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Bayesian and Non-Bayesian Estimations on the Exponentiated Gamma Distribution

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Abstract

In this paper, we consider the exponentiated gamma distribution as an important model of life time models and derive Bayesian and non-Bayesian estimators of the shape parameter, reliability and failure rate functions in the case of complete and type-II censored samples. The mean square errors of the estimates are computed. Comparisons are made between these estimators using Monte Carlo simulation study.

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1. Introduction

The exponentiated gamma (EG) distribution was introduced by Gupta et al. (1998) which has a probability density function (p.d.f.) of the form

$$f(t) = \theta\, t\, e^{-t}[1-e^{-t}(t+1)]^{\theta-1}, \qquad t>0,\ \theta>0, \tag{1.1}$$

and a cumulative distribution function (c.d.f.)

$$F(t) = [1-e^{-t}(t+1)]^{\theta}, \qquad t>0,\ \theta>0, \tag{1.2}$$

where θ is the shape parameter. It is important to mention that when $\theta=1$ the EG p.d.f. is that of gamma distribution with shape parameter $\alpha = 2$ and scale parameter

$\beta = 1$ i.e. G(2,1). For more details about this distribution, see Shawky and Bakoban (2008).

The reliability and failure rate functions of EG distribution are given, respectively, by

$$R(t) = 1 - [1 - e^{-t}(t+1)]^{\theta}, \qquad t > 0, \theta > 0, \tag{1.3}$$

$$h(t) = \theta\, t\, e^{-t}[1 - e^{-t}(t+1)]^{\theta - 1}\{1 - [1 - e^{-t}(t+1)]^{\theta}\}^{-1}, \qquad t > 0, \theta > 0. \tag{1.4}$$

Bayesian and non-Bayesian estimators were derived for the shape parameter, reliability and failure rate functions under complete and type-II censored samples from EG distribution.

In Bayesian estimation, we consider two types of loss functions. The first is the squared error loss function (quadratic loss) which is classified as a symmetric function and associates equal importance to the losses for overestimation and underestimation of equal magnitude. The second is the LINEX (linear-exponential) loss function which is asymmetric, was introduced by Varian (1975). These loss functions were widely used by several authors; among of them Rojo (1987), Basu and Ebrahimi (1991), Pandey (1997), Soliman (2000) and Nassar and Eissa (2004).

The quadratic loss for Bayes estimate of a parameter β, say, is the posterior mean assuming that exists, denoted by β_S. The LINEX loss function may be expressed as

$$l(\Delta) \propto e^{c\Delta} - c\Delta - 1, \; c \neq 0, \tag{1.5}$$

where $\Delta = \hat{\beta} - \beta$. The sign and magnitude of the shape parameter c reflects the direction and degree of asymmetry, respectively. (If $c > 0$, the overestimation is more serious than underestimation, and vice-versa). For c closed to zero, the LINEX loss is approximately squared error loss and therefore almost symmetric.

The posterior expectation of the LINEX loss function Equation (1.5) is

$$E_{\beta}[l(\hat{\beta} - \beta)] \propto \exp(c\hat{\beta})E_{\beta}[\exp(-c\beta)] - c(\hat{\beta} - E_{\beta}(\beta)) - 1, \tag{1.6}$$

where $E_{\beta}(.)$ denoting posterior expectation with respect to the posterior density of β. By a result of Zellner (1986), the (unique) Bayes estimator of β, denoted by $\hat{\beta}_L$ under the LINEX loss is the value $\hat{\beta}$ which minimizes (1.6), is given by

$$\hat{\beta}_L = -\frac{1}{c}\ln\{E_{\beta}[\exp(-c\beta)]\}. \tag{1.7}$$

provided that the expectation $E_{\beta}[\exp(-c\beta)]$ exists and is finite [Calabria and Pulcini (1996)].

We were interested with maximum likelihood estimation as a classical approach among non-Bayesian methods. The maximum likelihood is based on the information provided by empirical data. The invariant property was hold to obtain maximum likelihood estimators (MLE's) of reliability and failure rate functions.

In this paper, a discussion of the MLE's is considered in section 2. In section 3, Bayesian estimators is obtained. Finally, numerical illustration and comparisons are presented in section 4.

