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A Single Server Queue with Additional Optional Service in Batches and Server Vacation

R. Kalyanaraman

Department of Mathematics, Annamalai University
Annamalainagar-608 002, India
r.kalyan24@rediffmail.com

S. Pazhani Bala Murugan

Department of Mathematics, Annamalai University
Annamalainagar-608 002, India
spbmaths@yahoo.com

Abstract

A single server infinite capacity queueing system with Poisson arrival and exponential service time distribution along with second optional service in batches is considered. As soon as the server becomes idle he leaves for a vacation and the duration of the vacation follows an exponential distribution. In steady state, the probability generating function for queue length has been obtained. The average queue length have been found and numerical results are presented to test the feasibility of the queueing model.

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1 Introduction

In this paper, an $M/M/1$ queueing model with server vacation is taken. The single server, apart from providing the usual service one by one, also provides an additional optional service to the customers in batches of fixed size $b(\geq 1)$. The customers are queued up for the first service, which is essential for all the customers. The second service is optional which is demanded by some of the units whereas the others leave the system after the first service is over.

The steady state and the transient solutions for the $M/M/1$ model in various forms have been dealt with by a wide range of authors. To mention a few references we would name Bailey[1], Chaudhry et al[2], Cohen[3], Conolley[4], Daley[5], Feller[7], Jaiswal[8], Medhi[13], Neuts[14], Saaty[15], Takacs[16]. The idea of optional service has been dealt with by several authors including Madan[10, 11, 12]. Queues with vacations have been studied extensively in the past: a comprehensive survey can be found in Teghem[17] and Doshi[6].

The model treated in this paper would seem to have potential applicability in numerous situations. In banks, a teller may perform a manual accounting or computer task after each service, or after busy periods, each of which takes its own random amount of time to do. Here on completion of a service, a customer may find his service unsatisfactory and consequently may demand an additional service, the additional optional service given in batches.

The organization of the paper is as follows: The model under consideration is described in section 2. In section 3, we analyze the model by deriving the system steady state equations. Using the equations the probability generating function of queue length are obtained in section 4. In section 5, we obtained the general steady state solution. The operating characteristics are obtained in section 6 and a numerical study is carried out in section 7 to test the effect of the system performance.

2 Mathematical Model

We consider a single server queueing system in which customers arrive according to a Poisson process with rate λ. The server, apart from providing the usual service one by one, also provides an additional optional service to the units in batches of fixed size $b(\geq 1)$. The units arriving one by one queue up for the usual service, herein called the 'first service' which is essentially needed by all the units. The first service times are exponentially distributed with mean service time $\frac{1}{\mu_1}$. The second service is optional which is demanded by some of the units where as the others leave the system after their first service is over. After the first service the unit leaves the system with probability q or queues up for the second service with probability p, where $p+q=1$. As soon as there is a batch of $b(\geq 1)$ units ready for the second service the server suspends the first service and instantly starts the second service. The second service times are exponentially distributed with mean service time $\frac{1}{\mu_2}$. After the completion of the second service of a batch he immediately attends to the first service if there is a unit in the first queue. A batch of fewer than b units can be taken for the second service only if there is no unit requiring first service at that moment. If there is no unit waiting in the first queue as well as in the second queue then the server takes a vacation of random length, at the end of the vacation, if the server finds the system empty, he immediately takes another vacation and continuous in this manner until he finds atleast one waiting customer upon return from vacation. The vacation times are exponentially distributed with parameter θ.

3 Model Analysis

Basically the system will have three possible states: i) The server busy providing the first service, ii). The server busy providing the second service, and iii). The server is on vacation. Let $P_{m,n}^{(1)}$ be the steady state probability that there are m, n units waiting for the first and the second service respectively excluding one unit in the first service and let $P_{m,0}^{(2)}$ be the steady state probability that there are m units waiting for the first service excluding a batch being provided the second service. Let $P_{m,0}^{(0)}$ be the steady state probability that there are m unit waiting for the first service, no unit waiting for second service and the server is on vacation. The system has the following set of steady state equations:

$$(\lambda+\mu_1)P_{m,n}^{(1)} = \lambda P_{m-1,n}^{(1)} + \mu_1 q P_{m+1,n}^{(1)} + \mu_1 p P_{m+1,n-1}^{(1)}; \quad (m>0; 0<n<b) \tag{1}$$

