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Some Properties on Base-Countably-Mesocompact Spaces

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Abstract

In this paperbase-countable-mesocompactness is investigated. And the following results are obtained: (1) Let X be base-countably-mesocompact and X' be an F_σ subset of X. If X is normal,then X' is base-countably-mesocompact relative to X. (2) Let X be mesocompact. locally base-countably-mesocompact. If X is normal then X is base-countably-mesocompact. (3) Let Y be a base-countably-mesocompact space and $f : X \to Y$ be a base-countably-mesocompact map and $\omega(X) \geq \omega(Y)$. If Y is regular then X is base-mesocompact. (4) Let $f : X \to Y$ be a closed $Lindel\ddot{o}f$ *map and* Y be a base-countably-mesocompact. If X and $Y$are both regular and $\omega(X) \geq \omega(Y)$, then X is base-countably-mesocompact.

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1 Introduction and preliminaries

In his paper [1], J.E. Porter introduced the notion of base-paracompactness and obtained some analogous results of base-paracompactness to paracompactness; a space X is said to be base-paracompact if there is a base β for X of cardinality equal to the weight of X such that every open cover μ of X has a locally finite refinement β' by members of β. And he observed that metric spaces are base-paracmpact. Moreover he showed that paracompact ordered spaces of weight not greater than ω_1 are base-paracompact. At the end of his paper he raised four open questions. Many scholars showed great interest on base-paracompactness. In [2] Professor Y. Ge defined base-paracompact mapping and claimed that every base-paracompact mapping inversely preserves base-paracompactness if $\omega(X) \geq \omega(Y)$, which improved the same result of J.E. Porter about perfect mapping. At the same time he proved that every closed $Lindel\ddot{o}f$ mapping inversely preserves base-paracompactness if and only if X is regular and $\omega(X)$ is a regular cardinalitywhich gave some answers for the question on inverse images of base-paracompact spaces. In this paper we investigate base-countably-mesocompactness.

Throughout paper, all spaces are assumed to be Hausdorff spaces and all maps are continuous and surjective. The symbol $\boldsymbol{N}$ denotes the set of all natural numbers. The cardinality of $\boldsymbol{A}$ is denoted by $|A|$. For a space X, $\omega(X)$ stands for the weight of X. A^c and $IntA$ stand for the complement of A and the interior of A in X respectively. For any $A \subset X$, Let $\mu_A = \{U \in \mu : U \cap A \neq \phi\}$. For any $A \subset X$**,** the closure of A is denoted by $\overline{A}$**.** One may refer to [3] for undefined notations and terminology.

Definition 1.1 A space X is base-countably-mesocompact if there is a basis β for X with $|\beta| = \omega(X)$ such that every countable open cover μ of X admits a compact-finite refinement β' by members of β.

Definition 1. 2 A topological space X is said to be locally base-countably-mesocomact if for each $x \in X$ there exists a neighborhood G_x of x such that $\overline{G_x}$ is base-countbly-mesocompact.

Definition 1.3[4] Let A be a subset of X and μ a family of subset of X**.**μ is said to be locally finite at A in X if for every $x \in A$ there exists an open (in $\boldsymbol{X}$) neighborhood G_x of $\boldsymbol{x}$ such that G_x intersects at most finite members of μ**.**

μ is said to be compact-finite at A in X if $(\mu)_K$ is finite for every compact subset K of A.

Definition 1.4 A subspace M of X is called to be base-countably-mesocompact relative to X provided that there is a basis β for X with $|\beta| = \omega(X)$ such that every open cover (in X) of M has a compact-finite (in M) partial open refinement $\beta' \subset \beta$ such that $M \subset \cup\beta'$.

Definition 1.5 A map $f : X \to Y$ is called base-countably-mesocompact if there exists a basis β for X with $|\beta| = \omega(X)$ such that for every $y \in Y$ and a countable family μ of open subsets of X which covers $f^{-1}(y)$, there exists a neighborhood O_y of y and a refinement $\underline{\beta_y \text{ of } \mu,}$ where $\beta_y \subset \beta$ such that $f^{-1}(O_y) \subset \cup\beta_y$ and β_y is compact-finite at $\overline{f^{-1}(O_y)}$ in X.

