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Some New Properties of Wishart Distribution

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Abstract

We obtain the exact distributions of determinants and quotient of determinants of some submatrices of a Wishart distributed random matrix. We show an application of the obtained representations in testing hypotheses concerning the covariance matrix of multivariate normal distribution.

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1 Introduction

Let $\mathbf{W}$ be a random matrix with Wishart distribution $W_n(m, \mathrm{I}_n)$, where $m > n$ and I_n is the identity matrix of order n. The matrix $\mathbf{W}$ can be represented as a product (see [3])

$$\mathbf{W} = \mathbf{D}\,\mathbf{V}\,\mathbf{D}, \tag{1}$$

where $\mathbf{D}$ is a diagonal random matrix, $\mathbf{D} = diag(\sqrt{\tau_1}, \dots, \sqrt{\tau_n})$ and $\mathbf{V} = (\nu_{i,j})$ is a symmetric random matrix with units on the main diagonal. The random variables τ_i, $i = 1, \dots, n$ are mutually independent, independent of $\nu_{i,j}$, $1 \leq i < j \leq n$ and have chi - square distribution, $\tau_i \sim \chi^2(m)$, $i = 1, \dots, n$. The joint density function of $\nu_{i,j}$, $1 \leq i < j \leq n$ has the form

$$f(y_{i,j}, 1 \leq i < j \leq n) = \frac{\left[\Gamma\left(\frac{m}{2}\right)\right]^n}{\Gamma_n\left(\frac{m}{2}\right)} (\det Y)^{\frac{m-n-1}{2}}, \tag{2}$$

if Y is a positive definite matrix, $Y = \begin{pmatrix} 1 & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{1n} & \cdots & 1 \end{pmatrix}$. By $\Gamma_n(\alpha)$ is denoted the multivariate Gamma function,

$$\Gamma_n(\alpha) = \pi^{\frac{n(n-1)}{4}} \Gamma(\alpha) \Gamma\left(\alpha - \frac{1}{2}\right) \ldots \Gamma\left(\alpha - \frac{n-1}{2}\right).$$

Ignatov and Nikolova in [3], denote by $\psi(m,n)$ the joint distribution of $\nu_{i,j}$, $1 \le i < j \le n$ with density function of the form (2). For $n = 2$ the distribution $\psi(m,n)$ is a univariate distribution with density function

$$f(y) = \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m-1}{2}\right)\Gamma\left(\frac{1}{2}\right)} (1 - y^2)^{\frac{m-3}{2}}, \quad y \in (-1, 1).$$

Definition 1.1 *A random matrix* $\mathbf{V}$ *is said to have distribution* $\psi_n(m)$ *with parameters* n, m, $n < m$, *written as* $\mathbf{V} \sim \psi_n(m)$, *if* $\mathbf{V}$ *is a symmetric matrix of order* n *with units on the main diagonal and the joint distribution of the elements above the main diagonal is* $\psi(m,n)$.

Let $\mathbf{R}$ be the sample correlation matrix for a sample of size $m+1$ from n - variate normal distribution $N_n(\mu, \Sigma)$ with unknown mean vector μ. Suppose that Σ is a diagonal matrix with unknown positive diagonal elements. Then the distribution of the sample correlation matrix $\mathbf{R}$ is $\psi_n(m)$ (see [7]).

In the present paper we obtain some properties of the distribution $\psi_n(m)$ of the matrix $\mathbf{V}$ in (1). By equality (1), we get the corresponding properties of Wishart distribution. In an example, we show an application of the obtained representations in testing hypotheses concerning the covariance matrix of multivariate normal distribution.