$$(\lambda+\mu_1)P_{m,0}^{(1)} = \lambda P_{m-1,0}^{(1)} + \mu_1 q P_{m+1,0}^{(1)} + \mu_2 P_{m+1,0}^{(2)} + \theta P_{m+1,0}^{(0)}; (m>0) \tag{2}$$

$$(\lambda+\mu_1)P_{0,n}^{(1)} = \mu_1 q P_{1,n}^{(1)} + \mu_1 p P_{1,n-1}^{(1)}; \quad (0<n<b) \tag{3}$$

$$(\lambda+\mu_1)P_{0,0}^{(1)} = \mu_1 q P_{1,0}^{(1)} + \mu_2 P_{1,0}^{(2)} + \theta P_{1,0}^{(0)} \tag{4}$$

$$(\lambda+\mu_2)P_{m,0}^{(2)} = \lambda P_{m-1,0}^{(2)} + \mu_1 p P_{m,b-1}^{(1)}; \quad (m>0) \tag{5}$$

$$(\lambda+\mu_2)P_{0,0}^{(2)} = \mu_1 p \sum_{n=1}^{b-1} P_{0,n}^{(1)} \tag{6}$$

$$(\lambda+\theta)P_{m,0}^{(0)} = \lambda P_{m-1,0}^{(0)}; \quad (m>0) \tag{7}$$

$$\lambda P_{0,0}^{(0)} = \mu_1 q P_{0,0}^{(1)} + \mu_2 P_{0,0}^{(2)} \tag{8}$$

4 Probability Generating Functions of Queue Length

We define the following probability generating functions:

$P_n^{(1)}(\alpha) = \sum_{m=0}^{\infty} P_{m,n}^{(1)}\alpha^m;\ P_m^{(1)}(\beta) = \sum_{n=0}^{b-1} P_{m,n}^{(1)}\beta^n$

$P^{(1)}(\alpha,\beta) = \sum_{m=0}^{\infty} P_m^{(1)}(\beta)\alpha^m = \sum_{n=0}^{b-1} P_n^{(1)}(\alpha)\beta^n = \sum_{m=0}^{\infty}\sum_{n=0}^{b-1} P_{m,n}^{(1)}\alpha^m\beta^n$

$$P_0^{(2)}(\alpha) = \sum_{m=0}^{\infty} P_{m,0}^{(2)}\alpha^m \quad and \quad P_0^{(0)}(\alpha) = \sum_{m=0}^{\infty} P_{m,0}^{(0)}\alpha^m \tag{9}$$

Also, we define

$$\sum_{m=0}^{\infty} P_{m,0}^{(0)} = P^{(0)}; \quad \sum_{i=0}^{2} P_0^{(i)}(1) = 1 \tag{10}$$

Applying (10) in equation (7) and (8) we have,

$$\theta P^{(0)} = \theta P_{0,0}^{(0)} + \mu_1 q P_{0,0}^{(1)} + \mu_2 P_{0,0}^{(2)} \tag{11}$$

Performing, α times equation (3) + $\sum\limits_{m=1}^{\infty} \alpha^{m+1}$ times equation (1); α times equation (4) + $\sum\limits_{m=1}^{\infty} \alpha^{m+1}$ times equation (2); equation (6) + $\sum\limits_{m=1}^{\infty} \alpha^{m}$ times equation (5) and rearranging the terms we have

$$\{[\lambda(1-\alpha)+\mu_1]\alpha - \mu_1 q\} P_n^{(1)}(\alpha) = \mu_1 p P_{n-1}^{(1)}(\alpha) - \mu_1 q P_{0,n}^{(1)} - \mu_1 p P_{0,n-1}^{(1)}; \quad (0 < n \leq b-1)) \tag{12}$$

$$\{[\lambda(1-\alpha)+\mu_1]\alpha - \mu_1 q\} P_0^{(1)}(\alpha) = \mu_2 P_0^{(2)}(\alpha) + \theta P_0^{(0)}(\alpha) - \theta P^{(0)} \tag{13}$$

$$[\lambda(1-\alpha)+\mu_2] P_0^{(2)}(\alpha) = \mu_1 p P_{b-1}^{(1)}(\alpha) + \mu_1 p \sum_{n=1}^{b-2} P_{0,n}^{(1)} \tag{14}$$