Recall that a closed map $f : X \to Y$ is called closed map if $f^{-1}(y)$ is a $Lindel\ddot{o}f$ subset of X for every $y \in f^{-1}(y)$.

Lemma 1.6[1] Let β be basis for X with $|\beta| = \omega(X)$. Then there is a basis β' for X with $|\beta'| = \omega(X)$ and $\beta \subset \beta'$ which is closed under finite unionsfinites intersectionsand complements of closures .

The validity of the following lemmas is easy to verify.

Lemma1.7 Let $\mu = \{U_\lambda : \lambda \in \Lambda\}$ be a compact-finite family of subsets of space X . Then $\{U_\lambda : \lambda \in \Lambda'\}$ is compact-finite for every subset Λ' of Λ .

Let $\mu = \{U_\lambda : \lambda \in \Lambda\}$ be a compact-finite family of subsets of space X. Then $\{\nu_\lambda : \lambda \in \Lambda\}$ is compact-finite if $\nu_\lambda \subset U_\lambda$ for every $\lambda \in \Lambda$.

Lemma 1.8 If μ is locally finite at A in X. Then μ is compact-finite at $\overline{A}$ in X.

2 Main Results

Proposition 2.1 Every closed subset M of base-countably-mesocompact space X is base-countably-mesocompact relative to X.

Proof Let M be closed in X and μ a countable open family of subsets of X which covers M . Let β be a basis which witnesses base-countably-mesocompactness for X . Note that $\mu \cup \{X \backslash M\}$ is a countable open cover of X . By base-countably-mesocompactness there exists $\beta' \subset \beta$ such that β' is a compact-finite refinement of $\mu \cup \{X \backslash M\}$. Let $w = \{B \in \beta' : B \cap M \neq \phi\}$. Then w is a compact-finite partial open refinement of μ and $M \subset \cup w$.

Theorem 2.2 Let X be base-countably-mesocompact and M a closed subset of X with $\omega(X) = \omega(M)$ then M is base-countably-mesocompact.

Proof By Proposition 2.1 M is base-countably-mesocompact relative to X . Since $\omega(X) = \omega(M)$, M is base-countably-mescompact.

Corollary 2.3 Let X, Y be mesompact spaces with $\omega(X) \geq \omega(Y)$. if $X \times Y$ is base-countably-mesocompact, then so is X.

Proof By the definition of product space , X is homeomorphic to a closed subspace of $X \times Y$. Since $\omega(X) \geq \omega(Y), \omega(X) = \omega(X \times Y)$ holds, and $X \times Y$ is base-countably-mesocompact, so is X by theorem 2.2 .

Theorem 2.4 Let X be base-countably-mesocompact and X' be an F_σ subset of X. If X is normal, then X' is base-countably-mesocompact relative to X.

Proof Let X be base-countably-mesocompact and normal. Let β be a basis which witnesses base-countably-mesocompact for X with $|\beta| = \omega(X)$. By virtue of lemma 1.7 we can assume β is closed under finite intersections, finite unionsand complements of closures. Observe that $X' = \bigcup_{n=1}^{\infty} F_n$, where F_n is closed in X such that $F_n \subset Int_{X'} F_{n+1} \subset F_{n+1}$ ($Int_{X'} F_n$ denotes the interior of F_n relative to $X', n \in N$).