2 Preliminary Notes

Let $P(n, \Re)$ be the set of all real, symmetric, positive definite matrices of order n. Let us denote by $D(n, \Re)$ the set of all real, symmetric matrices of order n, with positive diagonal elements, which off-diagonal elements are in the interval (-1,1). There exist a bijection $h: \ D(n, \Re) \to P(n, \Re)$, considered in [4], [5] and [6]. The image of an arbitrary matrix $X = (x_{i,j})$ from $D(n, \Re)$ by the bijection h, is a matrix $Y = (y_{i,j})$ from $P(n, \Re)$, such that

$$y_{j,j} = x_{j,j}, \quad j = 1, \ldots, n, \tag{3}$$

$$y_{1,j} = x_{1,j}\sqrt{x_{1,1}x_{j,j}}, \quad j = 2, \ldots, n, \tag{4}$$

$$y_{i,j} = \sqrt{x_{i,i}x_{j,j}}\left[\sum_{r=1}^{i-1}\left(x_{r,i}x_{r,j}\prod_{q=1}^{r-1}\sqrt{(1-x_{q,i}^2)(1-x_{q,j}^2)}\right) + x_{i,j}\prod_{q=1}^{i-1}\sqrt{(1-x_{q,i}^2)(1-x_{q,j}^2)}\right], \quad 2 \leq i < j \leq n. \quad (5)$$

The next Proposition can be found in [4] and [6].

Proposition 2.1 *Let $\xi = (\xi_{i,j})$ be a random symmetric matrix of order n with units on the main diagonal. Suppose that $\xi_{i,j}$ are independent and $\xi_{i,j} \sim \psi(m-i+1, 2)$, $1 \leq i < j \leq n$. Let $\mathbf{V}$ be the matrix $\mathbf{V} = h(\xi)$, where h is the bijection, defined by (3)-(5). Then the matrix $\mathbf{V}$ has distribution $\psi_n(m)$.*

Let $A = (a_{i,j})$ be a real square matrix of order n. Let α and β be nonempty subsets of the set $\{1, \ldots, n\}$. By $A[\alpha, \beta]$ we denote the submatrix of A, composed of the rows with numbers from α and the columns with numbers from β. Denote by α^c the complement of the set α in $\{1, \ldots, n\}$, i.e. $\alpha^c = \{1, \ldots, n\} \backslash \alpha$. For the matrix $A[\alpha^c, \beta^c]$ we use the notation $A(\alpha, \beta)$. When $\beta \equiv \alpha$, $A[\alpha, \alpha]$ is denoted simply by $A[\alpha]$ and $A(\alpha, \alpha)$ by $A(\alpha)$.

Let $X \in D(n, \Re)$ and $Y = h(X)$, where h is the bijection, defined by (3)-(5). We get interesting relations between the elements of the matrices X and Y (see [4], [5] and [6]):

$$x_{i,j} = \frac{\det Y[\{1, \ldots, i\}, \{1, \ldots, i-1, j\}]}{\sqrt{\det Y[\{1, \ldots, i\}] \det Y[\{1, \ldots, i-1, j\}]}}, \quad 2 \leq i < j \leq n; \quad (6)$$

$$(1-x_{1,j}^2)(1-x_{2,j}^2)\ldots(1-x_{i,j}^2) = \frac{\det Y[\{1, \ldots, i, j\}]}{y_{j,j} \det Y[\{1, \ldots, i\}]}, \quad 1 \leq i < j \leq n; \quad (7)$$

$$\det Y[\{1, \ldots, i, j\}] = x_{1,1} \ldots x_{i,i} x_{j,j} \left(\prod_{1 \leq k < s \leq i} (1-x_{k,s}^2)\right)\left(\prod_{k=1}^{i}(1-x_{k,j}^2)\right), \quad (8)$$

$$1 \leq i < j \leq n.$$

3 Main Results

Theorem 3.1 *Let $\mathbf{V} \sim \psi_n(m)$. Then for all integer i and j, $2 \leq i < j \leq n$*

$$\det \mathbf{V}[\{1, \ldots, i\}, \{1, \ldots, i-1, j\}] \sim \zeta_1 \ldots \zeta_{i-1}\sqrt{\zeta_i \zeta_{i+1}},$$

where the random variables ζ_s*,* $s = 1, \dots, i+1$ *are independent,* $\zeta_1 \sim \psi(m - i + 1, 2)$ *and* ζ_s*,* $s = 2, \dots, i+1$ *are beta distributed,* $\zeta_s \sim \beta\left(\frac{m-s+1}{2}, \frac{s-1}{2}\right)$*,* $s = 2, \dots, i$*,* $\zeta_{i+1} \sim \beta\left(\frac{m-i+1}{2}, \frac{i-1}{2}\right)$*.*