2. Maximum Likelihood Estimation

Suppose a type-II censored sample $\underline{t} = (t_1, t_2, ..., t_r)$ where t_i is the i^{th} order statistics. This sample are obtained and recorded from EG(θ) distribution with p.d.f. and c.d.f. given, respectively, by (1.1) and (1.2). The likelihood function (LF) in this case can be written as:

$$\ell(\underline{t};\theta) \propto \theta^r e^{-T} (1 - V^{\theta})^{n-r}, \tag{2.1}$$

where

$$T = \sum_{i=1}^{r} [t_i - \ln t_i - (\theta - 1) \ln u_i],$$

$$u_i = 1 - e^{-t_i} (1 + t_i), \text{ and } V = 1 - e^{-t_r} (1 + t_r).$$

The logarithm of the LF is given by

$$L = \ln \ell(\underline{t};\theta) \propto r \ln \theta - T + (n-r) \ln (1 - V^{\theta}). \tag{2.2}$$

The MLE of θ, denoted by $\hat{\theta}_M$, is given by

$$\hat{\theta}_M = r / [(n-r)(V^{-\hat{\theta}_M} - 1)^{-1} \ln V - \sum_{i=1}^{r} \ln u_i] . \tag{2.3}$$

This equation is in implicit form, so it may be solved by using numerical iteration such as Newton Raphson method by using Mathematica 4.0. For a given t, the MLE's of $R(t)$ may be obtained by replacing θ by $\hat{\theta}_M$ in Equation (1.3), then MLE's of $H(t) = -\ln R(t)$ can be obtained.

3. Bayesian Estimation

The natural family of conjugate prior for θ is a gamma distribution with p.d.f.

$$g(\theta) = \frac{\delta^{\nu}}{\Gamma(\nu)} \theta^{\nu - 1} e^{-\delta\theta}, \quad \theta > 0, \nu > 0, \delta > 0. \tag{3.1}$$

From which the prior mean and variance of θ are given, respectively, by ν / δ and ν / δ^2.

Applying Bayes theorem, we obtain from Equations (2.1) and (3.1), the posterior density of θ as

$$g(\theta|\underline{t}) = \frac{(\delta + q)^{r+\nu}}{k\,\Gamma(r+\nu)} \theta^{r+\nu-1} e^{-(\delta+q)\theta} (1 - V^{\theta})^{n-r}, \quad \theta > 0, \nu > 0, \delta > 0, \tag{3.2}$$

where $k = \sum_{j=0}^{n-r} w(j)\left(1-\frac{jp}{\delta+q}\right)^{-(r+\nu)}$, (3.3)

$$w(j) = (-1)^j \binom{n-r}{j}, q = -\ln u, p = \ln V, u = \prod_{i=1}^{r}[1-e^{-t_i}(1+t_i)], \tag{3.4}$$

and

$$V = 1-e^{-t_r}(1+t_r).$$

Note that $r = n,$ the constant $k = 1$ and $g(\theta|\underline{t})$ approaches to the posterior density in a complete sample situation.

Estimation of θ:

The Bayes estimate $\hat{\theta}_s$ of θ relative to squared error loss function is given by

$$\hat{\theta}_S = k^{-1}\frac{r+\nu}{\delta+q}\xi_1, \tag{3.5}$$

where $\xi_1 = \sum_{j=0}^{n-r} w(j)\left(1-\frac{jp}{\delta+q}\right)^{-(r+\nu+1)}$.

Under LINEX loss function, the Bayes estimate $\hat{\theta}_L$ of θ using Equation (1.5) can be obtained as

$$\hat{\theta}_L = -\frac{1}{c}\ln\xi_2,\ c \neq 0, \tag{3.6}$$

where $\xi_2 = k^{-1}\sum_{j=0}^{n-r} w(j)\left(1+\frac{c-jp}{\delta+q}\right)^{-(r+\nu)}$.

Estimation of $R(t)$:

Consider the reliability $R = R(t)$ is a parameter itself. Replacing θ in terms of R by that of Equation (3.2), we obtain posterior density function of R as

$$g(R|\underline{t}) = k^{-1}\frac{Q^{r+\nu}}{\Gamma(r+\nu)}[\varphi_1(R)]^{r+\nu-1}e^{-(Q-1)\varphi_1(R)}(1-V^{Z\varphi_1(R)})^{n-r}, \quad 0<R<1, \tag{3.7}$$

where

$Q = Z(\delta+q), \varphi_1(R) = \ln(1-R)^{-1}, Z = 1/\ln z^{-1},$ and $z = z(t) = 1-e^{-t}(1+t).$ (3.8)

Assuming the quadratic loss is appropriate, the Bayes estimate of the reliability function R is

$$\hat{R}_S = k^{-1}\sum_{j=0}^{n-r} w(j)(\xi_3-\xi_4), \tag{3.9}$$

where $\xi_3 = \left(1-\frac{jpZ}{Q}\right)^{-(r+\nu)}, \xi_4 = \left(1-\frac{jpZ-1}{Q}\right)^{-(r+\nu)}$,

and k and $w(j)$ are given, respectively, by (3.3) and (3.4).
Under LINEX loss function, the Bayes estimate of R using Equation (1.5) is

$$\hat{R}_L = 1 - \frac{1}{c}\ln \xi_5, \quad c \neq 0, \tag{3.10}$$

where $\xi_5 = k^{-1} \sum_{s=0}^{\infty} \frac{c^s}{s!} \sum_{j=0}^{n-r} w(j) \left(1 + \frac{s - jpZ}{Q}\right)^{-(r+\nu)}$.

