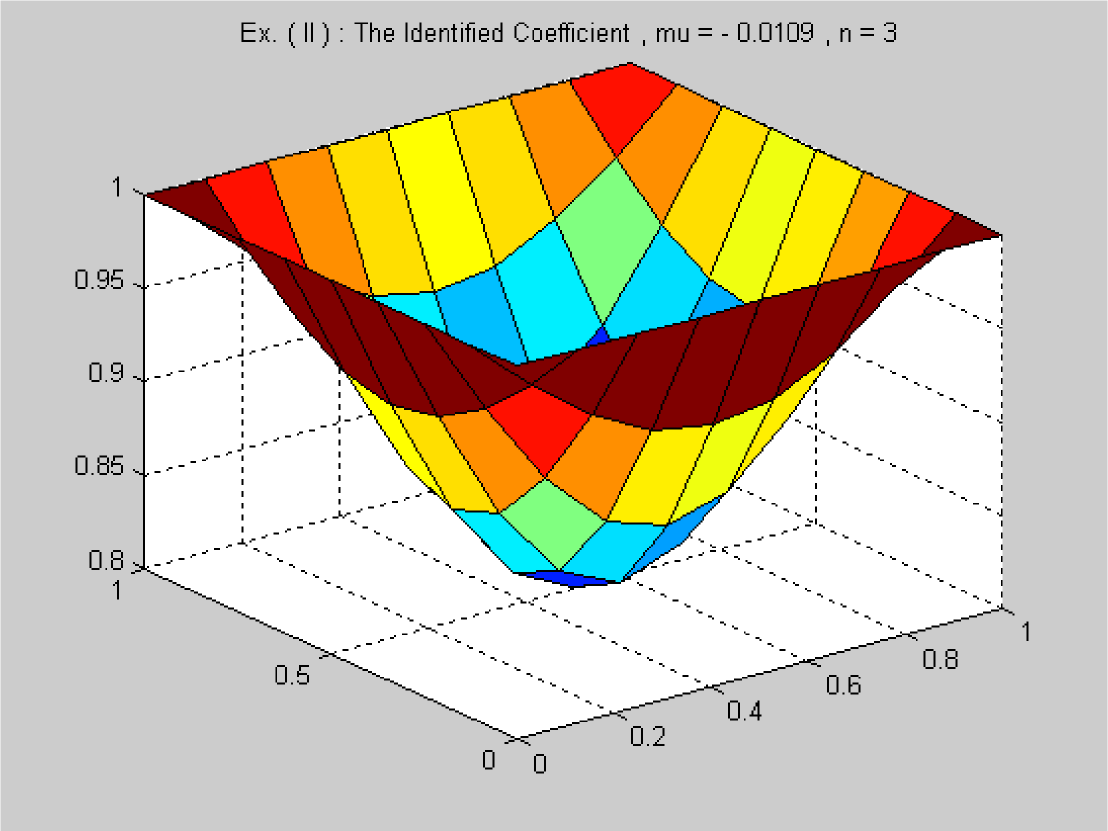

Ex.(I,2): The Identified Coefficient , mu = 0.0100 , n = 4
3
2.5
2
1.5
1
1
0.5
0
0
0.2
0.4
0.6
0.8
1

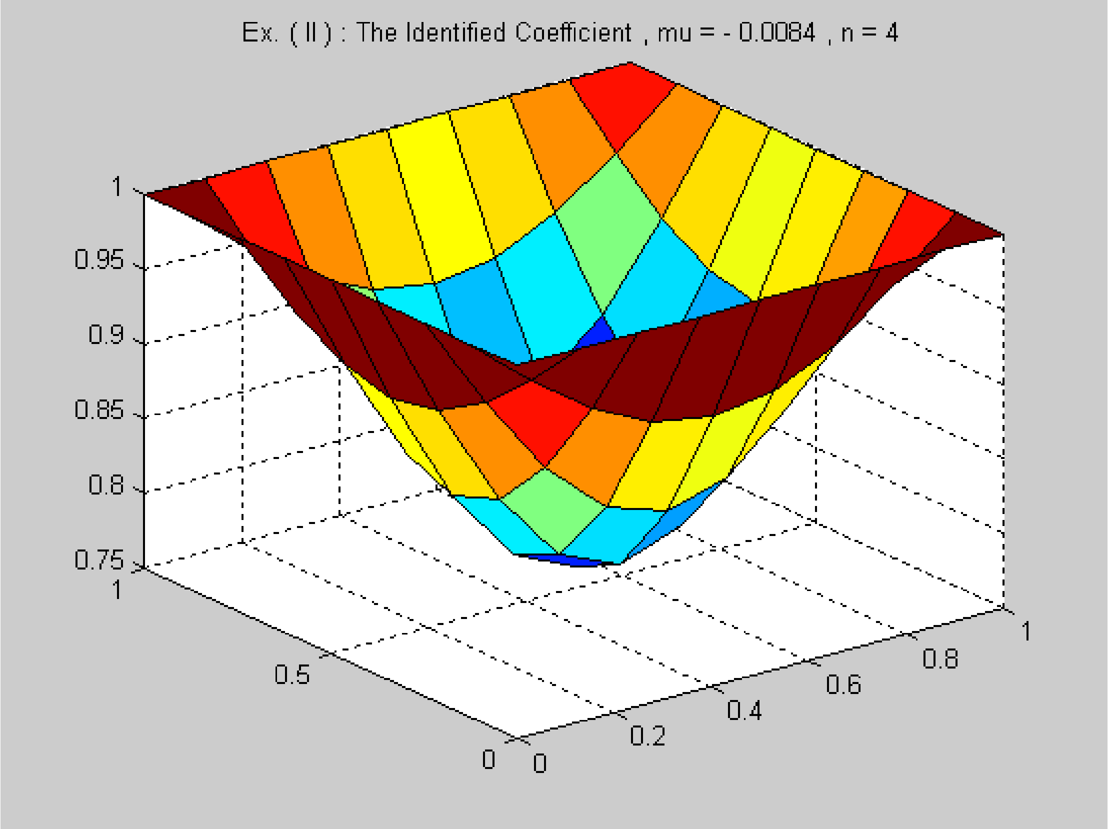

Ex.(I , 2) : The Identified Coefficient , mu = 0.0077 , n = 5
3
2.5
2
1.5
1
1
0.5
0
0
0.2
0.4
0.6
0.8
1

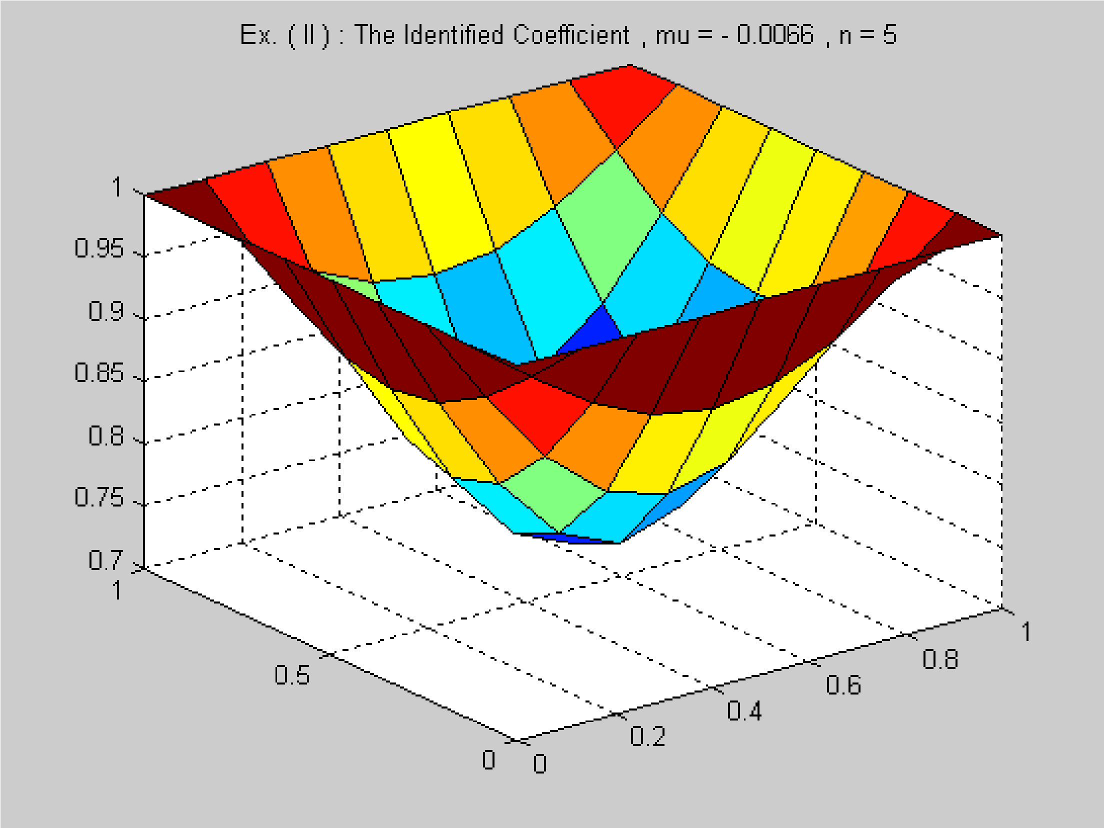

Ex.(I) : The Actual Coefficient
3
2.5
2
1.5
1
1
0.5
0
0
0.2
0.4
0.6
0.8
1


Ex. (II) : The Initial Coefficient , mu = -0.0199
2
1.5
1
0.5
0
1
0.5
0
0
0.2
0.4
0.6
0.8
1