Proof. Using Proposition 2.1 and the representations (6) and (8), for $2 \leq i < j \leq n$ we have

$$\begin{aligned}
&\det \mathbf{V}[\{1,\dots,i\},\{1,\dots,i-1,j\}] \\
&\qquad = \xi_{i,j}\sqrt{\det \mathbf{V}[\{1,\dots,i\}]\det \mathbf{V}[\{1,\dots,i-1,j\}]} \\
&\qquad = \xi_{i,j}\left(\prod_{1\leq k<s\leq i-1}(1-\xi_{k,s}^2)\right)\sqrt{\left(\prod_{k=1}^{i-1}(1-\xi_{k,i}^2)\right)\left(\prod_{k=1}^{i-1}(1-\xi_{k,j}^2)\right)} \\
&\qquad = \xi_{i,j}\left(\prod_{s=2}^{i-1}\left(\prod_{k=1}^{s-1}(1-\xi_{k,s}^2)\right)\right)\sqrt{\left(\prod_{k=1}^{i-1}(1-\xi_{k,i}^2)\right)\left(\prod_{k=1}^{i-1}(1-\xi_{k,j}^2)\right)}. \qquad (9)
\end{aligned}$$

Denote by ζ_s, $s = 1, \dots, i+1$ the random variables

$$\zeta_1 = \xi_{i,j}, \quad \zeta_s = \prod_{k=1}^{s-1}(1-\xi_{k,s}^2),\ s = 2,\dots,i, \quad \zeta_{i+1} = \prod_{k=1}^{i-1}(1-\xi_{k,j}^2).$$

The random variables $\xi_{i,j}$, $1 \leq i < j \leq n$ are independent, therefore ζ_s, $s = 1, \dots, i+1$ are independent, too. Since $\xi_{i,j} \sim \psi(m-i+1, 2)$, $1 \leq i < j \leq n$, it can be shown that $1 - \xi_{i,j}^2 \sim \beta\left(\frac{m-i}{2}, \frac{1}{2}\right)$, $1 \leq i < j \leq n$. It is known, that if π_1 and π_2 are independent random variables, $\pi_1 \sim \beta(\alpha, \gamma)$, $\pi_2 \sim \beta(\alpha + \gamma, \delta)$, then $\pi_1\pi_2 \sim \beta(\alpha, \gamma + \delta)$. Consequently,

$$(1-\xi_{i-1,j}^2)(1-\xi_{i,j}^2) \sim \beta\left(\frac{m-i}{2}, 1\right), \dots,$$

$$(1-\xi_{1,j}^2)\dots(1-\xi_{i,j}^2) \sim \beta\left(\frac{m-i}{2}, \frac{i}{2}\right). \ \square \qquad (10)$$

Corollary 3.1 *Let* $\mathbf{V} \sim \psi_n(m)$*. Then for all integer* i *and* j*,* $2 \leq i < j \leq n$*,*

$$\frac{\det \mathbf{V}[\{1,\dots,i\},\{1,\dots,i-1,j\}]}{\det \mathbf{V}[\{1,\dots,i-1\}]} \sim \zeta_1\sqrt{\zeta_2\,\zeta_3}$$

and

$$\frac{\det \mathbf{V}[\{1,\dots,i\},\{1,\dots,i-1,j\}]}{\det \mathbf{V}[\{1,\dots,i\}]} \sim \zeta_1\sqrt{\frac{\zeta_2}{\zeta_3}},$$

where the random variables ζ_1*,* ζ_2 *and* ζ_3 *are independent,* $\zeta_1 \sim \psi(m-i+1, 2)$*,* $\zeta_2, \zeta_3 \sim \beta\left(\frac{m-i+1}{2}, \frac{i-1}{2}\right)$*.*