Estimation of $H(t)$:

To derive the Bayes estimate of the cumulative failure rate function $H(t) = -\ln R(t)$, we first obtain the posterior density function of $H = H(t)$, which can be given by

$$g(H|\underline{t}) = k^{-1} \frac{Q^{r+\nu}}{\Gamma(r+\nu)} \frac{e^{-H}}{1-e^{-H}} [\varphi_2(H)]^{r+\nu-1} e^{-Q\varphi_2(H)} (1 - V^{Z\varphi_2(H)})^{n-r}, H > 0, \tag{3.11}$$

where $\varphi_2 = \ln(1 - e^{-H})^{-1}$.

The Bayes estimate of H relative to quadratic loss is

$$\hat{H}_S = k^{-1} \frac{Q^{r+\nu}}{\Gamma(r+\nu)} G_1, \tag{3.12}$$

where $G_1 = \int_0^{\infty} \ln(1-e^{-x})^{-1} x^{r+\nu-1} e^{-Qx} (1 - V^{Zx})^{n-r} \, dx$.

By using the binomial theorem and logarithmic series, after some simplification we obtain

$$G_1 = \sum_{j=0}^{n-r} \sum_{i=1}^{\infty} (-1)^j \binom{n-r}{j} \frac{\Gamma(r+\nu)}{i(Q+i-jZp)^{r+\nu}}.$$

When the LINEX loss function is appropriate, the Bayes estimate of H is

$$\hat{H}_L = -\frac{1}{c}\ln\left(\frac{k^{-1}Q^{r+\nu}}{\Gamma(r+\nu)} G_2\right), \; c \neq 0, \tag{3.13}$$

where $G_2 = \int_0^{\infty} (1-e^{-x})^c x^{r+\nu-1} e^{-Qx} (1 - V^{Zx})^{n-r} \, dx$.

By using the binomial theorem, after some simplification we obtain

$$G_2 = \sum_{j=0}^{n-r} \sum_{i=0}^{c} (-1)^{i+j} \binom{n-r}{j} \binom{c}{i} \frac{\Gamma(r+\nu)}{(Q+i-jZp)^{r+\nu}}.$$

4. Simulation Study

We obtained, in the above Sections, Bayesian and non-Bayesian estimates for the shape parameter θ, reliability, $R(t)$, and failure rate, $H(t)$, functions of the

EG(θ) distribution. We adopted the squared error loss and LINEX loss functions. The MLE's are also obtained.

In order to assess the statistical performances of these estimates, a simulation study is conducted. The mean square errors (MSE's) using generated random samples of different sizes are computed for each estimator. The random samples are generated as follows:

1. For given values of the prior parameters (ν, δ), generate a random value for θ from the gamma distribution whose density function given by Equation (3.1).

2. Using θ, obtained in step (1), and generate random samples of different sizes: $n = 15, 25, \text{and}\, 50$ from the EG(θ) distribution by using Newton-Raphson method to solve the following equation numerically by Mathematica 4.0 since the inverse transform method can not be applied for EG(θ):
$$U - [1 - e^{-T}(1+T)]^{\theta} = 0,$$
where T is EG(θ) and U is uniform(0,1) distributions. The computations are carried out for such sample sizes and censored samples of sizes: 12, 20, 40, respectively.

3. The MLE of the parameter θ, $\hat{\theta}_M$, is obtained by iteratively solving the Equation (2.3). The estimators $\hat{R}_M(t_0)$, and $\hat{H}_M(t_0)$ of the functions $R(t)$ and $H(t)$ are then computed at some values t_0.

4. The Bayes estimates relative to squared error loss, $\hat{\theta}_s, \hat{R}_s$, and $\hat{H}_s$ given, respectively, by Equations (3.5), (3.9) and (3.12) and relative to LINEX loss $\hat{\theta}_L, \hat{R}_L$, and $\hat{H}_L$ given, respectively, by Equations (3.6), (3.10) and (3.13), are all computed.

5. The above steps are repeated 1000 times and the biases and the mean square errors are computed for different sample sizes n and censoring sizes r, where the had-symbol ^ stands for an estimate $(\hat{\cdot})_M, (\hat{\cdot})_s$ and $(\hat{\cdot})_L$.

