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Generalized and Perturbed Elasticity System

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Abstract

In this work, we study the existence, the uniqueness and the regularity of the solution for some boundary value problems gouverned by perturbed and generalized dynamical Elasticity system.

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1. Notations.- Ω is a bounded and connected open set of $\mathbb{R}^n$ $(n = 2, 3)$ with boundary $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, a lipschitzian manifold of dimension $n-1$, were $\Gamma_i \subset \Gamma$, $i = 1, 2$, with $\text{mes}(\Gamma_1) > 0$ and $\Gamma_1 \cap \Gamma_2 = \phi$.

2. Position of the Problem.- We consider firstly the mathematical model of the perturbed elasticity system :

$$-L_p u + F(u),$$

were $F(u)$ is the perturbation and

$$L_p u = \mu \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) + (\lambda + \mu) \nabla (div(u)),$$

p, q are two real numbers such that $p \in]1, \infty[$ and $\frac{1}{p} + \frac{1}{q} = 1$,

λ and μ are the elasticity coefficients subjected to the constraint $\lambda + \mu \geq 0$ and $\lambda > 0$,

ν denotes the outgoing normal vector to Γ_2.

For $p=2$, we recover the classical dynamical Lamé system.

Given f and $\varphi=(\varphi_{i,j})_{1<i,j<n}$,such that $\varphi_{i,j}=\varphi_{j,i}\in C^{0,1}(\overline{\Omega})$ and $\varphi_{i,j}(x)>0, \forall\ x\in\Gamma_2$.We study the existence, the uniqueness and the regularity of the complex-valued solutions $u=u(x)$, $x\in\Omega,$for the following problems :

$$(p_1)\begin{cases} -L_p u+\ |u|^{P-2}\,u=f,\ \text{in }\Omega & (1,1)\\ u=0,\ \text{on }\Gamma_1 & (1,2)\\ \sum_{i=1}^{n}\left|\dfrac{\partial u}{\partial x_i}\right|^{p-2}\dfrac{\partial u}{\partial x_i}\eta_i+\varphi(x)\,|u|^{p-2}\,u=0,\ \text{on }\Gamma_2 & (1,3)\end{cases}$$

$$(p_2)\begin{cases} -L_p u+|u|^{P-2}\,u\ -|u|^{q-1}\,u=f,\ \text{in }\Omega & (2,1)\\ u=0,\ \text{on }\Gamma_1 & (2,2)\\ \sum_{i=1}^{n}\left|\dfrac{\partial u}{\partial x_i}\right|^{p-2}\dfrac{\partial u}{\partial x_i}\eta_i+\varphi(x)\,|u|^{p-2}\,u=0,\ \text{on }\Gamma_2 & (2,3)\end{cases}$$

Here $\sigma(u)=(\sigma_{ij}(u))_{1<i,j<n}$ is the matrix of the constraints tensor $\sigma_{ij}(u)=\lambda div(u)\delta_{ij}+2\mu\ \varepsilon_{ij}(u)$, were $\varepsilon_{ij}(u)=\frac{1}{2}(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}), 1\leq i,j\leq n$, are the components of the deformation tensor.

In this work, we consider the cas when Γ_1 , $\Gamma_2\neq\phi$ and $\varphi(x)\neq 0$ on Γ_2.

Remark 2.1.- The space $V=(H_0^1(\Omega))^n\cap(L^p(\Omega))^n$, were $p=\rho+2,$is separable (i.e. admits a countable dense subset).

In fact, V is identified, by the application $v\rightarrow\left\{v,\frac{\partial v}{\partial x_1},\frac{\partial v}{\partial x_2},...,\frac{\partial v}{\partial x_n}\right\}$, to a closed subspace of

$(L^p(\Omega))^n\times(L^2(\Omega))^n\times...\times(L^2(\Omega))^n$,separable and uniformly convex, in such way that it possible to project a countable dense set on this subspace.

3. stady of the problem (p_1)

The main result is

Theorem 3.1.- We suppose that

$$f\in(W^{-1,q}(\Omega))^n).$$

Then, there exist a function $u=u(x)$ solution of the problem (P_1) with :

$$u\in(W^{1,p}(\Omega))^n\,),$$

Before giving the proof, we make the following remarks :

Remarque 3.1.- The application defined on $(L^p(\Omega))^n$ by $u\longrightarrow|u|^{p-2}\,u$, is $(L^q(\Omega))^n$-valued, moreover it is continuous. To see that, if

$u\in(L^p(\Omega))^n$, $|u|^{p-2}\,u$ est mesurable and

$$\int_\Omega \left||u|^{p-2}u\right|^q dx = \int_\Omega |u|^p dx < \infty \Longrightarrow u \in (L^q(\Omega))^n.$$

We deduce that $\forall u \in (W^{1,p}(\Omega))^n$, $\forall\ i$, $1 \le i \le n$,

$$\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i} \in (L^q(\Omega))^n.$$