Proof. The Corollary follows from Theorem 3.1 and the representation (8). □

Lemma 3.1 *Let* $\mathbf{V} \sim \psi_n(m)$*. Let* i, j *be arbitrary integers,* $1 \le i < j \le n$*. Suppose that we interchange the places of the* i*'th and* j*'th rows in* $\mathbf{V}$ *and then interchange the places of the* i*'th and* j*'th columns. Then the obtained matrix* $\mathbf{V}'$ *is distributed again* $\psi_n(m)$*.*

Proof. The Lemma follows from the properties of determinants and positive definite matrices. □

Theorem 3.2 *Let* $\mathbf{V} \sim \psi_n(m)$*,* $n > 2$*. Then for* $1 \le i < j \le n$

$$\det \mathbf{V}(\{i\},\{j\}) \sim (-1)^{j-i-1}\zeta_1\zeta_2\dots\zeta_{n-2}\sqrt{\zeta_{n-1}\zeta_n},$$

where ζ_s*,* $s = 1,\dots,n$ *are independent,* $\zeta_1 \sim \psi(m-n+2, 2)$*,* $\zeta_s \sim \beta\left(\frac{m-s+1}{2}, \frac{s-1}{2}\right)$*,* $s = 2,\dots,n-1$ *and* $\zeta_n \sim \beta\left(\frac{m-n+2}{2}, \frac{n-2}{2}\right)$*.*

Proof. Let us put the i'th and j'th rows of $\mathbf{V}$ after its n'th row; the i'th and j'th columns after the n'th column. We get a new matrix $\mathbf{V}'$,

$$\mathbf{V}' = \begin{pmatrix} \mathbf{V}(\{i,j\}) & \mathbf{V}(\{i,j\},\{i\}^c) & \mathbf{V}(\{i,j\},\{j\}^c) \\ \mathbf{V}(\{i\}^c,\{i,j\}) & 1 & \nu_{i,j} \\ \mathbf{V}(\{j\}^c,\{i,j\}) & \nu_{i,j} & 1 \end{pmatrix}, \tag{11}$$

where α^c denotes the set $\{1,\dots,n\}\backslash\alpha$. Applying Lemma 3.1 several times we get that $\mathbf{V}' \sim \psi_n(m)$. It is not difficult to see that

$$\det \mathbf{V}(\{i\},\{j\}) = (-1)^{j-i-1}\det \mathbf{V}'(\{n-1\},\{n\}). \tag{12}$$

On the other hand,

$$\begin{aligned} \det \mathbf{V}'(\{n-1\},\{n\}) &= \det \mathbf{V}'(\{n\},\{n-1\}) \\ &= \det \mathbf{V}'[\{1,\dots,n-1\},\{1,\dots,n-2,n\}]. \end{aligned} \tag{13}$$

Now applying Theorem 3.1, we complete the proof. □

Corollary 3.2 *Let* $\mathbf{V} \sim \psi_n(m)$*,* $n > 2$*. Then the elements* $\nu^{i,j}$*,* $1 \le i < j \le n$ *of the inverse matrix* $\mathbf{V}^{-1}$ *are identically distributed*

$$\nu^{i,j} \sim -\frac{\zeta_1}{(1-\zeta_1^2)}\frac{1}{\sqrt{\zeta_2\zeta_3}},$$

where the random variables $\zeta_1, \zeta_2, \zeta_3$ *are independent,* $\zeta_1 \sim \psi(m-n+2, 2)$*,* $\zeta_2, \zeta_3 \sim \beta\left(\frac{m-n+2}{2}, \frac{n-2}{2}\right)$*.*