Performing, equation (8) + $\sum\limits_{m=1}^{\infty} \alpha^{m}$ times equation (7), we have

$$[\lambda(1-\alpha)+\theta] P_0^{(0)}(\alpha) = \theta P^{(0)} \tag{15}$$

Performing, equation (13) + $\sum\limits_{n=1}^{b-1} \beta^{n}$ times equation (12), and rearranging the terms and simplifying, we have

$$P^{(1)}(\alpha,\beta) = \frac{\mu_2 P_0^{(2)}(\alpha) - \theta P_0^{(0)}(\alpha) - \theta P^{(0)} - \mu_1 (q+p\beta) P_0^{(1)}(\beta) + \mu_1 q P_{0,0}^{(1)}}{[\lambda(1-\alpha)+\mu_1]\alpha - \mu_1(q+p\beta)} \tag{16}$$

We note that for $\beta = 0$, $P^{(1)}(\alpha,\beta) = P_0^{(1)}(\alpha)$ and $P_0^{(1)}(0) = P_{0,0}^{(1)}$. Thus for $\beta = 0$, equation (16) gives

$$P_0^{(1)}(\alpha) = \frac{\mu_2 P_0^{(2)}(\alpha) + \theta P_0^{(0)}(\alpha) - \theta P^{(0)}}{[\lambda(1-\alpha)+\mu_1]\alpha - \mu_1 q} \tag{17}$$

Equation (14) for $b = 1$ gives

$$P_0^{(2)}(\alpha) = \frac{\mu_1 p P_0^{(1)}(\alpha)}{\lambda(1-\alpha)+\mu_2} \tag{18}$$

Substituting equations (15) and (18) in (17) and simplifying we have

$$P_0^{(1)}(\alpha) = \frac{[\lambda(1-\alpha)+\mu_2]\lambda\theta P^{(0)}}{[\lambda(1-\alpha)+\theta][\lambda^2\alpha^2-\lambda(\lambda+\mu_1+\mu_2)\alpha+\mu_1(\lambda q+\mu_2)]} \tag{19}$$

Using equation (19) in (18), we have

$$P_0^{(2)}(\alpha) = \frac{\lambda\mu_1 p\theta P^{(0)}}{[\lambda(1-\alpha)+\theta][\lambda^2\alpha^2-\lambda(\lambda+\mu_1+\mu_2)\alpha+\mu_1(\lambda q+\mu_2)]} \tag{20}$$

The normalizing condition is
$P_0^{(0)}(1)+P_0^{(1)}(1)+P_0^{(2)}(1)=1$, Equations(15), (19) and (20) gives

$$P^{(0)} = 1-\frac{\lambda}{\mu_1}\left(\frac{\mu_1 p}{\mu_2}+1\right) \tag{21}$$

which implies that the utilisation factor is $\rho = \frac{\lambda}{\mu_1}\left(\frac{\mu_1 p}{\mu_2}+1\right)$ and the steady state condition is therefore

$$\frac{\lambda}{\mu_1}\left(\frac{\mu_1 p}{\mu_2}+1\right) < 1 \tag{22}$$

We observe that on setting $p=0$ in equation (22) reduces to $\rho=\frac{\lambda}{\mu_1}<1$ which is the usual well known steady state condition for the $M/M/1$ queue with multiple vacation.

5 The General Steady State Solution

In this section, we derive the general steady state solution for the particular case $b=1$. For this, we use the notation $P^{(0)}(\alpha), P^{(1)}(\alpha)$ and $P^{(2)}(\alpha)$ instead of $P_0^{(0)}(\alpha)$, $P_0^{(1)}(\alpha)$ and $P_0^{(2)}(\alpha)$ respectively.

Consider the equation (19), and applying the concept of partial fraction, we have

$$P^{(1)}(\alpha) = \frac{A_1}{\lambda(1-\alpha)+\theta}+\frac{A_2\alpha+A_3}{\lambda^2\alpha^2-\lambda(\lambda+\mu_1+\mu_2)\alpha+\mu_1(\lambda q+\mu_2)} \tag{23}$$

where

$$\begin{aligned} A_1 &= \frac{\lambda(\mu_2-\theta)\theta P^{(0)}}{(\lambda+\theta)(\theta-\mu_1-\mu_2)+\mu_1(\lambda q+\mu_2)} \\ A_2 &= \frac{\lambda^2(\mu_2-\theta)\theta P^{(0)}}{(\lambda+\theta)(\theta-\mu_1-\mu_2)+\mu_1(\lambda q+\mu_2)} \\ A_3 &= \frac{\lambda[\lambda(\theta-\mu_1-\mu_2)-\mu_2(\mu_2-\theta)+\lambda\mu_1 q]\theta P^{(0)}}{(\lambda+\theta)(\theta-\mu_1-\mu_2)+\mu_1(\lambda q+\mu_2)} \end{aligned} \tag{24}$$

