Applied Mathematical Sciences, Vol. 2, 2008, no. 58, 2859 - 2866

Expanded Lie Group Transformations

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Abstract

Continuous groups of transformation acting on the expnded space of variables, which includes the equation parameters in addition to independent and dependent variables, are considered. It is shown that use of the expanded transformations enables one to enrich the concept Lie theory of symmetry group for partial differential equations.

Mathematics Subject Classification: 22E70, 35A30, 58J70

Keywords: Lie group, Lie algebra, symmetry, optimal system, invariant, adjoint representation

1 Introduction

In applied group analysis, Lie theory of symmetry group for differential equations, constituted by Sophus Lie, is the most important solution method for the nonlinear problems in the field of applied maths. The fundamentals of Lie's theory are based on the invariance of the equation under transformation groups of independent and dependent variables, so called Lie groups. This approach is used to analyisis the symmetries of the differential equations and may be apoint, a contact, and a generalized or nonlocal symmetry. In the last century, the application of the Lie group method has been developed by a number of mathematicians. Ovsiannikov [7], Olver [15], Ibragimov [9], [10], Baumann [2] and Bluman and Anco [5] are some of the mathematicians who have enormous amount of studies in this field.

The existence of symmetries of diferential equations under Lie group of transformations often allows those equations to be reduced to simpler equations. One of the major accomlishment of Lie was to identify that the properties of global transformations of the group are completely and uniquely determined by the infinitesimal transformations around the identity transformation. This allows the nonlinear relations for the identification of invariance groups to be dealing with global transformation equations, we use differential operators, called the group generators, whose exponentiation generates the action of the group. The collection of these differentail operators forms the basis for the Lie algebra. There is a one-to-one correspondence between the Lie groups and the associated Lie algebras.

A basic problem concerning the group invariant solution is its classification. Since a Lie group (or Lie algebra) usually contain inifitely many subgroups (or subalgebras) of the same dimensional, a classification of them up to some equivalence relation is necessary. Ovsiannikov [7] given equivalent of two subalgebras of a given Lie algebra. Optimal system consists of representative elements of each equality class. Disussion on optimal systems can be found in [15], [7]. Some examples of optimal system can also be found in Ibragimov [10].

Among those transformation groups, an expanded Lie group transformation of a partial differential equation is a continuious group transformation which is acting on expanded space of variables that includes the equation parameters in addition to independent and dependant variables. The transformations can be found using the Lie infinitesimal criterion with groups of point transfomations, leaving aside problems involving generalized symmetry groups.

An expanded group of transformations represents a particular case of the equivalence group that presrves the class of partial differential equations which holds the same structure. The approach to find these equivalence transformation groups with th use of the Lie infinitesimal tecnique was introduced by Ovasiannikov [7] who suggested using the Lie infinitesimal principle in the properly extend space of variables which include dependent and independent variables, arbitrary functions and their derivatives. This idea, later, extended by Akhatov *et al* [1]. More recently, Burde [6] used the Lie groups of transformations in the expanded space of variables including equation parameters enables one to enrich the cocept of similarity reductions as applied to partial differential equations. And also he used these groups for finding changes of variables that remove some terms from the original equation.

In this paper, we have used an Expande Lie group transformation and similarity reduction to obtain the exact analytical solution of Generalized Boussinesq Equation.

2 Application of the expanded transformations

The example applies the tecnique to the Generalised Boussinesq (GBQ) Equation

$$u_{xxxx} + pu_t u_{xx} + qu_x u_{xt} + ru_x^2 u_{xx} + u_{tt} = 0 \tag{1}$$

where p,q and r are constants such that $r \neq 0$ and subscripts denote partial derivatives.

Clasical symmetry reductions of some special cases of equation (1) have been discussed by Schwarz [18], Clarkson [13], Kawamoto [16], Lou and Ni [17], Paquin and Winternitz [4]. Clasical symmetrys of some different type of equation (1) have been investigated by Clarkson and Priestly [14], Gandarias and Bruzon [8]. Clarkson and Kruskal [11] developed a Direct method (in the sequel referred as the Direct method) for finding symmetry reductions which is used to obtain previously unknown reductions of the Boussinesq Equation and Clarkson and Ludlow [12] derived smmetry reductions of GBQ by using the Direct method and said that those derived by using the Lie group method with one illustration. Recently Burde [6] showed that the use the Lie groups of transformation in the expanded space of variables including parameters equation improved the concept of similarity reductions as applied to partial differential equations.

In this paper our main motivation and starting point based on Burde's paper [6], is to demonstrate that the procedure of symmetry reduction implemented in the expanded space which reduces GBQ systematically to a previously unknown target ordinary differential equation by the suitable choice of expanded group transformaton.

