

On a Class of Triangle Interpolation Operators of Revised Bernstein Type

Xuegang Yuan

School of Science, Dalian Nationalities University
Dalian 116600, Liaoning, P.R. China
yxg1971@163.com

Abstract

In this paper, to improve the uniform convergence of the known Lagrange interpolation polynomials, a new triangle interpolation operator of Bernstein type is constructed by using the method of two revised nodes. It is proved that the constructed operator converges uniformly to arbitrary continuous functions with period on the whole axis. The best approximation order of the operator is then obtained. Finally, that the highest convergence order of the operator cannot exceed is proved. The problem put forward by Bernstein is answered satisfactorily.

Mathematics Subject Classification: 41A28

Keywords: triangle interpolation operator of Bernstein type; two revised nodes; uniform convergence; the best approximation order

1. Introduction

Let $C_{2\pi}^j$, ($0 \leq j \leq r$) be a space with continuous functions, in which all the elements are of period 2π and have continuous derivatives of order j . Denote

$$\theta_k^{(n)} = \frac{2k+1}{N}\pi, \quad k = 0, 1, 2, \dots, 2n \quad (1)$$

by a set of equidistant nodes, where $N = 2n + 1$. The triangle interpolation polynomial of Lagrange type of $f(\theta) \in C_{2\pi}$, which is equal to $f(\theta_k^{(n)})$ at the equidistant nodes $\theta_k^{(n)}$ ($k = 0, 1, \dots, 2n$), is given by

$$L_n(f; \theta) = \sum_{k=0}^{2n} f(\theta_k^{(n)}) \phi_k^{(n)}(\theta), \quad (2)$$

where

$$\phi_k^{(n)}(\theta) = \frac{\sin \frac{N}{2}(\theta - \theta_k^{(n)})}{N \sin \frac{1}{2}(\theta - \theta_k^{(n)})}, \quad k = 0, 1, 2, \dots, 2n \tag{3}$$

are the basic Lagrange interpolation functions based on the interpolation nodes $\theta_k^{(n)}$.

Since the Lagrange interpolation polynomials do not converge uniformly to arbitrary continuous functions, and so, for many scholars, one of the interesting works is how to improve the convergence and the convergence order of the interpolation polynomials. Bernstein S.N., a Soviet Russia mathematician, put forward the following question: for any given continuous functions $f(x)$ and for any λ , ($1 < \lambda < 2$), whether there exists an interpolation polynomial with the degree $M(< \lambda N)$, such that it is equal to $f(x)$ at the given N points and converges to $f(x)$ uniformly as $N \rightarrow \infty$. In 1932, he constructed an operator $Q_n(f; x)^{[1]}$ and answered the question what he had put forward, moreover, he has proved that the operator converges uniformly to arbitrary continuous functions. Further references on this aspect are [2~4].

In this paper, a new triangle interpolation operator of Bernstein type is constructed by using the method of two revised nodes, and the problem put forward by Bernstein is answered satisfactorily.

The operator $G_n(f; r, \theta)$ is constructed as follows:

For the given even number $2l$, the nodes $\theta_0^{(n)}, \theta_1^{(n)}, \dots, \theta_{2n}^{(n)}$ are divided into s groups by $2l$, namely, $2n + 1 = 2ls + q$, ($0 \leq q < 2l$), where l and s are positive numbers and $l \geq 2$. Furthermore, at the nodes $2lt - 1$, ($t = 1, 2, \dots, s$) and $2lt$ of each group, let

$$B_{2lt-1} = f(\theta_{2lt-1}^{(n)}) + \sum_{p=1}^l \Delta_h^{r+1} f(\theta_{2l(t-1)+2p-1}^{(n)}) = f(\theta_{2lt-1}^{(n)}) + B_{2lt-1}^*, \tag{4}$$

$$B_{2lt} = f(\theta_{2lt}^{(n)}) + \sum_{p=1}^l \Delta_h^{r+1} f(\theta_{2l(t-1)+2p}^{(n)}) = f(\theta_{2lt}^{(n)}) + B_{2lt}^*, \tag{5}$$

where

$$\Delta_h^{r+1} f(\theta_k^{(n)}) = \frac{1}{2^{r+1}} \sum_{i=0}^{r+1} (-1)^{\alpha+1+i} \binom{r+1}{i} f(\theta_{k+i+\alpha}^{(n)}) \tag{6}$$

and $\alpha = \lceil \frac{r+1}{2} \rceil$, $[a]$ is the integer part of a , $\binom{r+1}{i} = \frac{(r+1)!}{i!(r+1-i)!}$, $t = 1, 2, \dots, s$,

$\Delta_h^{r+1} f(\theta_k^{(n)})$ is the $r + 1$ -th difference of the function $f(\theta)$ with step $h = \frac{2\pi}{2n+1}$ at the node $\theta_k^{(n)}$.