The computational (our) results were computed by using Mathematica 4.0. In all above cases the prior parameters chosen as $\nu = 3$ and $\delta = 3$, which yield the generated value of $\theta = 1.9033$ as the true value. The true values of $R(t)$ and $H(t)$, when $t = t_0 = 0.5$, are computed to be $R(0.5) = 0.989732$ and $H(0.5) = 0.010321$. The bias (first entries) and MSE's (second entries) are displayed in Tables 1-3. The computations are achieved under complete and censored samples.

Table (1): Estimated bias and MSE's (second entries) of various estimators of θ, for different sample sizes.

n	r	$\hat{\theta}_M$	$\hat{\theta}_S$	$\hat{\theta}_L$, c=2.5
15	12	0.106335	- 0.203213	- 0.371594
		0.311831	0.141229	0.204783
15	15	0.134140	- 0.193669	- 0.375481
		0.317469	0.141157	0.205972
25	20	0.078750	- 0.118097	- 0.251537
		0.170777	0.102803	0.127528
25	25	0.096342	- 0.142511	- 0.272212
		0.193124	0.100493	0.132202
50	40	0.042833	- 0.068795	- 0.147306
		0.080929	0.057873	0.064259
50	50	0.049002	- 0.064418	- 0.142683
		0.078521	0.058514	0.065716

Table (2): Estimated bias and MSE's (second entries) of various estimators of $R(t)$, for different sample sizes.

n	r	$\hat{R}_M$	$\hat{R}_S$	$\hat{R}_L$, c=-25
15	12	- 0.004007	- 0.019849	- 0.013528
		0.000218	0.000714	0.000361
15	15	- 0.003695	- 0.020112	- 0.013742
		0.000225	0.000705	0.000355
25	20	- 0.002233	- 0.012524	- 0.009446
		0.000120	0.000308	0.000192
25	25	- 0.001984	- 0.011150	- 0.008294
		0.000113	0.000289	0.000181
50	40	- 0.001100	- 0.005898	- 0.004860
		0.000052	0.000110	0.000080
50	50	- 0.000831	- 0.006174	- 0.005070
		0.000046	0.000111	0.000085

Table (3): Estimated bias and MSE's (second entries) of various estimators of $H(t)$, for different sample sizes.

n	r	$\hat{H}_M$	$\hat{H}_S$	$\hat{H}_L$, c=25
15	12	0.004148	0.020230	0.013400
		0.000231	0.000765	0.000364
15	15	0.003837	0.019766	0.013079
		0.000238	0.000723	0.000344
25	20	0.002307	0.012366	0.009129
		0.000127	0.000336	0.000203
25	25	0.002051	0.013613	0.010162
		0.000118	0.000378	0.000229
50	40	0.001127	0.006168	0.005372
		0.000054	0.000106	0.000095
50	50	0.000852	0.006212	0.005064
		0.000047	0.000110	0.000084

5. Concluding Remarks

In this paper we have presented the Bayesian and non-Bayesian estimates of the shape parameter θ, reliability, $R(t)$, and failure rate, $H(t)$, functions of the lifetimes follow the EG(θ) distribution. The estimation are conducted on the basis of complete and type-II censored samples. Bayes estimators, under squared error loss and LINEX loss functions, are derived. The MLE's are also obtained.

Our observations about the results are stated in the following points:

1- Table 1 shows that the Bayes estimates under the quadratic loss function have the smallest estimated MSE's as compared with the estimates under LINEX loss function or MLE's. This is True for both complete and censored samples. It is immediate to note that MSE's decrease as sample size increases. On the other hand the Bayes estimates under the LINEX loss function and quadratic loss function are underestimation, but MLE's are overestimation. This is true for both complete and censored samples.

2- Table 2 shows that the MLE's have the smallest estimated MSE's as compared with the Bayes estimates under LINEX loss function or quadratic loss function. This is True for both complete and censored samples. It is immediate to note that MSE's decrease as a complete sample size increases. On the other hand the Bayes estimates under the LINEX loss function have the smallest estimated MSE as compared with the estimates under quadratic loss function. This is True for both complete and censored samples. Also, all estimates are underestimation.

3- Table 3 shows that the MLE's have the smallest estimated MSE's as compared with the Bayes estimates under LINEX loss function or quadratic loss function. This is True for both complete and censored samples. It is immediate to note that MSE's decrease as a complete sample size increases. Also, the Bayes estimates under the LINEX loss function have the smallest estimated MSE's as compared with the estimates under quadratic loss function. This is True for both complete and censored samples. On the other hand all estimates are overestimation.

From the previous observations, we suggest to use Bayes approach under quadratic loss function for estimating the shape parameter of EG distribution, while the MLE's are better for the reliability and failure rate functions.

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