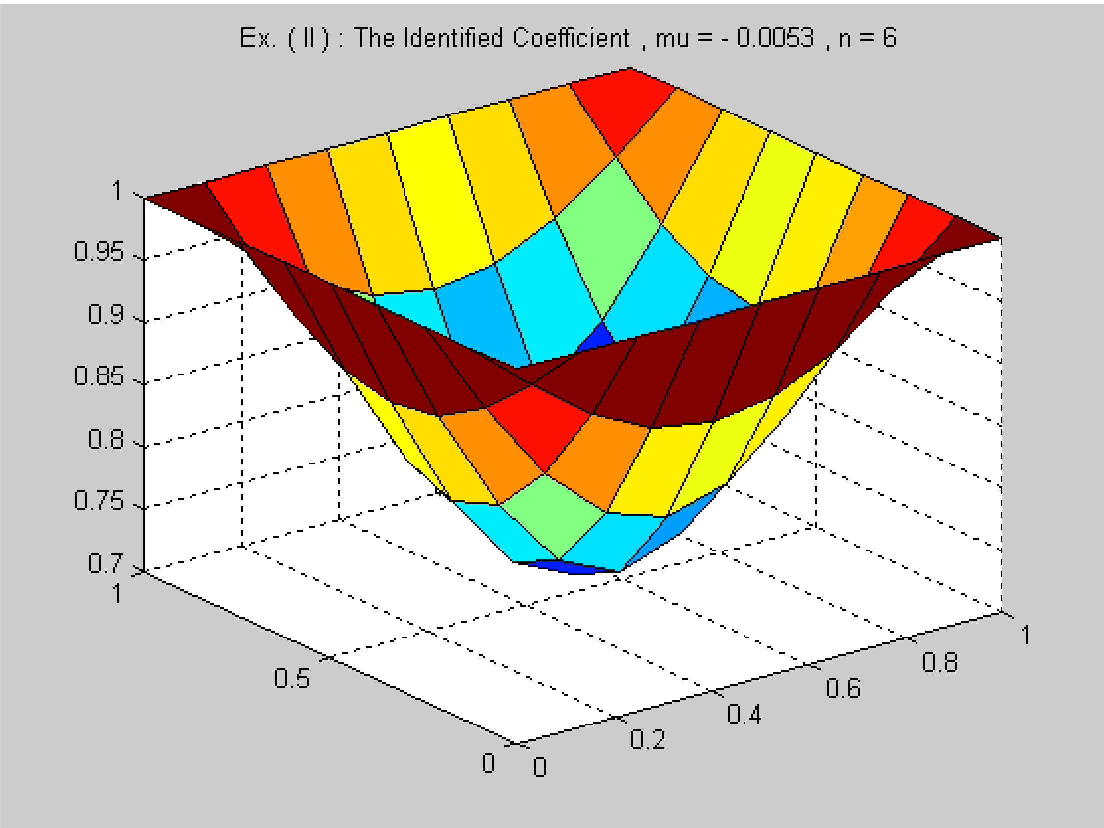

Ex. (II) : The Identified Coefficient , mu = - 0.0145 , n = 2
1
0.95
0.9
0.85
0.8
1
0.5
0
0
0.2
0.4
0.6
0.8
1


Ex. (II) : The Identified Coefficient , mu = - 0.0109 , n = 3
1
0.95
0.9
0.85
0.8
1
0.5
0
0
0.2
0.4
0.6
0.8
1

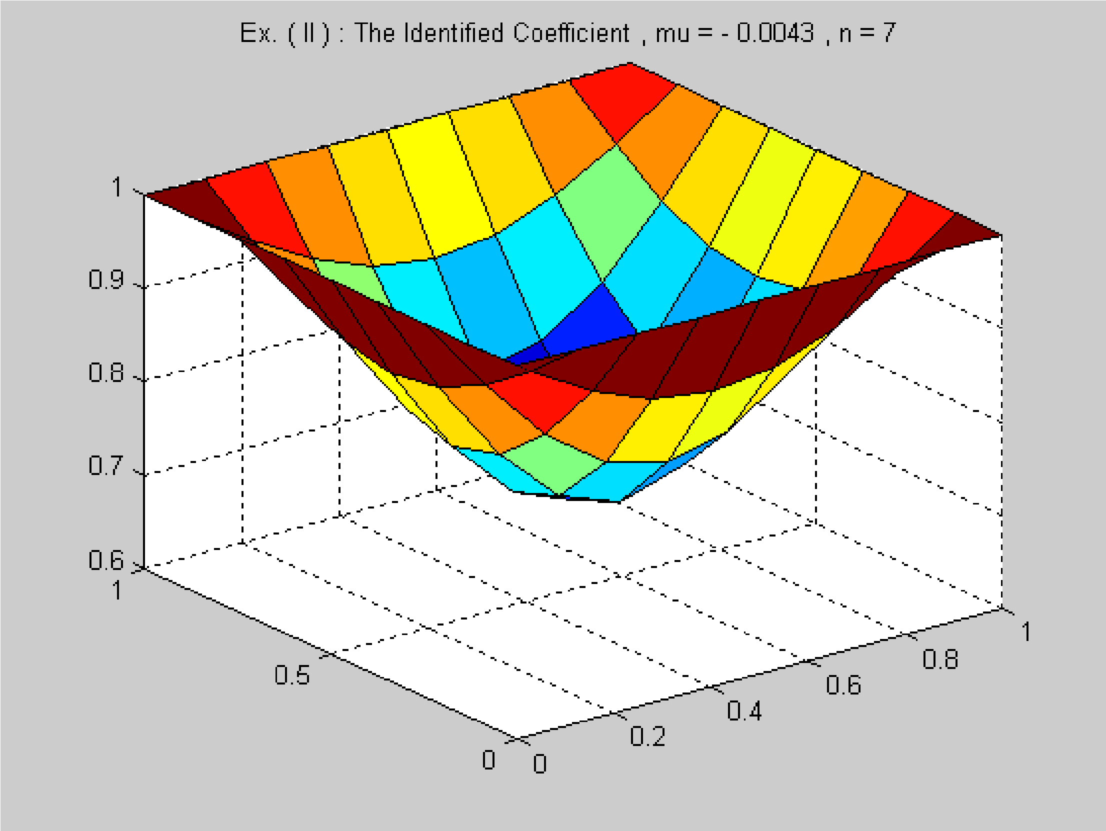

Ex. (II) : The Identified Coefficient , mu = - 0.0084 , n = 4
1
0.95
0.9
0.85
0.8
0.75
1
0.5
0
0
0.2
0.4
0.6
0.8
1