Proof. Let $\mathbf{V}'$ be the matrix in (11). Then $\det \mathbf{V} = \det \mathbf{V}'$. From (12) and (13) we get

$$\begin{aligned}\nu^{i,j} &= (-1)^{j-i}\frac{\det \mathbf{V}(\{i\},\{j\})}{\det \mathbf{V}} = -\frac{\det \mathbf{V}'(\{n-1\},\{n\})}{\det \mathbf{V}'}\\ &= -\frac{\det \mathbf{V}'[\{1,\dots,n-1\},\{1,\dots,n-2,n\}]}{\det \mathbf{V}'}.\end{aligned}$$

The matrix $\mathbf{V}'$ is distributed $\psi_n(m)$. Hence, from Proposition 2.1 and equalities (6) and (8) it follows that

$$\begin{aligned}&\frac{\det \mathbf{V}'[\{1,\dots,n-1\},\{1,\dots,n-2,n\}]}{\det \mathbf{V}'}\\ &\quad= \frac{\xi_{n-1,n}\sqrt{\det \mathbf{V}'[\{1,\dots,n-1\}]\det \mathbf{V}'[\{1,\dots,n-2,n\}]}}{\det \mathbf{V}'}\\ &\quad= \frac{\xi_{n-1,n}\left(\prod_{1\le k<s\le n-2}(1-\xi_{k,s}^2)\right)\sqrt{\left(\prod_{k=1}^{n-2}(1-\xi_{k,n-1}^2)\right)\left(\prod_{k=1}^{n-2}(1-\xi_{k,n}^2)\right)}}{\left(\prod_{1\le k<s\le n-1}(1-\xi_{k,s}^2)\right)\left(\prod_{k=1}^{n-1}(1-\xi_{k,n}^2)\right)}\\ &\quad= \frac{\xi_{n-1,n}}{(1-\xi_{n-1,n}^2)}\frac{1}{\sqrt{\left(\prod_{k=1}^{n-2}(1-\xi_{k,n-1}^2)\right)\left(\prod_{k=1}^{n-2}(1-\xi_{k,n}^2)\right)}},\end{aligned}$$

where $\xi_{i,j}$, $1\le i<j\le n$ are independent and $\xi_{i,j}\sim\psi(m-i+1,2)$. Let us denote by ζ_1, ζ_2 and ζ_3 the random variables

$$\zeta_1 = \xi_{n-1,n},\quad \zeta_2 = \prod_{k=1}^{n-2}(1-\xi_{k,n-1}^2),\quad \zeta_3 = \prod_{k=1}^{n-2}(1-\xi_{k,n}^2).$$

Then ζ_1,ζ_2,ζ_3 are independent and $\zeta_1\sim\psi(m-n+2,2)$. From (10) we have that $\zeta_2,\zeta_3\sim\beta\left(\frac{m-n+2}{2},\ \frac{n-2}{2}\right)$. $\square$

Using the representation (1), the statements, proved for the distribution $\psi_n(m)$ of $\mathbf{V}$, can be easily reformulated for the distribution $W_n(m,\mathrm{I}_n)$. We shall use several times the following known property of the Gamma and Beta distributions.

Proposition 3.1 *Let π_1 and π_2 be independent random variables, π_1 is Gamma distributed $G(\alpha,\lambda)$ and $\pi_2\sim\beta(\alpha-\delta,\delta)$. Then $\pi_1\pi_2\sim G(\alpha-\delta,\lambda)$.*

Theorem 3.3 *Let $\mathbf{W}\sim W_n(m,\mathrm{I}_n)$. Then for $2\le i<j\le n$*

$$\det \mathbf{W}[\{1,\dots,i\},\{1,\dots,i-1,j\}]\sim\zeta_1\dots\zeta_i\sqrt{\zeta_{i+1}\zeta_{i+2}},$$

where the random variables ζ_s, $s = 1, \ldots, i+2$ *are independent,* $\zeta_1 \sim \psi(m-i+1, 2)$, $\zeta_s \sim \chi^2(m-s+2)$, $s = 2, \ldots, i+1$, $\zeta_{i+2} \sim \chi^2(m-i+1)$.