So the expression of $G_n(f; r, \theta)$ is given by

$$G_n(f; r, \theta) = \sum_{k=0}^{2n} B_k \phi_k^{(n)}(\theta), \tag{7}$$

where B_k is given by Eq.(4) as $k = 2lt - 1 (t = 1, 2, \dots, s)$ and is given by Eq.(5) as $k = 2lt$, otherwise, $B_k = f(\theta_k^{(n)})$.

2. Main Results and their Proofs

For the operator $G_n(f; r, \theta)$, the following results are valid.

Theorem 1 For any $f(\theta) \in C_{2\pi}$, the following limit expression is valid.

$$\lim_{n \rightarrow \infty} G_n(f; r, \theta) = f(\theta). \tag{8}$$

Theorem 2 For any $f(\theta) \in C_{2\pi}^j, (0 \leq j \leq r)$, we have

$$|G_n(f; r, \theta) - f(\theta)| = O\left(\frac{1}{n^j} \omega(f^{(j)}, \frac{1}{n})\right), \tag{9}$$

where O means

$$\lim_{n \rightarrow \infty} \frac{|G_n(f; r, \theta) - f(\theta)|}{\frac{1}{n^j} \omega(f^{(j)}, \frac{1}{n})} = C \neq 0$$

and is independent of $n, \theta, f, f', \dots, f^{(j)}, \omega(f^{(j)}, \delta)$ is the modulus of continuity of $f^{(j)}(x)$.

Theorem 3 For arbitrary functions with any derivatives, the highest convergence order of $G_n(f; r, \theta)$ can not exceed $1/n^{r+1}$.

Obviously, if Theorem 2 is proved to be valid, it is easy to see that Theorem 1 is also valid.

Proof of Theorem 2

Noting that $2n + 1 = 2ls + q, 0 \leq q < 2l$, using Eqs.(4), (5) and some complicate reductions, we have

$$\begin{aligned} G_n(f; r, \theta) - f(\theta) &= \sum_{k=0}^{2n} B_k \phi_k^{(n)}(\theta) - f(\theta) \\ &= \left(\sum_{k=0}^{2n} f(\theta_k^{(n)}) \phi_k^{(n)}(\theta) - f(\theta) \right) + \sum_{t=1}^s \left(B_{2lt-1}^* \phi_{2lt-1}^{(n)}(\theta) + B_{2lt}^* \phi_{2lt}^{(n)}(\theta) \right) \\ &= \sum_{v=1}^4 e_v, \end{aligned} \tag{10}$$

where

$$e_1 = \sum_{k=0}^{2n} \left(f(\theta_k^{(n)}) + \Delta_h^{r+1} f(\theta_k^{(n)}) \right) \phi_k^{(n)}(\theta) - f(\theta), \tag{11}$$

$$e_2 = \sum_{t=1}^s \sum_{p=1}^{l-1} \Delta_h^{r+1} f(\theta_{2l(t-1)+2p-1}^{(n)}) \left(\phi_{2lt-1}^{(n)}(\theta) - \phi_{2l(t-1)+2p-1}^{(n)}(\theta) \right), \tag{12}$$

$$e_3 = \sum_{t=1}^s \sum_{p=1}^{l-1} \Delta_h^{r+1} f(\theta_{2l(t-1)+2p}^{(n)}) \left(\phi_{2lt}^{(n)}(\theta) - \phi_{2l(t-1)+2p}^{(n)}(\theta) \right), \tag{13}$$

$$e_4 = (-1) \sum_{p=2ls+1}^{n+1} \Delta_h^{r+1} f(\theta_p^{(n)}) \phi_p^{(n)}(\theta). \tag{14}$$

We first consider e_1 , namely, Eq.(11).

Let P_n is a set composed of triangle polynomials, whose degree does not greater than n . For any $f(\theta) \in C_{2\pi}^j$, ($0 \leq j \leq r$), there exists a triangle polynomial $p_n(\theta)$ of degree n in P_n such that

$$|p_n(\theta) - f(\theta)| = O\left(\frac{1}{n^j} \omega(f^{(j)}, \frac{1}{n}\right)^{[5]}. \tag{15}$$