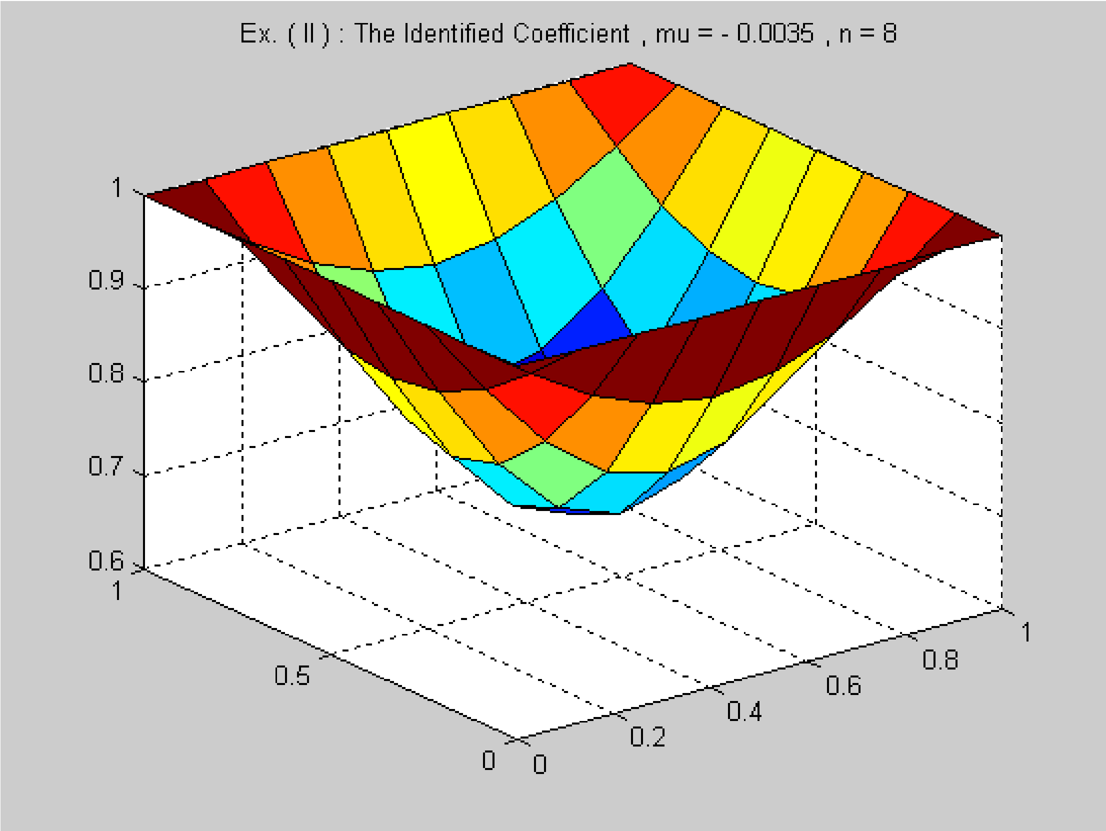


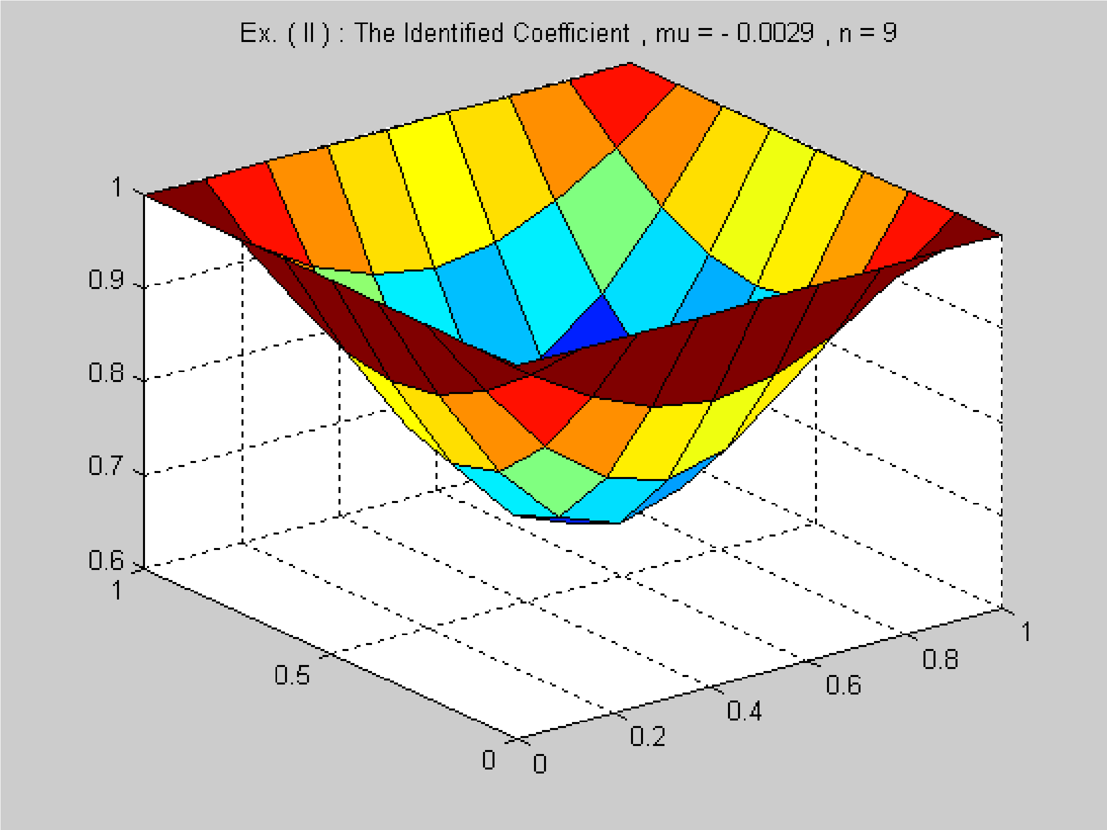
Ex. (II) : The Identified Coefficient , mu = - 0.0066 , n = 5
1
0.95
0.9
0.85
0.8
0.75
0.7
1
0.5
0
0
0.2
0.4
0.6
0.8
1

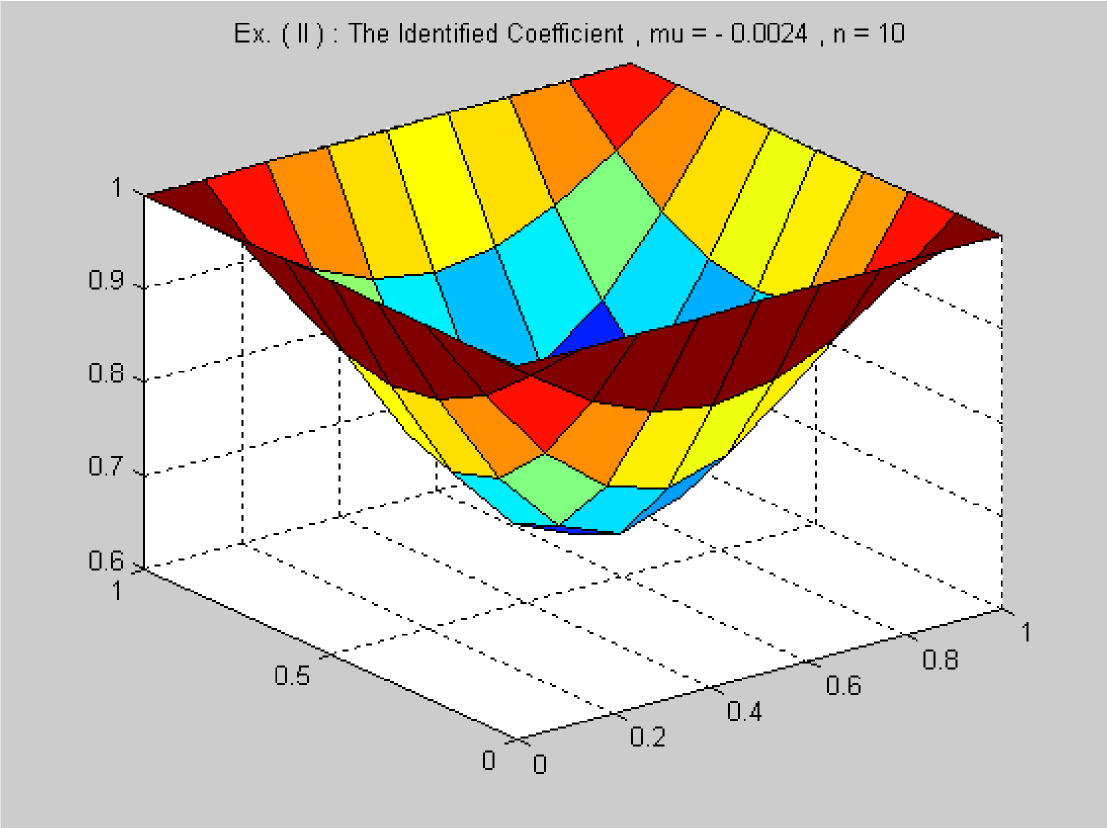


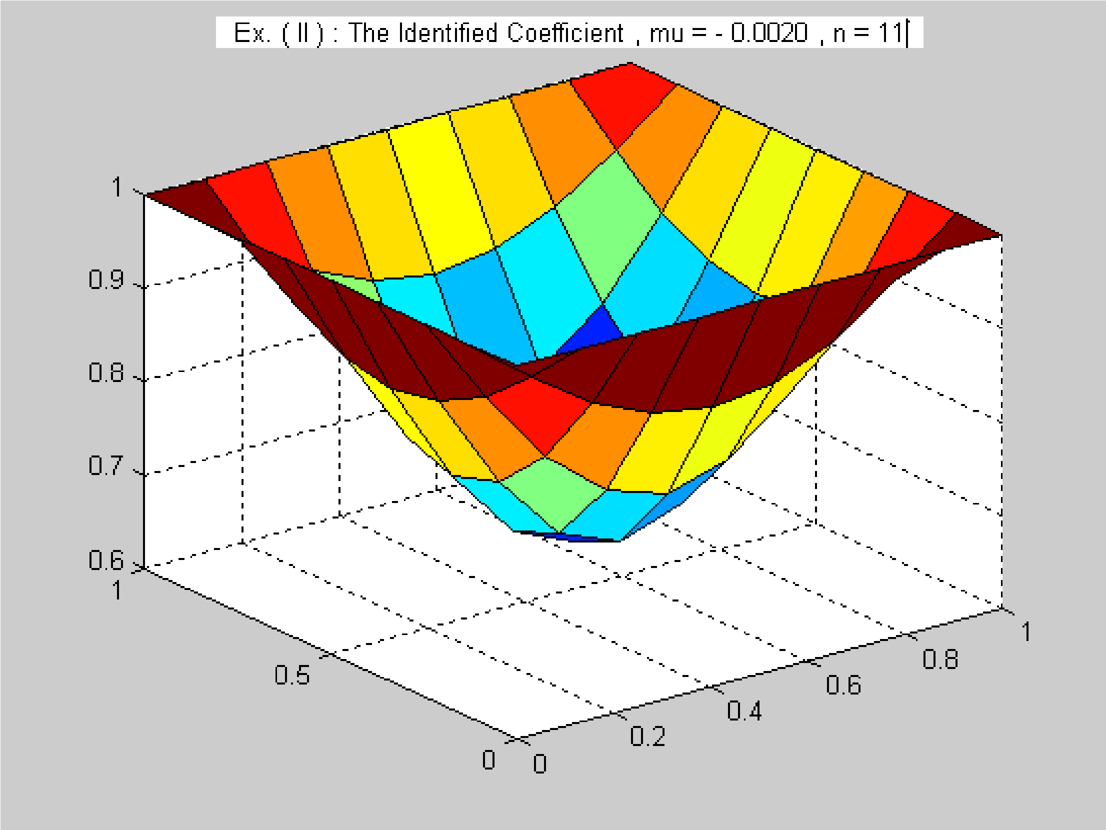
Ex. (II) : The Identified Coefficient , mu = - 0.0053 , n = 6
1
0.95
0.9
0.85
0.8
0.75
0.7
1
0.5
0
0
0.2
0.4
0.6
0.8
1