So, it is possible to define the real-valued application :

$$\left((W^{1,p}(\Omega))^n\right)^2 \longrightarrow \mathbb{R},\ (u,v) \longmapsto a_p(u,v).$$

$$\begin{aligned} a_p(u,v) \quad = \quad & = \mu \sum_{i=1}^n \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\ dx + (\lambda+\mu)\int_\Omega (divu)\,(divv)\ dx \\ & + \mu \int_{\Gamma_2} \varphi(x)\,|u|^{p-2}\, u\gamma_0(v)\ \ \eta_i\ d\sigma - (\lambda+\mu)\int_{\Gamma_2} \gamma_0(divu)\,\gamma_0(v)\ \ \eta_i\ d\sigma \end{aligned}$$

For any u in $(W^{1,p}(\Omega))^n$, the application $(W^{1,p}(\Omega))^n \longrightarrow \mathbb{R}$, $v \longrightarrow a_p(u,v)$, is a continuous linear form. then *c.f.* [5] there exist a unique element $A_p(u)$ of $(W^{-1,q}(\Omega))^n$, such that

$$a(u,v)_p = \langle A_p(u), v\rangle\,,\ \forall v \in (W_0^{1,p}(\Omega))^n.$$

and we define the real-valued application :

$$(L^p(\Omega))^n \times (L^p(\Omega))^n \to IR$$

$$a_1(u,v) = \int_\Omega |u|^{p-2}\, uvdx$$

For any u in $(L^p(\Omega))^n$, the application $(L^p(\Omega))^n \longrightarrow \mathbb{R}$, $v \longrightarrow a_p(u,v)$, is a continuous linear form. then *c.f.* [5] there exist a unique element $A_1(u)$ of $(L^q(\Omega))^n$, such that

$$a_1(u,v) = \langle A_1(u), v\rangle\,, \forall v \in (L^p(\Omega))^n$$

The following proposition gives some properties of A_p and A_1 :

Proposition 2.1.- The operators A_p and A_1 are boundeds, hemicontinuous and monotones.

Demonstration: Using the expression of the norm in dual space and Lebesgue's dominated convergence theorem, we prove that A_p and A_1 are boundeds and hemicontinuous. From the convexity of the real application $t \longrightarrow |t|^p$, we deduce the monotonicity of A_p and A_1.

Proposition 2.2.- The operator $A : (W^{1,p}(\Omega))^n \longrightarrow (W^{-1,q}(\Omega))^n, A\,(u) = A_p\,(u) + A_1\,(u)$ is coercitive.

Proposition 2.3.- The problem (P_1) and the variational problem $(P_1.V)$:

$$a_p(u,v) + a_1(u,v) = (f,v)\,,\ \forall v \in (W^{1,p}(\Omega))^n,$$

are equivalent.

Demonstration: Indeed, it suffices to observe the variationnal equality is then equivalent to

$$-L_p u + |u|^{p-2}\, u = f \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \Gamma_1$$
$$\sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\eta_i + \varphi\,(x)\,|u|^{p-2}\,u = 0, \text{ on } \Gamma_2$$

because $(D(\Omega))^n$ is dense in $(W_0^{1,p}(\Omega))^n$.

Let us return to the demonstration of **Theorème3.1.**

(i) Construction of approximated solutions :

We look for $u_m = \sum_{i=1}^{n} \lambda_i v_i$ solution of the following problem $(P_{1,m})$:
$\forall j,\ 1 \leq j \leq m$:

$$a_p(u_m, v_j) + a_1(u_m, v_j) = (f, v_j)$$

We obtain a second order nonlinear differential system. Let be the function

$$F : \mathbb{R}^m \longrightarrow \mathbb{R}^m$$
$$F(\lambda_1, ..., \lambda_m) = \left(\left\langle A(\sum_{i=1}^{n} \lambda_i v_i), v_j\right\rangle - ((f, v_j) + (-\varphi(x)(u_m, v_j)).\right)_{1 \leq j \leq m}$$

(ii) Establishment of priori estimates.-

- Of the coercivité of to one deducts that $||u_m||$ is a bounded;
- The operator has a bounded $\Longrightarrow (A(u_m))_{\ m\in\mathbb{N}}$ is a bounded in $V\prime$;
- $\exists u \in V, \exists \chi \in V' \Longrightarrow \begin{cases} u_p \rightharpoonup u, \ \sigma(V,V'), \\ A(u_m) \rightharpoonup \chi, \ \sigma(V',V). \end{cases}$

(iii) Passage to the limit via compactness.

- The monotony and the hemicontinuous $\Longrightarrow \chi = A(u)$.

What finishes the demonstration of the **Theorem 3.1.**

Theorem 3.2.

The solution of the problem (P_1) is unique.

demonstration : it suffieces to observe that $\sum\limits_{i=1}^{n}\int\limits_{\Omega}\left(\left|\frac{\partial u_1}{\partial x_i}\right|^{p-2}\frac{\partial u_1}{\partial x_i}-\left|\frac{\partial u_2}{\partial x_i}\right|^{p-2}\frac{\partial u_2}{\partial x_i}\right)\left(\frac{\partial u_1}{\partial x_i}-\frac{\partial u_2}{\partial x_i}\right)dx = \|u_1-u_2\|^p, \forall u_1,u_2 \in (W^{1,p}(\Omega))^n$.