Let ν be a countable open (in X) cover of X' . Put $\nu_1 = \{V \in \nu : V \cap F_1 \neq \phi\}$. The cover $\nu_1 \cup \{X - F_1\}$ of X has a compact-finite refinement β'_1 by members of β . Put $w_1 = \{B \in \beta'_1 : B \cap F_1 \neq \phi\}$. Proceeded by induction. Suppose that w_j and β'_j have been constructed for every $j < n$ such that $\beta'_j \subset \beta$

and β'_j is compact-finite and $w_j = \left\{B - \cup_{i<j}F_i : B \in \beta'_j, B \cap F_j \neq \phi\right\}$. Put $\nu_n = \{V \in \nu : V \cap F_n \neq \phi\}$. Notice that $\nu_n \cup \{X - F_n\}$ covers X, by induction hypothesis, there exists $\beta'_n \subset \beta$ such that β'_n is compact-finite. For every $n \in N$ put $w_n = \{B - \cup_{i<n}F_i : B \in \beta'_n, B \cap F_j \neq \phi\}$. $B - \cup_{i<n}F_i = B \cap (\cup_{i<n}F_i)^c = B \cap \left(\cap_{i<n}F_i^C\right)$. By the property of the basis we can find an $B'_i \in \beta$ such that $B'_i \subset F_i^c$. Since β is closed under finite intersections. $\cap_{i<n}B_i$ is an element of β. Put $B' = \cap_{i<n}B_i$ and $w'_n = \{B \cap B' : B \in \beta'_n\}$. Put $w = \cup_{n\in N}w_n$ and $w' = \cup_{n\in N}w'_n$. Since β'_n is compact-finite, w_n and w_n^{-1} are both compact-finite.

Claim . w' is a partial refinement of ν.

In fact, for any $x \in X'$ let n be the first integer such that $x \in F_n$, then $x \notin \cup_{i<n}F_i$. Then there exist an element B' of β such that $x \in B' \subset (U_{i<n}F_i)^c$. Since β'_n refines $\nu_n \cup \{X - F_n\}$ then there exists $B \in \beta'_n$ such that $x \in B$. Thus $x \in B \cap B'$, hence w' is an open cover of X'. Clearly for any $W' \in w'$ there exists an element $V \in v$ such that $W' \in V$.

Claim. w is compact-finite.

In fact $X' = \cup_{n\in N}Int_{X'}F_n, Int_{X'}F_n \subset Int_{X'}F_{n+1}$. For any compact set $K \subset X'$, there exists $n_0 \in N$ such that $K \subset Int_{X'}F_{n_0}$. For any $m > n_0$, we have

$$K \cap w_m = K \cap \{B - (\cup_{i<m}F_i) : B \in \beta'_m, B \cap F_m \neq \phi\}$$

$$= K \cap \{B \cap (\cup_{i<m}F_i)^c : B \in \beta'_m, B \cap F_m \neq \phi\}$$

$$\subset (Int_{X'}F_{n_0}) \cap \left(\cap_{i<m}F_i^C\right) = IntF_{n_0} \cap (F_{n_0})^C \subset F_{n_0} \cap F_{n_0}^C = \phi$$

Hence w is compact-finite.

It is clear that there exists $W \in w$ such that $W' \subset W$ for every $W' \in w'$. By lemma 1.8 w' is compact-finite.

Corollary 2.5 If X is a normal mesocompact space and the union of closed basd-coumtably-mesocompact spaces relative to X, then X is base-countably-mesocompact.

Proof Let $X = \cup_{i\in N}F_i$, where F_i is closed and base-countably-mesocompact relative to X . Then X is an F_σ subset of itself. By theorem 2.4 X is base-countably-mesocompact relative to X, hence X is base-countably-mesocompact.

Theorem 2.6 Let X be mesocompact , locally base-countably-mesocompact . If X is normal then X is base-countably-mesocompact.

Proof By the definite of locally base-countably-mesocompactness, for each $x \in X$, pick out a neighborhood G_x of x, such that $\overline{G_x}$ is base-countably-mesocompact. Let $g = \{G_x : x \in X\}$, By mesocompact of X, g has a compact-finite refinement μ such that $|\mu| \leq \omega(X)$. Since X is normal, μ has a shrinking v. For each $V \in v$, there exists a $G_V \in g$ such that $\overline{V} \subset G_V$ and hence $\overline{V}$ is base-mesocompact relative to G_V. For each $V \in v$ let β_v be a basis for G_V which admit base-countable-mesocompactness relative to $\overline{V}$. Let $\beta = \cup_{V\in v}\beta_V$ and w be a countable open cover of X. Let $w_V = \left\{W : W \cap \overline{V} \neq \phi\right\}$, then w_V is an open cover of $\overline{V}$. There exists a compact-finite (in X) refinement β'_v of w_V such that $\cup\beta'_V \subset G_V$. Observe that β'_v is compact-finite in X. Hence $\beta' = \cup_{V\in v}\beta'_V$ is a compact-finite refinement of w.