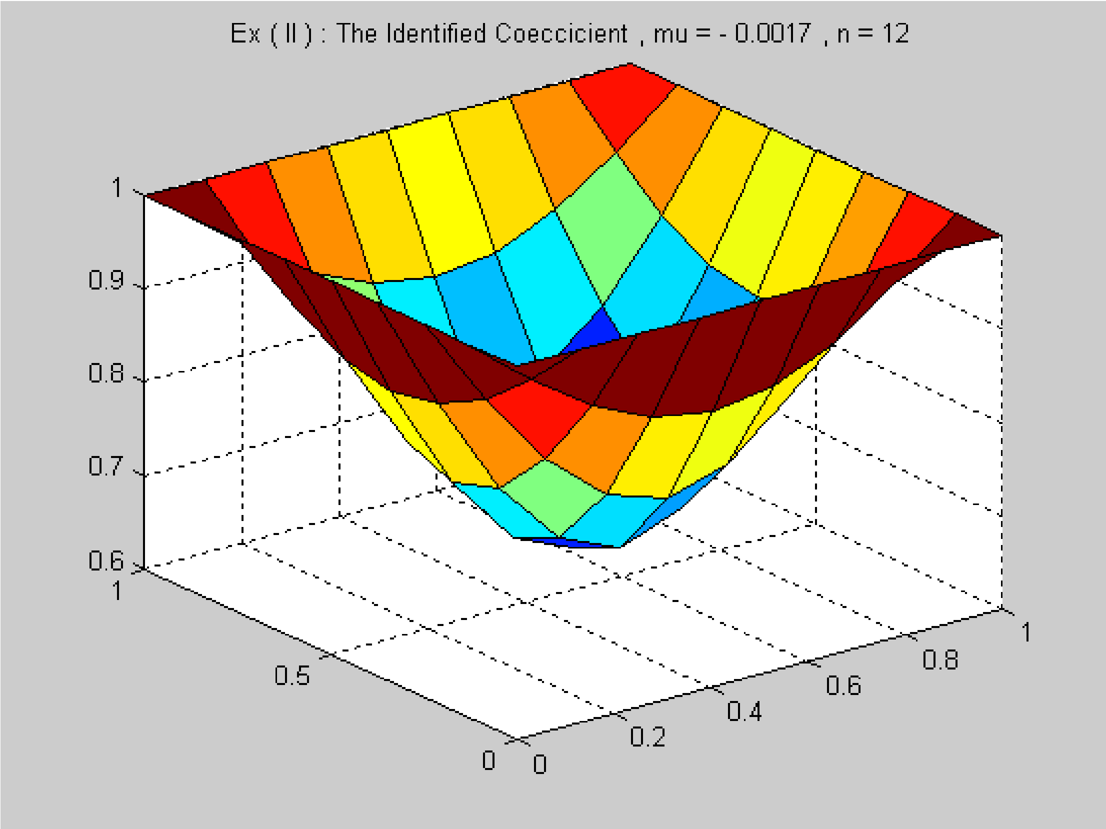

Ex. (II) : The Identified Coefficient , mu = - 0.0043 , n = 7
1
0.9
0.8
0.7
0.6
1
0.5
0
0
0.2
0.4
0.6
0.8
1

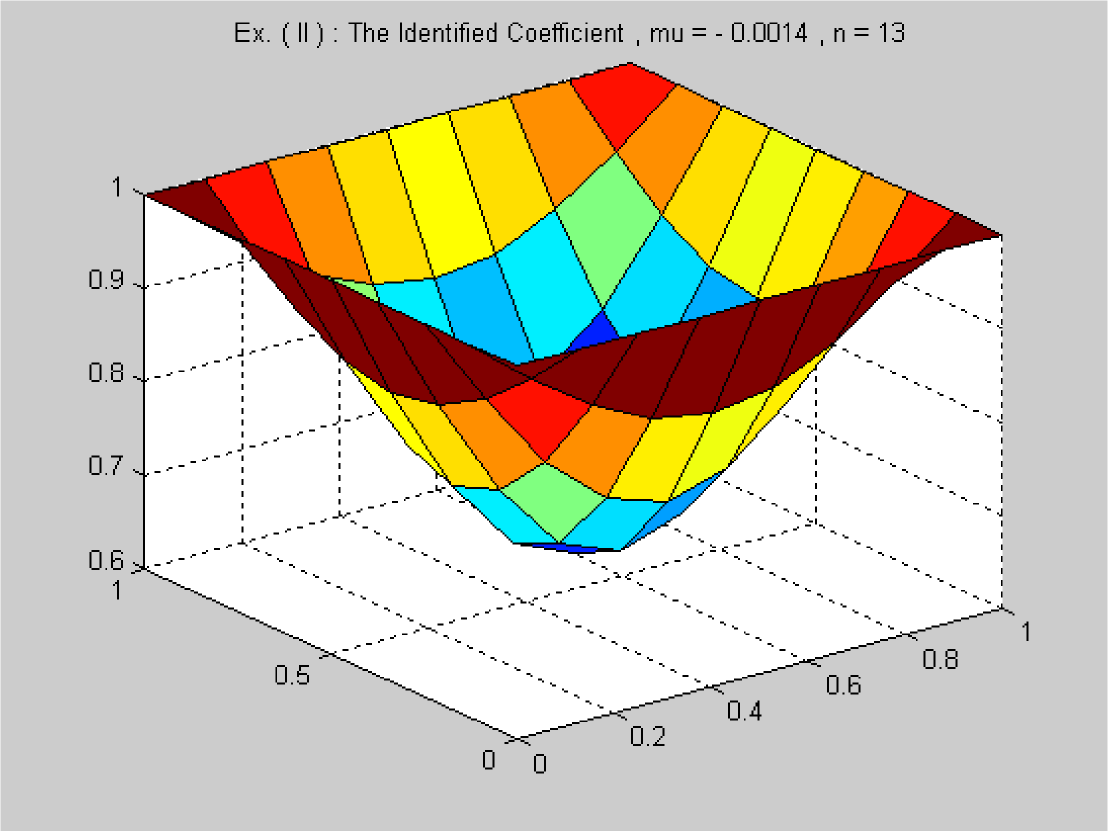

Ex. (II) : The Identified Coefficient , mu = - 0.0035 , n = 8
1
0.9
0.8
0.7
0.6
1
0.5
0
0
0.2
0.4
0.6
0.8
1

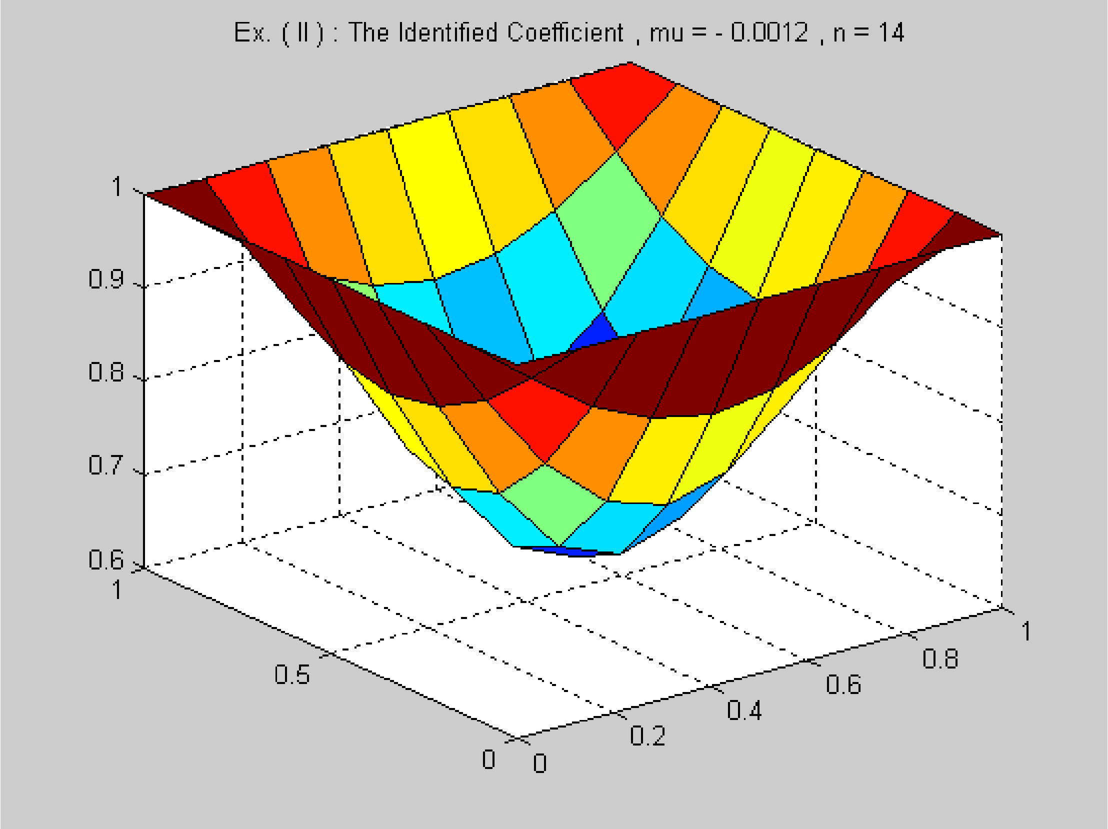

Ex. (II) : The Identified Coefficient , mu = - 0.0029 , n = 9
1
0.9
0.8
0.7
0.6
1
0.5
0
0
0.2
0.4
0.6
0.8
1