The denominator in the first term of the right hand side of the equation (23) has only one zero, namely,

$$\alpha_1 = \frac{\lambda + \theta}{\lambda} \tag{25}$$

Also, the denominator in the second term of the right hand side of equation (23) has two zeros given by

$$\alpha_2,\ \alpha_3 = \frac{\lambda(\lambda + \mu_1 + \mu_2) \pm \sqrt{\lambda^2(\lambda + \mu_1 + \mu_2)^2 - 4\lambda^2\mu_1(\lambda q + \mu_2)}}{2\lambda^2}$$

So that

$$\alpha_2.\alpha_3 = \frac{\mu_1(\lambda q + \mu_2)}{\lambda^2}\ and\ \alpha_2 + \alpha_3 = \frac{\lambda + \mu_1 + \mu_2}{\lambda} \tag{26}$$

Using (25) and (26), equation (23) can be rewritten as

$$P^{(1)}(\alpha) = \frac{A_1}{\lambda\alpha_1}\left(1 - \frac{\alpha}{\alpha_1}\right)^{-1} + \frac{A_3}{\lambda^2\alpha_2\alpha_3}\left(1 + \frac{A_2}{A_3}\alpha\right)\left(1 - \frac{\alpha}{\alpha_2}\right)^{-1}\left(1 - \frac{\alpha}{\alpha_3}\right)^{-1} \tag{27}$$

$$= \tfrac{A_1}{\lambda\alpha_1}\left[1 + \tfrac{\alpha}{\alpha_1} + \left(\tfrac{\alpha}{\alpha_1}\right)^2 + \dots\right] + \tfrac{A_3}{\lambda^2\alpha_2\alpha_3}\left(1 + \tfrac{A_2}{A_3}\alpha\right)\left[1 + \tfrac{\alpha}{\alpha_2} + \left(\tfrac{\alpha}{\alpha_2}\right)^2 + \dots\right]\left[1 + \tfrac{\alpha}{\alpha_3} + \left(\tfrac{\alpha}{\alpha_3}\right)^2 + \dots\right]$$

$$= \tfrac{A_1}{\lambda\alpha_1}\left[1 + \tfrac{\alpha}{\alpha_1} + \left(\tfrac{\alpha}{\alpha_1}\right)^2 + \dots\right] + \tfrac{A_3}{\lambda^2\alpha_2\alpha_3}\left(1 + \tfrac{A_2}{A_3}\alpha\right)\left[1 + \left(\tfrac{1}{\alpha_2} + \tfrac{1}{\alpha_3}\right)\alpha + \left(\tfrac{1}{\alpha_2^2} + \tfrac{1}{\alpha_2\alpha_3} + \tfrac{1}{\alpha_3^2}\right)\alpha^2 + \dots\right]$$

$$= \tfrac{A_1}{\lambda\alpha_1}\left[1 + \tfrac{\alpha}{\alpha_1} + \left(\tfrac{\alpha}{\alpha_1}\right)^2 + \dots\right] + \tfrac{A_3}{\lambda^2\alpha_2\alpha_3}[1 + \left(\tfrac{1}{\alpha_2} + \tfrac{1}{\alpha_3}\right)\alpha + \left(\tfrac{1}{\alpha_2^2} + \tfrac{1}{\alpha_2\alpha_3} + \tfrac{1}{\alpha_3^2}\right)\alpha^2 + \dots + \tfrac{A_2}{A_3}\alpha + \tfrac{A_2}{A_3}\left(\tfrac{1}{\alpha_2} + \tfrac{1}{\alpha_3}\right)\alpha^2 + \dots]$$

Finally, equation (27) can be expressed as

$$P^{(1)}(\alpha) = \frac{A_1}{\lambda\alpha_1}\left[1 + \frac{\alpha}{\alpha_1} + \left(\frac{\alpha}{\alpha_1}\right)^2 + \dots\right] + \frac{A_3}{\lambda^2\alpha_2\alpha_3}[1 + \left(\frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{A_2}{A_3}\right)\alpha + \left(\frac{1}{\alpha_2^2} + \frac{1}{\alpha_2\alpha_3} + \frac{1}{\alpha_3^2} + \frac{A_2}{A_3}.\left(\frac{1}{\alpha_2} + \frac{1}{\alpha_3}\right)\right)\alpha^2 + \dots] \tag{28}$$