Proof. From the representation (1) it can be seen that

$$\det \mathbf{W}[\{1, \ldots, i\}, \{1, \ldots, i-1, j\}] \\ = \tau_1 \ldots \tau_{i-1} \sqrt{\tau_i \tau_j} \det \mathbf{V}[\{1, \ldots, i\}, \{1, \ldots, i-1, j\}].$$

The random variables $\tau_1, \ldots, \tau_n$ are independent and $\tau_i \sim \chi^2(m)$. From Theorem 3.1 we have that

$$\det \mathbf{V}[\{1, \ldots, i\}, \{1, \ldots, i-1, j\}] \sim \zeta_1 \ldots \zeta_{i-1} \sqrt{\zeta_i \zeta_{i+1}},$$

where ζ_s, $s = 1, \ldots, i+1$ are independent, $\zeta_1 \sim \psi(m-i+1, 2)$, $\zeta_s \sim \beta\left(\frac{m-s+1}{2}, \frac{s-1}{2}\right)$, $s = 2, \ldots, i$, $\zeta_{i+1} \sim \beta\left(\frac{m-i+1}{2}, \frac{i-1}{2}\right)$. Now applying Proposition 3.1, we complete the proof. $\square$

Corollary 3.3 *Let* $\mathbf{W} \sim W_n(m, \mathrm{I}_n)$. *Then for all integer* i *and* j, $2 \le i < j \le n$

$$\frac{\det \mathbf{W}[\{1, \ldots, i\}, \{1, \ldots, i-1, j\}]}{\det \mathbf{W}[\{1, \ldots, i-1\}]} \sim \zeta_1 \sqrt{\zeta_2\, \zeta_3}$$

and

$$\frac{\det \mathbf{W}[\{1, \ldots, i\}, \{1, \ldots, i-1, j\}]}{\det \mathbf{W}[\{1, \ldots, i\}]} \sim \zeta_1 \sqrt{\frac{\zeta_2}{\zeta_3}},$$

where $\zeta_1, \zeta_2, \zeta_3$ *are independent,* $\zeta_1 \sim \psi(m-i+1, 2)$, $\zeta_2, \zeta_3 \sim \chi^2(m-i+1)$.

Proof. From the representation (1) it can be seen that

$$\frac{\det \mathbf{W}[\{1, \ldots, i\}, \{1, \ldots, i-1, j\}]}{\det \mathbf{W}[\{1, \ldots, i-1\}]} = \sqrt{\tau_i \tau_j} \frac{\det \mathbf{V}[\{1, \ldots, i\}, \{1, \ldots, i-1, j\}]}{\det \mathbf{V}[\{1, \ldots, i-1\}]},$$

$$\frac{\det \mathbf{W}[\{1, \ldots, i\}, \{1, \ldots, i-1, j\}]}{\det \mathbf{W}[\{1, \ldots, i\}]} = \sqrt{\frac{\tau_j}{\tau_i}} \frac{\det \mathbf{V}[\{1, \ldots, i\}, \{1, \ldots, i-1, j\}]}{\det \mathbf{V}[\{1, \ldots, i\}]}.$$

Now using Corollary 3.1 and Proposition 3.1, the Theorem follows. $\square$

Theorem 3.4 *Let* $\mathbf{W} \sim W_n(m, \mathrm{I}_n)$. *Then for all integer* i *and* j, $1 \le i < j \le n$

$$\det \mathbf{W}(\{i\}, \{j\}) \sim (-1)^{j-i-1} \zeta_1 \zeta_2 \ldots \zeta_{n-1} \sqrt{\zeta_n \zeta_{n+1}},$$

where the random variables ζ_k, $k = 1, \ldots, n+1$ *are independent,* $\zeta_1 \sim \psi(m-n+2, 2)$, $\zeta_k \sim \chi^2(m-k+2)$, $k = 2, \ldots, n$, $\zeta_{n+1} \sim \chi^2(m-n+2)$.