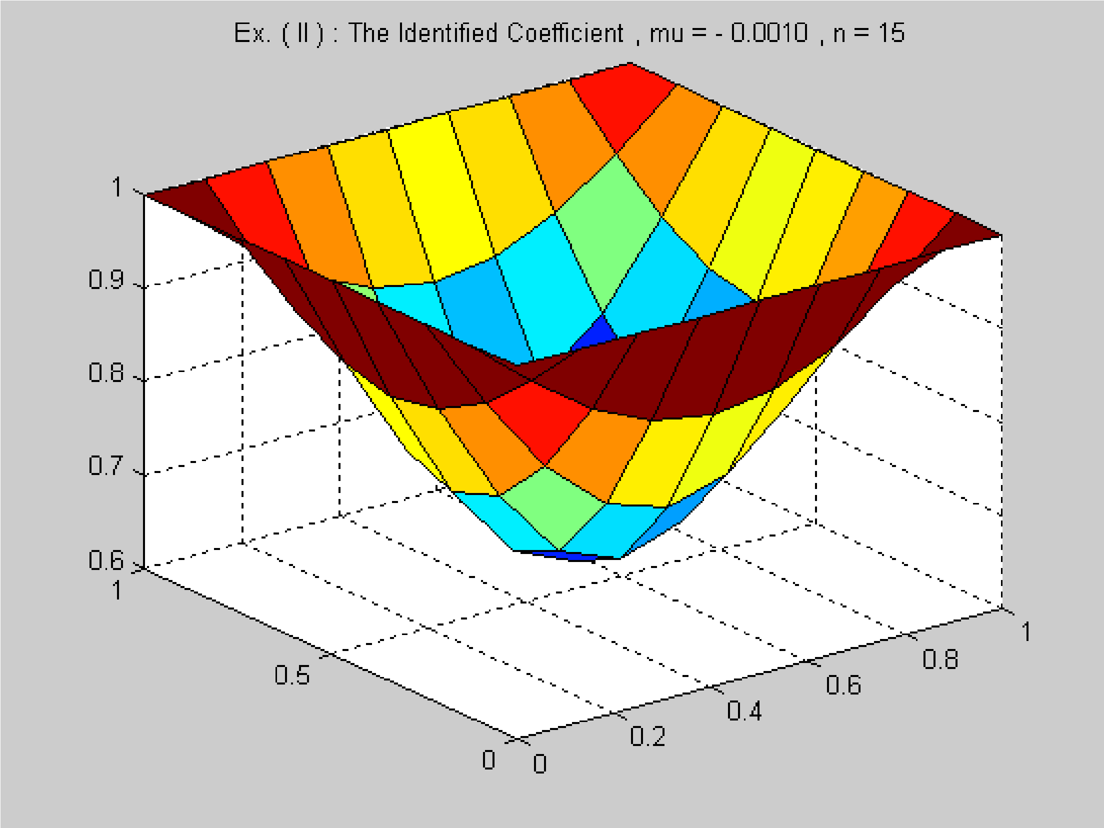

Ex. (II) : The Identified Coefficient , mu = - 0.0024 , n = 10
1
0.9
0.8
0.7
0.6
1
0.5
0
0
0.2
0.4
0.6
0.8
1

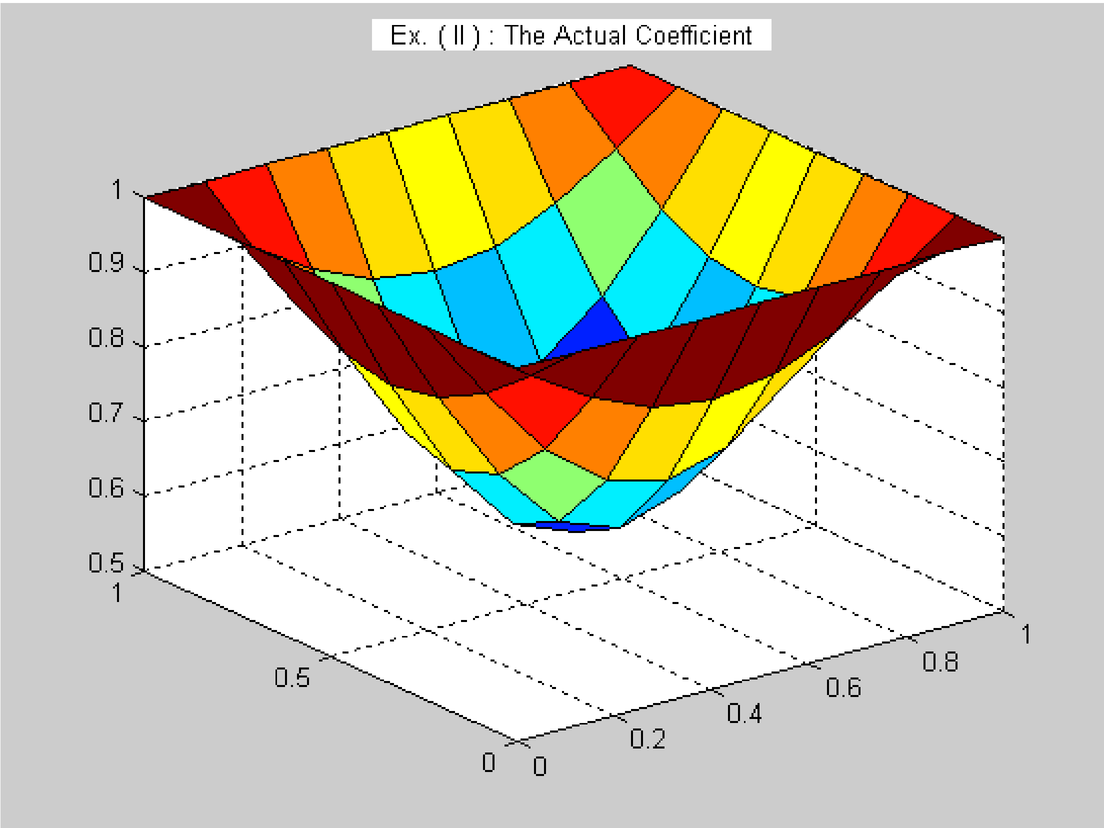

Ex. (II) : The Identified Coefficient , mu = - 0.0020 , n = 11
1
0.9
0.8
0.7
0.6
1
0.5
0
0
0.2
0.4
0.6
0.8
1

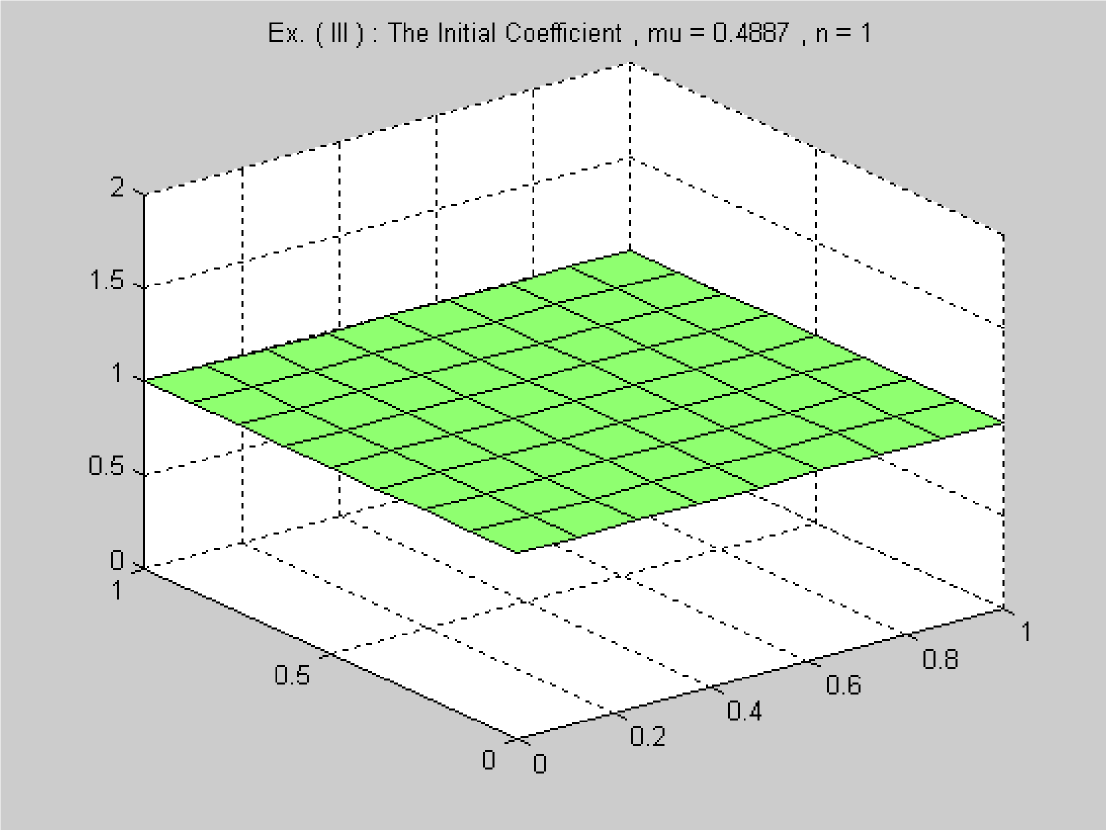

Ex (II) : The Identified Coeccicient , mu = - 0.0017 , n = 12
1
0.9
0.8
0.7
0.6
1
0.5
0
0
0.2
0.4
0.6
0.8
1