Similarly, from equations (20) and (15)we get,

$$P^{(2)}(\alpha) = \frac{B_1}{\lambda\alpha_1}\left[1 + \frac{\alpha}{\alpha_1} + \left(\frac{\alpha}{\alpha_1}\right)^2 + \dots\right] + \frac{B_3}{\lambda^2\alpha_2\alpha_3}[1 + \left(\frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{B_2}{B_3}\right)\alpha + \left(\frac{1}{\alpha_2^2} + \frac{1}{\alpha_2\alpha_3} + \frac{1}{\alpha_3^2} + \frac{B_2}{B_3}.\left(\frac{1}{\alpha_2} + \frac{1}{\alpha_3}\right)\right)\alpha^2 + \dots] \tag{29}$$

where

$$B_1 = \frac{\lambda\mu_1 p\theta P^{(0)}}{(\lambda+\theta)(\theta-\mu_1-\mu_2)+\mu_1(\lambda q+\mu_2)}$$
$$B_2 = \frac{\lambda^2\mu_1 p\theta P^{(0)}}{(\lambda+\theta)(\theta-\mu_1-\mu_2)+\mu_1(\lambda q+\mu_2)}$$
$$B_3 = \frac{\lambda\mu_1 p\theta P^{(0)}(\theta-\mu_1-\mu_2)}{(\lambda+\theta)(\theta-\mu_1-\mu_2)+\mu_1(\lambda q+\mu_2)} \quad (30)$$

and

$$P^{(0)}(\alpha) = \frac{\theta P^{(0)}}{\lambda\alpha_1}\left[1+\frac{\alpha}{\alpha_1}+\left(\frac{\alpha}{\alpha_1}\right)^2+\ldots\right] \quad (31)$$

The solutions $P_m^{(1)}$, $P_m^{(2)}$ and $P_m^{(0)}$ for $m = 0, 1, 2, \ldots$ can be obtained by picking up the coefficients of various powers of α in the right hand expressions of equations (28), (29) and (31) respectively. Thus, the general solution is

$$P_m^{(1)} = \frac{A_1}{\lambda\alpha_1}\left(\frac{1}{\alpha_1}\right)^m + \frac{A_3}{\lambda^2\alpha_2\alpha_3}\sum_{i=0}^{m}\frac{1}{\alpha_2^{m-i}\alpha_3^i} + \frac{A_2}{\lambda^2\alpha_2\alpha_3}\sum_{i=0}^{m-1}\frac{1}{\alpha_2^{m-1-i}\alpha_3^i} \quad (32)$$
$$P_m^{(2)} = \frac{B_1}{\lambda\alpha_1}\left(\frac{1}{\alpha_1}\right)^m + \frac{B_3}{\lambda^2\alpha_2\alpha_3}\sum_{i=0}^{m}\frac{1}{\alpha_2^{m-i}\alpha_3^i} + \frac{B_2}{\lambda^2\alpha_2\alpha_3}\sum_{i=0}^{m-1}\frac{1}{\alpha_2^{m-1-i}\alpha_3^i} \quad (33)$$
$$P_m^{(0)} = \frac{\theta P^{(0)}}{\lambda\alpha_1}.\left(\frac{1}{\alpha_1}\right)^m \quad (34)$$

where A_1, A_2, A_3, B_1, B_2 and B_3 are given in equation (24) and (30). It can be seen that, the solutions given in (32), (33), (34) and (21) satisfies the system equations from (1) to (8) for $n = 0$ and $b = 1$.

Particular case

If $\theta = 0$ in equation (1)-(8), then the above system coincides with the system that was considered by Madan[12].