Proof. From the representation (1) it can be seen that

$$\det \mathbf{W}(\{i\},\{j\}) = \tau_1 \dots \tau_{i-1}\tau_{i+1} \dots \tau_{j-1}\tau_{j+1} \dots \tau_n \sqrt{\tau_i\tau_j} \det \mathbf{V}(\{i\},\{j\}).$$

According to Theorem 3.2,

$$\det \mathbf{V}(\{i\},\{j\}) \sim (-1)^{j-i-1}\zeta_1\zeta_2 \dots \zeta_{n-2}\sqrt{\zeta_{n-1}\zeta_n},$$

where ζ_s, $s = 1, \dots, n$ are independent, $\zeta_1 \sim \psi(m-n+2, 2)$, $\zeta_s \sim \beta\left(\frac{m-s+1}{2}, \frac{s-1}{2}\right)$, $s = 2, \dots, n-1$ and $\zeta_n \sim \beta\left(\frac{m-n+2}{2}, \frac{n-2}{2}\right)$. Now applying Proposition 3.1, we complete the proof. $\square$

Corollary 3.4 *Let* $\mathbf{W} \sim W_n(m, \mathrm{I}_n)$. *Then the element* $w^{i,j}$ *on i'th row and j'th column of the inverse matrix* $\mathbf{W}^{-1}$, $1 \le i < j \le n$, *is distributed*

$$w^{i,j} \sim \frac{-\zeta_1}{(1-\zeta_1^2)}\frac{1}{\sqrt{\zeta_2\zeta_3}},$$

where the random variables ζ_1, ζ_2, ζ_3 *are independent,* $\zeta_1 \sim \psi(m-n+2, 2)$, $\zeta_2, \zeta_3 \sim \chi^2(m-n+2)$.

Proof. From the representation (1) it can be seen that

$$w^{i,j} = \frac{(-1)^{j-i}\det \mathbf{W}(\{i\},\{j\})}{\det \mathbf{W}} = \frac{(-1)^{j-i}}{\sqrt{\tau_i\tau_j}}\frac{\det \mathbf{V}(\{i\},\{j\})}{\det \mathbf{V}} = \frac{\nu^{i,j}}{\sqrt{\tau_i\tau_j}},$$

where $\nu^{i,j}$ is the (i,j) element of the matrix $\mathbf{V}^{-1}$. Now applying Corollary 3.2 and Proposition 3.1, we complete the proof. $\square$

The next Proposition (see [4], [6]) follows from Proposition 2.1, the representation (1) and equalities (3)-(5).

Proposition 3.2 *Let* $\xi = (\xi_{i,j})$ *be a random symmetric matrix of order n. Suppose that* $\xi_{i,j}$, $1 \le i \le j \le n$ *are independent,* $\xi_{i,j} \sim \psi(m-i+1, 2)$ *for* $1 \le i < j \le n$ *and* $\xi_{i,i} \sim \chi^2(m)$, $i = 1, \dots, n$. *Let* $\mathbf{W}$ *be the matrix* $\mathbf{W} = h(\xi)$, *where h is the bijection, defined by (3)-(5). Then the matrix* $\mathbf{W}$ *has distribution* $W_n(m, \mathrm{I}_n)$.

Let $\mathbf{W} \sim W_n(m, \mathrm{I}_n)$. From Proposition 3.2 we have that $\mathbf{W} = h(\xi)$, where $\xi = (\xi_{i,j})$ is a random symmetric matrix, $\xi_{i,j} \sim \psi(m-i+1, 2)$ for $1 \le i < j \le n$ and $\xi_{i,i} \sim \chi^2(m)$, $i = 1, \dots, n$. From equation (8) for $i = n-1$ and $j = n$, we get that

$$\det \mathbf{W} = \xi_{1,1} \dots \xi_{n,n} \left(\prod_{1 \le k < s \le n} (1 - \xi_{k,s}^2) \right). \tag{14}$$

Using (3) we obtain that

$$tr\mathbf{W} = \xi_{1,1} + \cdots + \xi_{n,n}. \tag{15}$$

The Wishart distribution arises frequently in multivariate statistical analysis. In an example below we show an application of the obtained representations.