To illustrate the process we introduce a cofficient (parameter) into the equation "artifically", for example, in front of the last term equation (1), i.e.

$$u_{xxxx} + pu_t u_{xx} + qu_x u_{xt} + ru_x^2 u_{xx} + au_{tt} = 0. \tag{2}$$

It may appear that introducing this kind of coeffient makes useless the physics of the problem, but one may always set a=1 in the final stage. Currently for convenience we choose the coefficients

$$p = q, \quad r = \frac{q^2}{2a}$$

and rewrite equation (2) as

$$u_{xxxx} + qu_t u_{xx} + qu_x u_{xt} + \frac{q^2}{2a} u_x^2 u_{xx} + au_{tt} = 0. \tag{3}$$

To apply the classical Lie symmetry group method to equation (3), we perform symmetry analysis. Let us consider a one-parameter Lie group of infinitesimal transformation

$$\begin{aligned} x &\to x + \varepsilon\xi_1(x,t,u,q,a) + O(\varepsilon^2) \\ t &\to t + \varepsilon\xi_2(x,t,u,q,a) + O(\varepsilon^2) \\ t &\to t + \varepsilon\xi_2(x,t,u,q,a) + O(\varepsilon^2) \end{aligned} \tag{4}$$

$$q \to q + \varepsilon Q(x,t,u,q,a) + O(\varepsilon^2)$$
$$a \to a + \varepsilon A(x,t,u,q,a) + O(\varepsilon^2)$$

where ε is group parameter in the expanded (x,t,u,q,a) space. The vector field associated with the above group of transformations can be written as

$$X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u} + Q(q)\frac{\partial}{\partial q} + A(a)\frac{\partial}{\partial a}. \tag{5}$$

This is symmetry generator and invariance of equation (3) under transformation (4). The associated Lie algebra of the infinitesimal system involves the set of vector fields of this form.

The symmetry condition

$$pr^{(4)} X\Delta \mid_{\triangle} = 0$$

yields an overdetermined system of PDE for the unknown functions ξ_1, ξ_2 and η where Δ is the manifold defined by (3) in jet space $J^{(3)}$ and $pr^{(4)}X$ is the fourth prolongation of X. We obtain this system by using package MathLie [2] and this system can solve if provide following system

$$(\xi_1)_u = 0, (\xi_2)_u = 0, (\xi_2)_x = 0, (\eta)_{uu} = 0$$
$$-6(\xi_1)_{xx} + 4(\eta)_{xu} = 0$$
$$-q(\xi_1)_{xx} + 3q(\eta)_{xu} = 0.$$

Then we obtain $(\xi_1)_{xx} = (\eta)_{xu} = 0$ and so $(\eta)_t = (\xi_1)_{tt} = 0$. Thus the determining equations can be obtained as:

$$\frac{A}{a} - 2(\xi_2)_t + (\eta)_u = 0$$
$$\frac{Q}{q} - 2(\xi_1)_x - (\xi_2)_t + 2(\eta)_u = 0$$
$$-\frac{A}{2a} + \frac{Q}{q} - 2(\xi_1)_x + \frac{3}{2}(\eta)_u = 0$$
$$-4(\xi_1)_x + (\eta)_u = 0$$
$$-2a(\xi_1)_t + q(\eta)_x = 0$$
$$-a(\xi_2)_{tt} + q(\eta)_{xx} = 0$$
$$-2a(\xi_1)_{xt} + q(\eta)_{xx} = 0$$

The resulting system of equations easily be solved to give the infinitesimals

$$\xi_1 = C_1xt + C_0x + C_3t + C_4$$

$$\xi_2 = C_1 + t^2 + 3C_0 t + C_2$$
$$\eta = 4C_0 u + \frac{a}{q} C_1 x^2 + \frac{2a}{q} C_3 x + C_5 \tag{6}$$
$$Q = -3qC_0$$
$$A = 2aC_0$$

which includes the determination of the generators A and Q. Here it is worth to note that not only the generators A and Q but also the coefficients C_0, C_1, C_2, C_3, C_4 and C_5 are depended on the parameters a and q. The symmetry variables are then found by solving the characteristc equations

$$\frac{dx}{\xi_1} = \frac{dt}{\xi_2} = \frac{du}{\eta} = \frac{da}{A} = \frac{dq}{Q}$$

and then, substituting the resulting expression into (3), one obtains the reduced equation. Hence, symmetry generator of equation (3) is

$$X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u} + Q \frac{\partial}{\partial q} + A \frac{\partial}{\partial a}.$$