6 Operating Characteristics

Let $L_q^{(0)}$ denote the average queue length of the waiting units when the server is on vacation, $L_q^{(1)}$ and $L_q^{(2)}$ respectively denote the average queue lengths of the waiting units when the server is busy with the first and the second service respectively. Then for $i = 0, 1, 2$.

$$L_q^{(i)} = \sum_{m=0}^{\infty} mP_m^{(i)} = \frac{d}{d\alpha}P^{(i)}(\alpha) \ at \ \alpha = 1 \quad (35)$$

Using (35) in (15), (19) and (20) we have

$$L_q^{(0)} = \frac{\lambda}{\theta}\left[1-\frac{\lambda}{\mu_1}\left(\frac{\mu_1 p}{\mu_2}+1\right)\right] \quad (36)$$

$$L_q^{(1)} = \left[\frac{\lambda^2}{\mu_1^2}\left[\frac{\lambda\mu_1 p}{\mu_2^2}+1\right]+\frac{\lambda^2}{\mu_1}.\frac{P^{(0)}}{\theta}\right] / \left[1-\frac{\lambda}{\mu_1}\left(\frac{\mu_1 p}{\mu_2}+1\right)\right] \quad (37)$$

$$L_q^{(2)} = \left[\frac{\lambda^2 p}{\mu_1\mu_2^2}\left[\mu_1+\mu_2-\lambda\right]+\frac{\lambda^2 p}{\mu_2}.\frac{P^{(0)}}{\theta}\right] / \left[1-\frac{\lambda}{\mu_1}\left(\frac{\mu_1 p}{\mu_2}+1\right)\right] \quad (38)$$

Finally, the average queue length is

$$\begin{aligned} L_q &= L_q^{(0)}+L_q^{(1)}+L_q^{(2)} \\ &= \frac{\frac{\lambda}{\theta}\left[1-\frac{\lambda}{\mu_1}\left(\frac{\mu_1 p}{\mu_2}+1\right)\right]^2+\left[\frac{\lambda^2}{\mu_1^2}\left[\frac{\lambda\mu_1 p}{\mu_2^2}+1\right]+\frac{\lambda^2}{\mu_1}\frac{P^{(0)}}{\theta}\right]+\left[\frac{\lambda^2 p}{\mu_1\mu_2^2}[\mu_1+\mu_2-\lambda]+\frac{\lambda^2 p}{\mu_2}\frac{P^{(0)}}{\theta}\right]}{1-\frac{\lambda}{\mu_1}\left(\frac{\mu_1 p}{\mu_2}+1\right)} \end{aligned} \quad (39)$$

For the particular case when no units demand second service, we have on setting $p = 0$ in equation (39)

$$L_q = \frac{\frac{\lambda^2}{\mu_1^2}}{1-\frac{\lambda}{\mu_1}}+\frac{\lambda}{1-\frac{\lambda}{\mu_1}}.\frac{p^{(0)}}{\theta} \quad (40)$$

Which is the well known result for the $M/M/1$ queue with multiple vacation.
Using Little's formula $L_q = \lambda W_q$, we can find that W_q, the average waiting time in the queue, is given by

$$W_q = \frac{\frac{1}{\theta}\left[1-\frac{\lambda}{\mu_1}\left(\frac{\mu_1 p}{\mu_2}+1\right)\right]^2+\left[\frac{\lambda}{\mu_1^2}\left[\frac{\lambda\mu_1 p}{\mu_2^2}+1\right]+\frac{\lambda}{\mu_1}\frac{P^{(0)}}{\theta}\right]+\left[\frac{\lambda p}{\mu_1\mu_2^2}[\mu_1+\mu_2-\lambda]+\frac{\lambda p}{\mu_2}\frac{P^{(0)}}{\theta}\right]}{1-\frac{\lambda}{\mu_1}\left(\frac{\mu_1 p}{\mu_2}+1\right)} \quad (41)$$

For the particular case when $p = 0$ equation (41) reduces to

$$W_q = \frac{\frac{\lambda}{\mu_1^2}}{1-\frac{\lambda}{\mu_1}}+\frac{1}{1-\frac{\lambda}{\mu_1}}\frac{p^{(0)}}{\theta} \quad (42)$$

which is a well-known steady state result for $M/M/1$ queue with multiple vacation.

7 Numerical Study

In this section we numerically analyse the model defined in this paper for the arbitrary values of λ, μ_1, μ_2 and various values of p and θ taken at the intervals of [0,1]. The dependence is then shown graphically in figures 1 to 5. Figure 1 clearly shows the linear dependence of the utilization factor ρ versus p whereas figure 2 and 3 shows respectively the dependence of L_q and W_q versus p by increasing curves. Figure 4 and 5 shows respectively the dependence of L_q and W_q versus θ by decreasing curves.