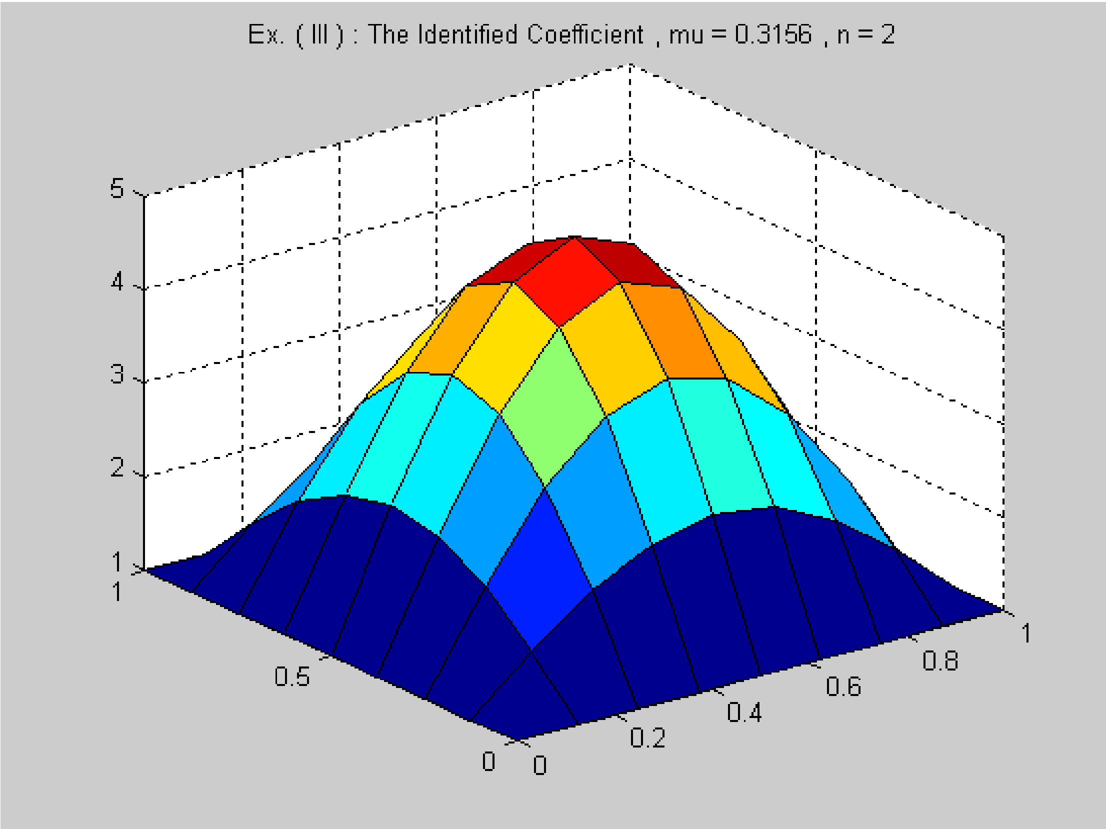

Ex. (II) : The Identified Coefficient , mu = - 0.0014 , n = 13
1
0.9
0.8
0.7
0.6
1
0.5
0
0
0.2
0.4
0.6
0.8
1

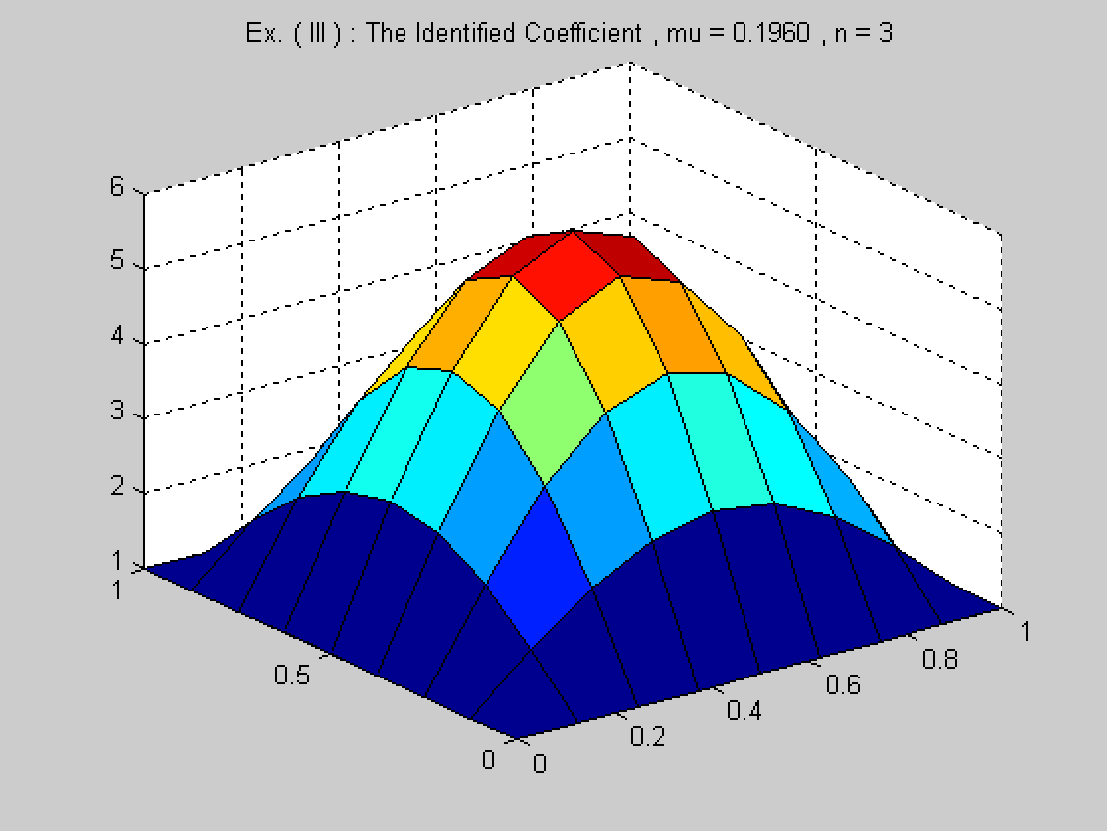

Ex. (II) : The Identified Coefficient , mu = - 0.0012 , n = 14
1
0.9
0.8
0.7
0.6
1
0.5
0
0
0.2
0.4
0.6
0.8
1

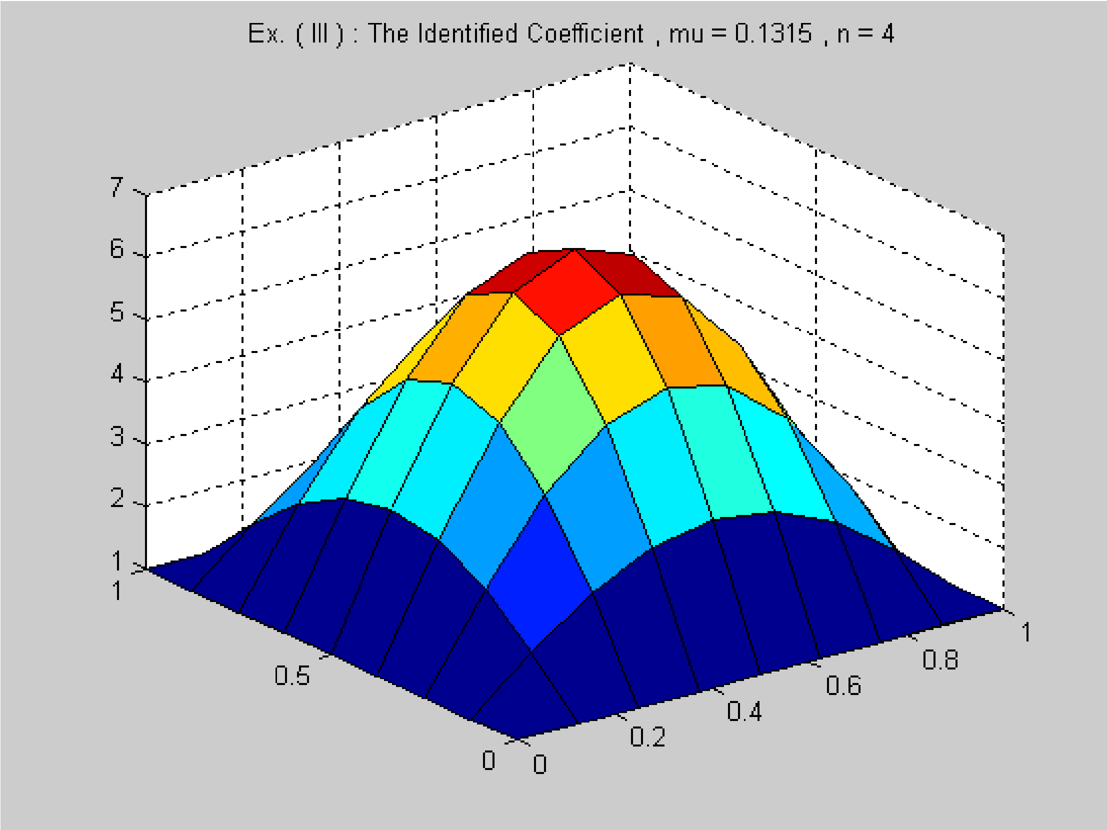

Ex. (II) : The Identified Coefficient , mu = - 0.0010 , n = 15
1
0.9
0.8
0.7
0.6
1
0.5
0
0
0.2
0.4
0.6
0.8
1