4. stady of the problem (p_2)

The main result is

Theorem 4.1.- We suppose that

$$f \in (W^{-1,q}(\Omega))^n).$$

Then, there exist a function $u = u(x)$ solution of the problem (P_2) with :

$$u \in (W^{1,p}(\Omega))^n\ ),$$

Before giving the proof, we make the following remarks :

Remarque 3.1.-

We define the real-valued applications :

$$\left((W^{1,p}(\Omega))^n\right)^2 \longrightarrow \mathbb{R},\ (u,v) \longmapsto a_p(u,v).$$

$$\begin{aligned} d_p(u,v) &= \mu\sum_{i=1}^{n}\int\limits_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\ dx + (\lambda+\mu)\int\limits_{\Omega}(divu)(divv)\ dx \\ &+\mu\int\limits_{\Gamma_2}\varphi(x)\,|u|^{p-2}\,u\gamma_0(v)\ \ \eta_i\ d\sigma - (\lambda+\mu)\int\limits_{\Gamma_2}\gamma_0(divu)\,\gamma_0(v)\ \ \eta_i\ d\sigma \end{aligned}$$

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For any u in $(W^{1,p}(\Omega))^n$, the application $(W^{1,p}(\Omega))^n \longrightarrow \mathbb{R},\ v \longrightarrow a_p(u,v)$, is a continuous linear form. then *c.f.* [5] there exist a unique element $D_p(u)$ of $(W^{-1,q}(\Omega))^n$, such that

$$d(u,v)_p = \langle D_p(u), v\rangle\,, \forall v \in (W_0^{1,p}(\Omega))^n.$$

and :

$$(L^p(\Omega))^n \times (L^p(\Omega))^n \rightarrow IR$$

$$\begin{array}{rcl} a_1(u,v) & = & \int_\Omega |u|^{p-2}\, uvdx = \langle A_1(u), v\rangle\,, \forall v \in (L^p(\Omega))^n \\ \text{such that } A_1(u) & \in & (L^q(\Omega))^n \end{array}$$

and :

$$(L^p(\Omega))^n \times (L^p(\Omega))^n \rightarrow IR$$

$$\begin{array}{rcl} a_2(u,v) & = & \int_\Omega |u|^{q-1}\, uvdx = \langle A_2(u), v\rangle\,, \forall v \in (L^p(\Omega))^n \\ \text{such that } A_2(u) & \in & (L^q(\Omega))^n \end{array}$$

The following proposition gives some properties of D_p, A_1 and A_2 :

Proposition 2.1.- The operators D_p, A_1 and A_2 are boundeds, hemicontinuous and monotones.

Demonstration: Using the expression of the norm in dual space and Lebesgue's dominated convergence theorem, we prove that D_p, A_1 and A_2 are boundeds and hemicontinuous. From the convexity of the real application $t \longrightarrow |t|^p$, we deduce the monotonicity of D_p, A_1 and A_2.

Proposition 2.2.- The operator $D : (W^{1,p}(\Omega))^n \longrightarrow (W^{-1,q}(\Omega))^n, A\,(u) = D_p\,(u) + A_1\,(u) + A_2\,(u)$ is coercitive.

Proposition 2.3.- The problem (P_2) and the variational problem $(P_2.V)$:

$$d_p(u,v) + a_1(u,v) + a_2(u,v) = (f,v),\ \forall v \in (W^{1,p}(\Omega))^n,$$

are equivalent.

Demonstration: Indeed, it suffices to observe the variationnal equality is then equivalent to

$$-L_p u + |u|^{P-2} u \; - |u|^{q-1} u = f, \text{ in } \Omega$$
$$u = 0, \text{ on } \Gamma_1$$
$$\sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \eta_i + \varphi(x) |u|^{p-2} u = 0, \text{ on } \Gamma_2$$

because $(D(\Omega))^n$ is dense in $(W_0^{1,p}(\Omega))^n$.
Let us return to the demonstration of **Theorème4.1.**

(i) Construction of approximated solutions :
We look for $u_m = \sum_{i=1}^{n} \lambda_i v_i$ solution of the following problem $(P_{2,m})$:
$\forall j,\ 1 \leq j \leq m$:

$$d_p(u_m, v_j) + a_1(u_m, v_j) + a_2(u_m, v_j) = (f, v_j)$$

We obtain a second order nonlinear differential system. Let be the function

$$F : \mathbb{R}^m \longrightarrow \mathbb{R}^m$$
$$F(\lambda_1, ..., \lambda_m) = \left(\left\langle A(\sum_{i=1}^{n} \lambda_i v_i), v_j \right\rangle - ((f, v_j) + (-\varphi(x)(u_m, v_j)). \right)_{1 \leq j \leq m}$$

(ii) Establishment of priori estimates.-
- Of the coercivité of to one deducts that $||u_m||$ is a bounded;
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(iii) Passage to the limit via compactness.
- The monotony and the hemicontinuous $\Longrightarrow \chi = D(u)$.
What finishes the demonstration of the **Theorem 4.1.**

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