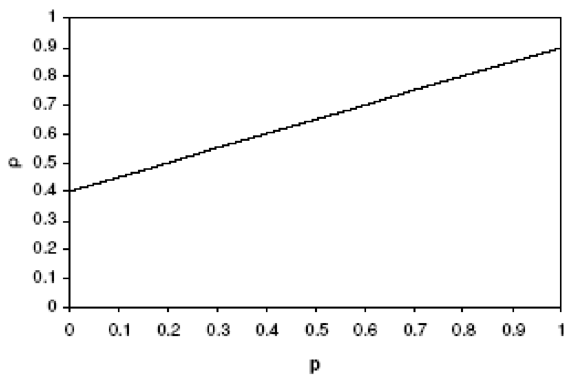

Figure 1. p versus ρ

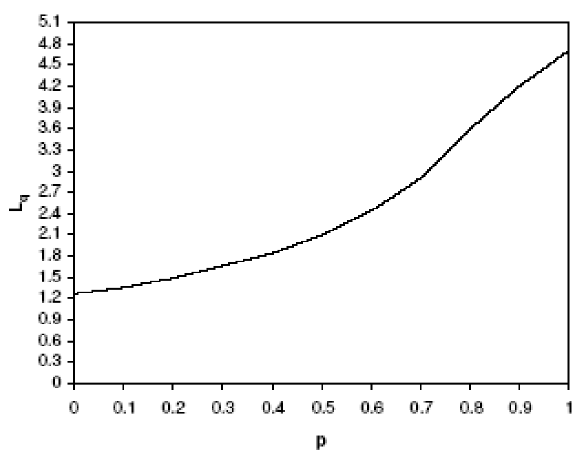

Figure 2. p versus L_q

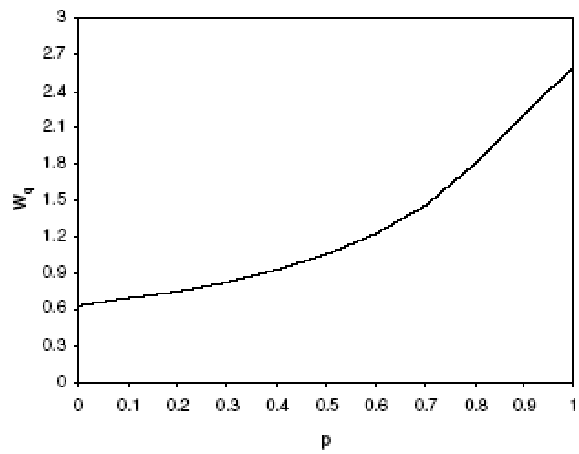

Figure 3. p versus W_q

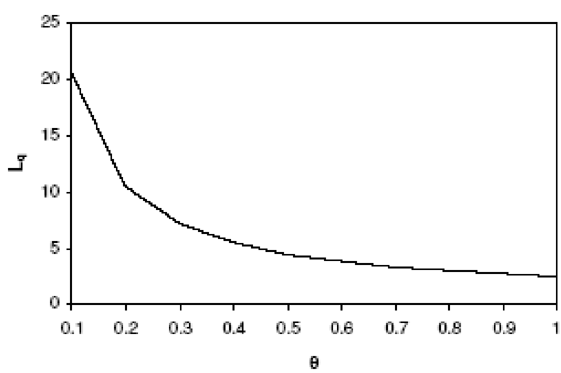

Figure 4. θ versus L_q

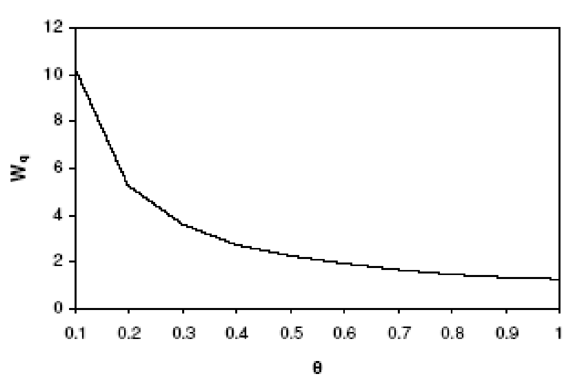

Figure 5. θ versus W_q

8 Conclusion

In the fore going analysis, an $M/M/1$ queue with additional optional service in batches and with server vacation is considered to obtain queue length distribution and mean queue length. Extensive numerical work has been carried out to observe the trends of the operating characters of the system.

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