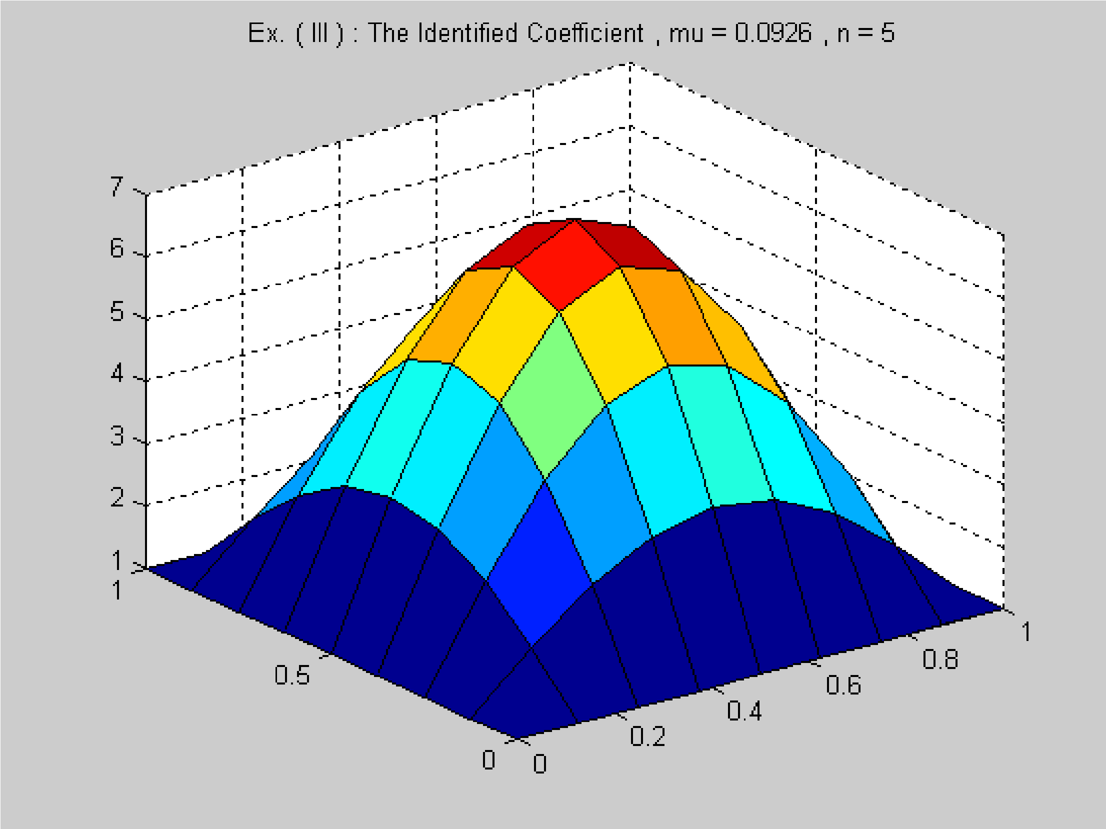

Ex. (II) : The Actual Coefficient
1
0.9
0.8
0.7
0.6
0.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1

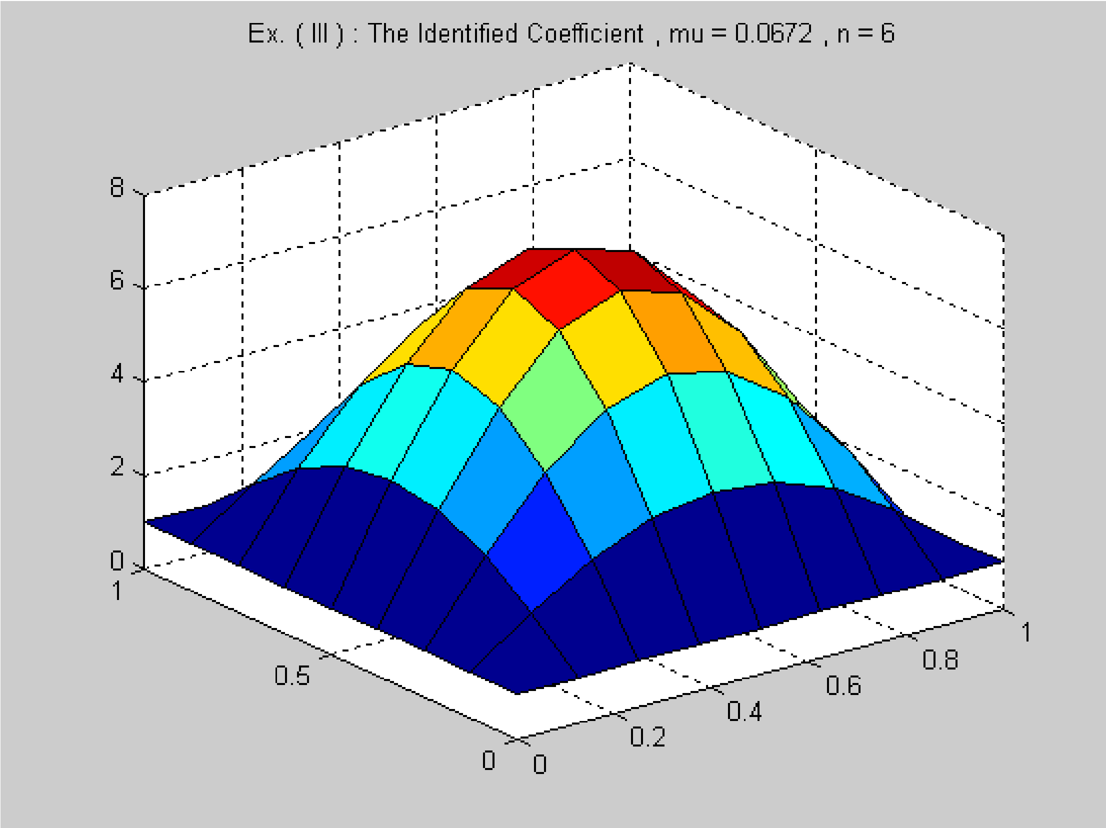

Ex. (III) : The Initial Coefficient , mu = 0.4887 , n = 1
2
1.5
1
0.5
0
1
0.5
0
0
0.2
0.4
0.6
0.8
1

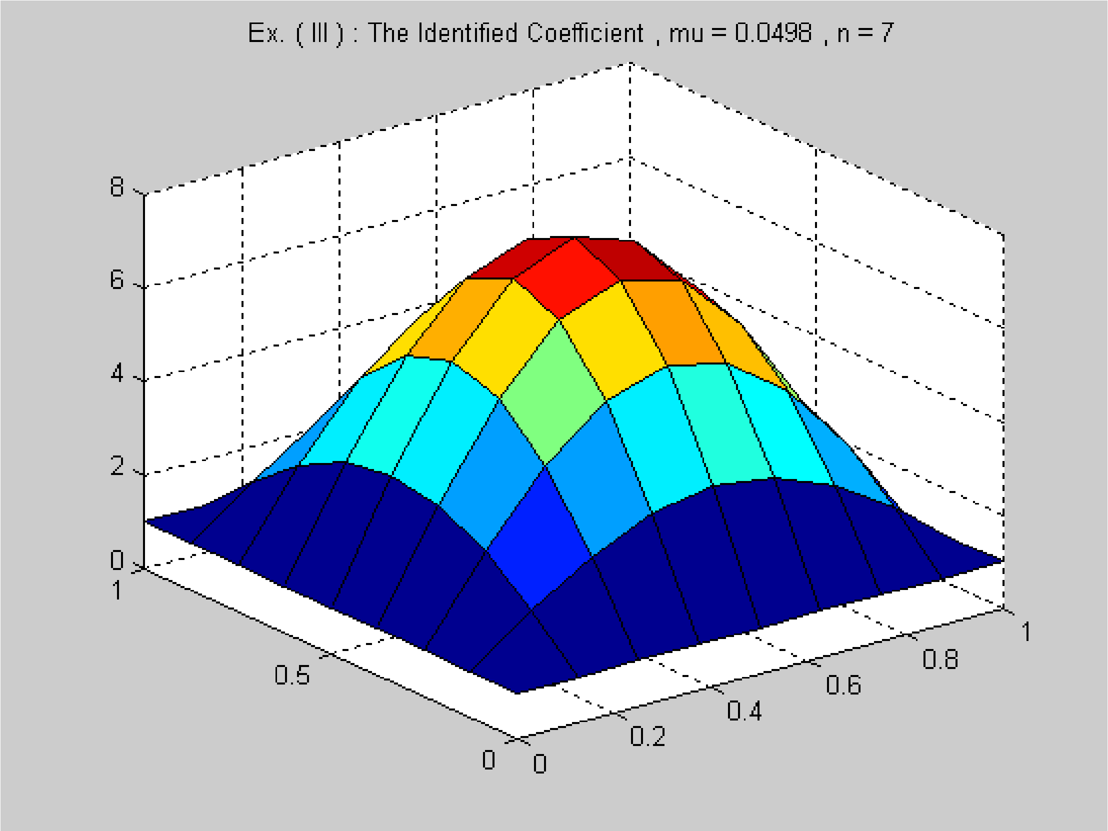

Ex. (III) : The Identified Coefficient , mu = 0.3156 , n = 2
5
4
3
2
1
1
0.5
0
0
0.2
0.4
0.6
0.8
1