Theorem 2.7 Let Y be a base-countably-mesocompact space and $f : X \to Y$ be a base-countably-mesocompact map and $\omega(X) \geq \omega(Y)$. If Y is regular then X is base-mesocompact.

Proof Let β_Y be a basis for Y which witnesses base-countable-mesocompactness. Let β_X be a basis for X with $|\beta_x| = \omega(X)$ which witnesses base-countable-mesocompactne f . Put $\beta = \{B \cap f^{-1}(B') : B \in \beta_X, B' \in \beta_Y\}$. Since $\omega(X) \geq \omega(Y)$, clearly $|\beta| = |\beta_X| = \omega(X)$ holds.

Prove β is a basis for X which witnesses base-countable-mesocompactness as follows. Let μ be a countable open cover of X. Since β_X witnesses base-countable-mesocompactness for f, for every $y \in Y$, there exists a neighborhood O_y of y and a partial refinement $\underline{\beta_y \text{ of } \mu,}$ where $\beta_y \subset \beta_X$, such that $f^{-1}(O_y) \subset \cup\beta_y$ and β_y is compact-finite at $\overline{f^{-1}(O_y)}$ in X . By regular of Y there exists a neighborhood G_y of y such that $\overline{G_y} \subset O_y$. Then $f^{-1}(y) \subset f^{-1}(G_y) \subset f^{-1}\left(\overline{G_y}\right) \subset f^{-1}(O_y) \subset \cup\beta_y$. Let $v = \{G_y : y \in Y\}$, then v is an open cover of Y. By base-countable-mesocompactness of Y, v has compact-finite open refinement $\beta'_y \subset \beta_Y$, say $\beta'_y = \{B'_\alpha : \alpha \in \Gamma\}$. For every $\alpha \in \Gamma$, pick $y_\alpha \in Y$ such that $B'_\alpha \subset G_{y\alpha}$. Put $\beta' = \{B_{y\alpha} \cap f^{-1}(B'_\alpha) : B_{y\alpha} \in \beta_{y\alpha}, B'_\alpha \in \beta'_Y, \alpha \in \Gamma\}$. Then $\beta' \subset \beta$ and β' is an open refinement of μ.

Claim. $\{f^{-1}(B'_\alpha) : B'_\alpha \in \beta'_Y, \alpha \in \Gamma\}$ is compact-finite.

Suppose to the contrary, there is a compact subset K of X which intersects infinite members of $\{f^{-1}(B'_\alpha) : B'_\alpha \in \beta'_Y, \alpha \in \Gamma\}$. Without loss of generality, assume countably infinite , say $f^{-1}(B'_1), f^{-1}(B'_2), \cdots$. Then $K \cap f^{-1}(B'_i) \neq \phi, i = 1, 2, \cdots$. Since f is a continuous map, $f(K)$ is a compact subset of Y. Observe that $f(f^{-1}(B'_i)) = B'_i$. Thus $f(K) \cap f(f^{-1}(B'_i)) = f(K) \cap B'_i \neq \phi, i = 1, 2, \cdots$, which contradict the compact-finiteness of β'_Y.

Claim β' is compact-finite.