The presence of these arbitrary constants lead to a finite-dimensional Lie algebra of symmetries. A general element of this algebra is written as

$$U = a_0 X_0 + a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4 + a_5 X_5$$

where

$$\begin{aligned} X_0 &= x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} + 4u\frac{\partial}{\partial u} - 3q\frac{\partial}{\partial q} + 2a\frac{\partial}{\partial a}, \\ X_1 &= xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} + \frac{a}{q}x^2\frac{\partial}{\partial u}, \quad X_2 = \frac{\partial}{\partial t}, \\ X_3 &= t\frac{\partial}{\partial x} + \frac{2a}{q}x\frac{\partial}{\partial u}, \quad X_4 = \frac{\partial}{\partial x}, \quad X_5 = \frac{\partial}{\partial u} \end{aligned}$$

construct a basis of vector space. The Lie algebra of infinitesimal symmetries for equation (3) is spanned by these base vectors. Lie algebra admited by (3)

$$\begin{aligned} L_6 &= \{X_0 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} + 4u\frac{\partial}{\partial u} - 3q\frac{\partial}{\partial q} + 2a\frac{\partial}{\partial a}, \ X_1 = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} + \frac{a}{q}x^2\frac{\partial}{\partial u}, \\ X_2 &= \frac{\partial}{\partial t}, X_3 = t\frac{\partial}{\partial x} + \frac{2a}{q}x\frac{\partial}{\partial u}, \quad X_4 = \frac{\partial}{\partial x}, \quad X_5 = \frac{\partial}{\partial u}\}. \end{aligned}$$

3 One-Parameter Optimal System

We can find a family similarity solutions using one-dimensional subalgebras which spannd by $X_0,\ X_1,\ X_2,\ X_3,\ X_4,\ \ X_5$.In this set there are similarity solutions which result from other similarity solutions of the same set applying a transformation of the symmetry group. It would be profitable to heve a minimal list of similarity solutions such that with these elements one can get all other similarity solutions via transformation. Such a minimal list is called an one-parameter optimal system and their elements are essentially different similarity soltions.

The construction of the one-paremeter optimal system of one-dimensional subalgebras can be made by using a global matrix of the adjoint transformations as suggested by Ovsiannikov [7]. In this paper we follow, instead, the method by Olver [15] which uses a slightly different technique. The problem of deriving an optimal system of subalgebras spanned by these operators is equivalent to find an optimal system of Lie symmetries (or group invariant solutions). This is possible because there is a connection of the Lie group and adjoint representation of Lie algebra. We constitute adjoint representation for to define an equivalence relation on one-dimensional subalgebras which generate by X_0, X_1, X_2, X_3, X_4, X_5.

$$Ad(\exp(\varepsilon X_i))X_j = X_j - \varepsilon[X_i, X_j] + \frac{1}{2}\varepsilon^2[X_i[X_i, X_j]] - ...$$

where $[X_i, X_j]$ is the usual commutator, given by

$$[X_i, X_j] = X_iX_j - X_jX_i.$$

Let $L_{1,1}$ and $L_{1,2}$ are one-dimensional subalgebras of L_6 and G is lie group with the corresponding Lie algebra L_6 . If there exists an inner automorphism $Ad(g)\epsilon Int(G)(g\epsilon G)$ for the related subalgebras

$$L_{1,1} = Ad(g)(L_{1,2})$$

then similarity solution of $L_{1,1}$ is equivalent similarity solution of $L_{1,2}$.Thus we need subalgebras of G with the corresponding one-dimensional subalgebras which spannd by X_0, X_1, X_2, X_3, X_4, X_5 for to constitute adjoint table. The one-paremeter subgroups which corresponding generators X_0, X_1, X_2, X_4, X_5 are

$$G_0 : (x, t, u, q, a) \longrightarrow (e^{\varepsilon_0}x, e^{3\varepsilon_0}t, e4^{\varepsilon_0}u, e^{-3\varepsilon_0}q, e^{2\varepsilon_0}a)$$

$$G_1 : (x, t, u, q, a) \longrightarrow (x, t + \varepsilon_2, u, q, a)$$

$$G_2 : (x, t, u, q, a) \longrightarrow (x - e^{-\varepsilon_1}, t - \frac{1}{\varepsilon_1}, u - \frac{a}{q}e^{-2}\varepsilon_1, q, a)$$

$$G_4 : (x, t, u, q, a) \longrightarrow (x + \varepsilon_4, t, u, q, a)$$
$$G_4 : (x, t, u, q, a) \longrightarrow (x, t, u + \varepsilon_5, q, a).$$

But the one-paremeter subgroup which corresponding generator is not determined with to solve ordinary differential equation system which is

$$\frac{d\overline{x}}{d\varepsilon} = \overline{t}, \quad \frac{d\overline{t}}{d\varepsilon} = 0, \ \frac{d\overline{u}}{d\varepsilon} = \frac{2a}{q}x, \ \frac{d\overline{q}}{d\varepsilon} = 0, \ \frac{d\overline{a}}{d\varepsilon} = 0$$

with $\overline{x}\mid_{\varepsilon=0}= x$, $\overline{t}\mid_{\varepsilon=0}= t$, $\overline{u}\mid_{\varepsilon=0}= u$, $\overline{q}\mid_{\varepsilon=0}= q$, $\overline{a}\mid_{\varepsilon=0}= a$ [9]. Thus optimal system is not determined.

4 Concluding remarks

In this paper, we have demonstrated that new application of the Lie group method to partial differential equations may be arise due to the use of expanded transformation groups.

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Received: August 13, 2008