From the properties of Lagrange interpolation polynomial, we know that

$$\sum_{k=0}^{2n} \left(p_n(\theta_k^{(n)}) + \Delta_h^{r+1} p_n(\theta_k^{(n)}) \right) \phi_k^{(n)}(\theta) = p_n(\theta) + \Delta_h^{r+1} p_n(\theta), \tag{16}$$

and so

$$\begin{aligned} e_1 &= \sum_{k=0}^{2n} \left(f(\theta_k^{(n)}) + \Delta_h^{r+1} f(\theta_k^{(n)}) \right) \phi_k^{(n)}(\theta) - f(\theta) \\ &= \sum_{k=0}^{2n} \left(f(\theta_k^{(n)}) - p_n(\theta_k^{(n)}) \right) \left(\frac{1}{2^{r+1}} \sum_{i=0}^{r+1} \binom{r+1}{i} \left(\phi_k^{(n)}(\theta) + (-1)^{\alpha+1+i} \phi_{k+i-\alpha}^{(n)}(\theta) \right) \right) \\ &\quad + \left\{ (p_n(\theta) - f(\theta)) + \left(\Delta_h^{r+1} p_n(\theta) - \Delta_h^{r+1} f(\theta) \right) \right\} + \Delta_h^{r+1} f(\theta) \\ &= d_1 + d_2 + d_3. \end{aligned} \tag{17}$$

For d_1 , the following expression is valid by using the method in [2], i.e.,

$$\sum_{k=0}^{2n} \left(\frac{1}{2^{r+1}} \sum_{i=0}^{r+1} \binom{r+1}{i} \left| \phi_k^{(n)}(\theta) + (-1)^{\alpha+1+i} \phi_{k+i-\alpha}^{(n)}(\theta) \right| \right) = O(1). \tag{18}$$

Furthermore, from Eq.(12), we have

$$d_1 = O\left(\frac{1}{n^j} \omega(f^{(j)}, \frac{1}{n}\right), d_2 = O\left(\frac{1}{n^j} \omega(f^{(j)}, \frac{1}{n}\right). \tag{19}$$

For any $f(\theta) \in C_{2\pi}^j$, ($0 \leq j \leq r$), using the relation between derivative and equidistant difference, we obtain

$$\Delta_h^j f(\theta) = - \left(-\frac{2\pi}{2n+1} \right)^j f^{(j)}(\xi_\theta), \tag{20}$$

where ξ_θ lies between $\theta - (1 - s)h$ and $\theta - (1 + j - s)h$. Thus, for d_3 , using the similar method in [8] (i.e., Eq.(39)), we have

$$d_3 = O\left(\frac{1}{n^j}\omega(f^{(j)}, \frac{1}{n})\right). \quad (21)$$

On combination of d_1, d_2, d_3 , it leads to

$$e_1 = O\left(\frac{1}{n^j}\omega(f^{(j)}, \frac{1}{n})\right). \quad (22)$$

For e_2, e_3, e_4 , using the similar method in [4], it is easy to show that

$$\sum_{t=1}^s \sum_{p=1}^{l-1} |\phi_q^{(n)}(\theta) - \phi_j^{(n)}(\theta)| = O(1), \quad \sum_{t=1}^s \sum_{p=1}^{l-1} |\phi_{q+1}^{(n)}(\theta) - \phi_{j+1}^{(n)}(\theta)| = O(1), \quad (23)$$

where $q = 2lt - 1, j = 2l(t - 1) + 2p - 1$. Again using Eq.(12), we get

$$e_v = O\left(\frac{1}{n^j}\omega(f^{(j)}, \frac{1}{n})\right), \quad v = 2, 3, 4. \quad (24)$$

Consequently, Theorem 2 is proved.

Proof of Theorem 3

In fact, from the properties of Lagrange interpolation polynomial we have

$$G_n(f; r, \theta_{2lt}^{(n)}) = B_{2lt}^*, \quad t = 1, 2, \dots, s.$$

Taking $f_0(\theta) = \cos\theta$ and $l = 1$, we obtain

$$\begin{aligned} |G_n(f_0; r, \theta_{2t}^{(n)}) - f_0(\theta_{2t}^{(n)})| &= |B_{2t}^*| = |\Delta_h^{r+1} \cos \theta_{2t}^{(n)}| \\ &= \left(\sin \frac{\pi}{2n+1}\right)^{r+1} \cdot \left|\sin\left(\theta_{2t}^{(n)} + \frac{\pi}{2n+1}\right)\right|. \end{aligned}$$

This means that Theorem 3 is valid as n is sufficiently large.

References

- [1] S.N. Bernstein. On a class of modifying Lagrange interpolation formula. Acad Nauk; S.N.Bernstein collect works, 2(1954) 130-140 (in Russian).
- [2] J.X. He, On an interpolation polynomial of S.N.Berstein Type, Acta Math. Hungar., 70(1996) 293-303.
- [3] X.G. Yuan, D.H. Wang, Approximation to Continuous Functions by a Kind of Interpolation Operator, Northeast Math. J., 17(2001) 39-44
- [4] X.G. Yuan, Y.B. Chang, On linear combination of two classes of triangle summation operators of Bernstein type, Applied Mathematical Science, 1 (2007) 1571-1580
- [5] W. Cheney, W. Light, A course in approximation theory, Thomson Learning, 2000.

Received: April 11, 2008