For every compact subset K of X, there exists a finite $\Gamma_K \subset \Gamma$ such that $K \cap f^{-1}(B'_\alpha) = \phi$ for every $\alpha \in \Gamma - \Gamma_K$. Let $\alpha \in \Gamma_K$. For every $x \in K$, if $x \notin f^{-1}\left(\overline{B'_\alpha}\right)$ then there exists a neighborhood V_α of x such that $V_\alpha \cap f^{-1}\overline{(B'_\alpha)} = \phi$, thus $V_\alpha \cap f^{-1}(B'_\alpha) = \phi$. If $x \in f^{-1}\left(\overline{B'_\alpha}\right)$, then $x \in f^{-1}\left(\overline{G}_\alpha\right) \subset f^{-1}(O_{y_\alpha})$. Since $\underline{\beta_{y_\alpha}}$ is compact-finite at $\overline{f^{-1}(O_{y_\alpha})}$ in X, there exists a compact subset V_α of $\overline{f^{-1}(O_{y_\alpha})}$ such that $x \in V_\alpha$ and $(\beta_{y_\alpha})_{V\alpha}$ is finite . Thus a neighborhood V_α of x is constructed as above for each $\alpha \in \Gamma_K$. Observe that $K \subset \cup_{\alpha\in\Gamma K}V_\alpha$. Hence $(\beta')_k \subset (\beta')_{\cup_{\alpha\in\Gamma_k}V_\alpha} \subset \cup_{\alpha\in\Gamma_k}(\beta')_{V_\alpha} \subset \cup_{\alpha\in\Gamma_k}(\beta_{y\alpha})_{V_\alpha}$. Then $(\beta')_k$ is finite.

The proof of the following corollaries is trivial.

Corollary 2.8 Let $f : X \to Y$ be a perfect map. X is base-countable-mesocompact if Y is regular and base-countable-mesocompact.

Remark The condition regularity of Y can be committed in corollary 2.8.

Corollary 2.9 Let X be base-countable-mesocompact and Y be a compact space . Then $X \times Y$ is base-countable-mesocompact.

Proof The projection $p : X \times Y \to X$ is a perfect map. Since X is base-countably-mesocompact, $X \times Y$ is base-countably-mesocompact by the preceding remark.

Proposition 2.10 Let $f : X \to Y$ be a closed $Lindel\ddot{o}f$ map . If X is regular then f is base-countable-mesocompact.

Proof Let β be a basis for X with $|\beta| = \omega(X)$. By virtue of lemma 1.7 we can assume β is closed under finite intersection, finite unions, and complements of closures. Let μ is countable cover of $f^{-1}(y)$ for every $y \in Y$. There exists $U \in \mu$ such that $x \in U$ for every $x \in f^{-1}(y)$. Then there exist B'_x, such that $x \in B'_x \subset \overline{B''_x} \subset U$ by regularity of X. Since $f^{-1}(y)$ is a $Lindel\ddot{o}f$ subset, there exists countable family $\{B'_{xn} : n \in N\} \subset \{B'_x : x \in f^{-1}(y)\}$ such that $f^{-1}(y) \subset \cup_{x\in N} B'_{xn}$. Let $B_1 = B''_{x1}$, $B_n = B''_{xn} - \cup\{\overline{B'_{xi}} : i < n\}$. Let $\beta_y = \{B_n : n \in N\}$.

Since β is closed under finite intersection and unions and complements of closures, we have that $\beta_y \subset \beta$ and $f^{-1}(O_y) \subset \cup\beta_y$ and β_y is a partial refinement of μ. Let $x \in f^{-1}(O_y)$. Then there exists $i \in N$ such that $x \in B'_{xi}$, thus B'_{xi} is a neighborhood of x and $B'_{xi} \cap B_n = \phi$ for each $n > i$. Hence β_y is locally finite at $f^{-1}(O_y)$ in X. By lemma 1. 8 β_y is compact-finite at $\overline{f^{-1}(O_y)}$ in X.

Theorem 2.11 Let $f : X \to Y$ be a closed $Lindel\ddot{o}f$ map and Y be a base-countably-mesocompact. If X and Y are both regular and $\omega(X) \geq \omega(Y)$, then X is base-countably-mesocompact.

Proof It is easy to prove it by theorem 2.7 and proposition 2.10.

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