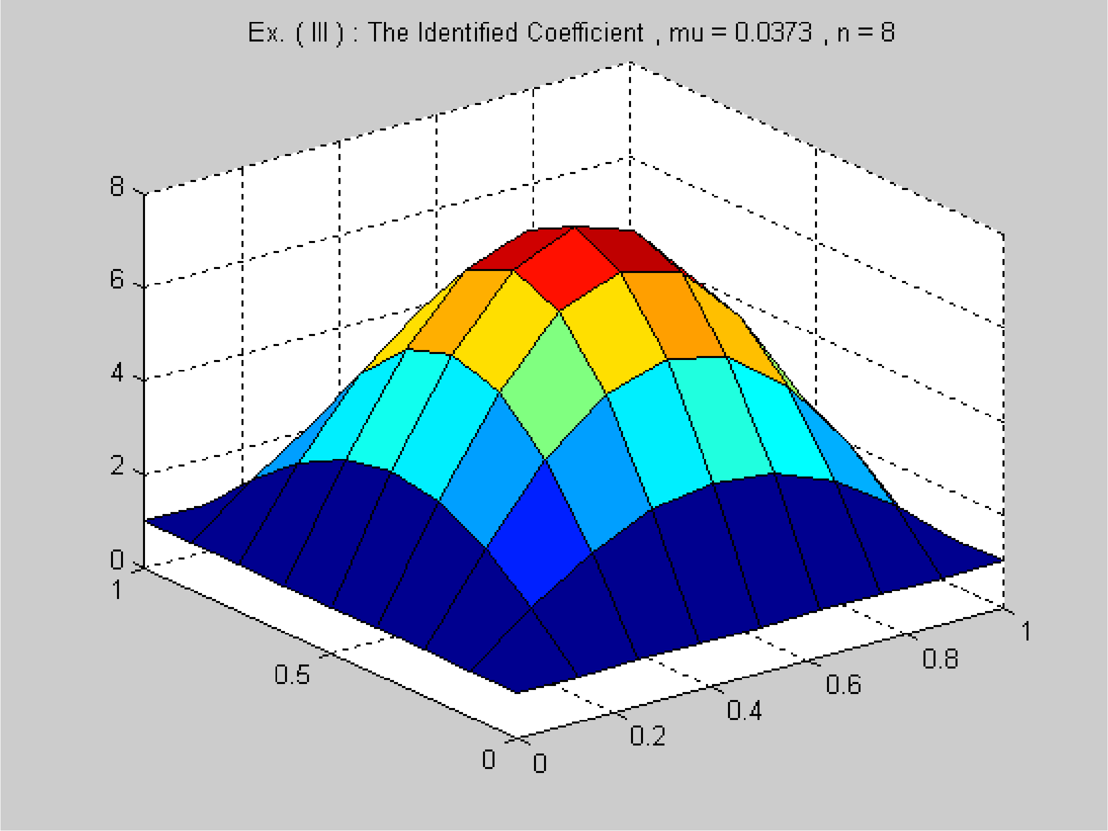

Ex. (III) : The Identified Coefficient , mu = 0.1960 , n = 3
6
5
4
3
2
1
1
0.5
0
0
0.2
0.4
0.6
0.8
1

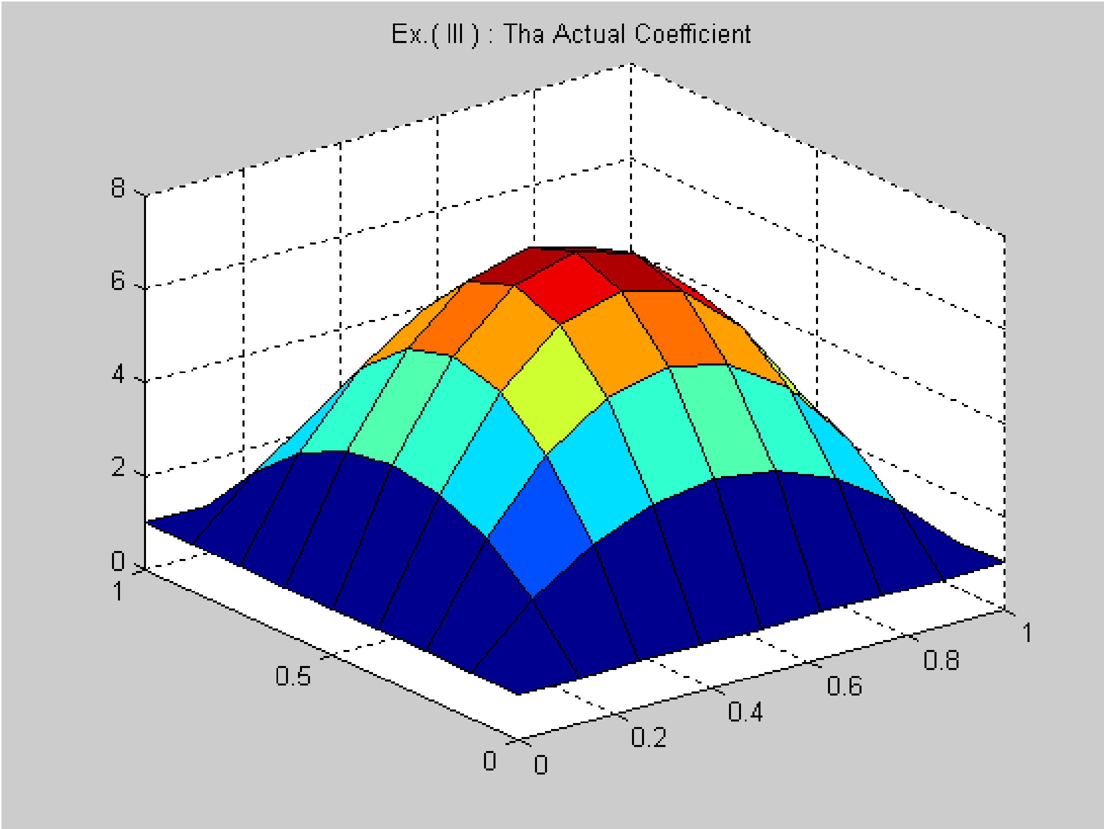

Ex. (III) : The Identified Coefficient , mu = 0.1315 , n = 4
7
6
5
4
3
2
1
1
0.5
0
0
0.2
0.4
0.6
0.8
1


Ex. (III) : The Identified Coefficient , mu = 0.0926 , n = 5
7
6
5
4
3
2
1
1
0.5
0
0
0.2
0.4
0.6
0.8
1


Ex. (III) : The Identified Coefficient , mu = 0.0672 , n = 6
8
6
4
2
0
1
0.5
0
0
0.2
0.4
0.6
0.8
1


Ex. (III) : The Identified Coefficient , mu = 0.0498 , n = 7
8
6
4
2
0
1
0.5
0
0
0.2
0.4
0.6
0.8
1


Ex. (III) : The Identified Coefficient , mu = 0.0373 , n = 8
8
6
4
2
0
1
0.5
0
0
0.2
0.4
0.6
0.8
1



Comments

Regarding to the previous results, in examples (I) and (III) the flux $g_1(x,y)$ corresponding to the initial coefficient $A_1(x,y)$ is greater than the actual flux, corresponding to the actual coefficient $A(x,y)$. So, the sequence $\{\mu_n\} \equiv \{mu\}_n$ is a positive decreasing convergent sequence to zero and as $\mu_n \xrightarrow{decreasing} 0$, $A_n \xrightarrow{decreasing} A$, according to lemma (4.1, (i)). While in example (II), $g_1(x,y) \prec g(x,y)$. So, $\{\mu_n\}$ is a negative increasing convergent sequence to zero and as $\mu_n \xrightarrow{increasing} 0$, $A_n \xrightarrow{increasing} A$, according to lemma (4.1, (ii)).

In this method, it is required to know the value of the actual coefficient on the boundary. Thanks to Nachman in [10] who make it reasonable by using the boundary conditions.

By choosing, a reasonable eigen values and the corresponding eigen functions (different from the Laplacian operator), this method may be also work for coefficients with less smoothness, such as piecewise continuous. [2]

Regarding to the idea of this method, it was just born and no one can say that we solved all the theoretical concepts about it. However, we can say that the method may work.

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