Example 3.1 *Let $\mathbf{x}_i = (x_{i,1}, \ldots x_{i,n})^t$, $i = 1, \ldots, m$ be a random sample of size m $(m > n)$ from n - variate normal distribution with unknown mean vector μ and unknown positive definite covariance matrix Σ. We are interested in testing the null hypothesis $H_0 : \ \Sigma = \Sigma_0$ against the alternatives $H_1 : \ \Sigma \neq \Sigma_0$, where Σ_0 is a fixed positive definite matrix. By a linear transformation of the observations (see [2]), we can reduce the task to testing the hypotheses $H_0 : \ \Sigma = \mathrm{I}_n$ against $H_1 : \ \Sigma \neq \mathrm{I}_n$. Let us denote the transformed observations by $\mathbf{y}_i$, $i = 1, \ldots, m$. The likelihood ratio criterion for testing $H_0 : \ \Sigma = \mathrm{I}_n$ is given by (see [2])*

$$\lambda = \left(\frac{e}{m}\right)^{\frac{mn}{2}} (\det \mathbf{S})^{\frac{m}{2}} \, e^{-\frac{1}{2} tr\mathbf{S}}, \tag{16}$$

where $\mathbf{S} = \sum_{i=1}^{n} (\mathbf{y}_i - \bar{\mathbf{y}})(\mathbf{y}_i - \bar{\mathbf{y}})^t$, $\bar{\mathbf{y}} = \frac{1}{m} \sum_{i=1}^{m} \mathbf{y}_i$. Under H_0, the distribution of the sample covariance matrix $\mathbf{S}$ is $W_n(m-1, \mathrm{I}_n)$. Using (14) and (15), λ can be written in the form

$$\lambda = \left(\frac{e}{m}\right)^{\frac{mn}{2}} \left(\prod_{1 \le k < s \le n} (1 - \xi_{k,s}^2)\right)^{\frac{m}{2}} \left(\prod_{k=1}^{n} \xi_{k,k}^{\frac{m}{2}} \, e^{-\frac{1}{2}\xi_{k,k}}\right),$$

where $\xi_{k,s}$, $1 \le k \le s \le n$ are independent, $\xi_{k,s} \sim \psi(m-k, 2)$ for $k \neq s$ and $\xi_{k,k} \sim \chi^2(m-1)$. From Proposition 2.1 and equality (8) we have that if $\mathbf{V} \sim \psi_n(m-1)$ then

$$\det \mathbf{V} = \prod_{1 \le k < s \le n} (1 - \xi_{k,s}^2).$$

From (10) it follows that

$$\prod_{1 \le k < s \le n} (1 - \xi_{k,s}^2) \sim \zeta_1 \ldots \zeta_{n-1},$$

where ζ_i, $i = 1, \ldots, n-1$ are independent and $\zeta_i \sim \beta\left(\frac{m-i-1}{2}, \frac{i}{2}\right)$. Consequently,

$$\lambda \sim \left(\frac{e}{m}\right)^{\frac{mn}{2}} (\zeta_1 \ldots \zeta_{n-1})^{\frac{m}{2}} \, \nu_1 \ldots \nu_n, \tag{17}$$

where ν_i, $i = 1, \dots, n$ *are mutually independent, independent of* ζ_i, $i = 1, \dots, n-1$ *and are identically distributed,*

$$\nu_i \sim \xi^{\frac{m}{2}} e^{-\frac{1}{2}\xi}, \quad \xi \sim \chi^2(m-1).$$

The relation (17) is an exact representation of λ *as a product of independent random variables. Using (16), for simulating of each value of* λ *we have to generate a* $n \times n$ *covariance matrix and calculate its determinant. With the representation (17), we get each value of* λ *by* $2n-1$ *independent realizations of chi-square and beta random variables.*

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