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Finite Dimensional Approximations for Mixed Variational Inequalities with J-Pseudomonotone Operators

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Abstract

In this paper, we construct and study the convergence of the finite dimensional approximation for the mixed variational inequalities with J-pseudomonotone operators and convex nondifferentiable functionals in real uniformly smooth Banach spaces which admit a weakly sequentially continuous duality map.

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1 Introduction and Preliminaries

Let V be a real Banach space with uniformly convex dual space V^*. we denote by $\langle .,. \rangle$ the duality pairing between V and V^*. The modulus of smoothness of V is defined by:

$$\rho_V(\tau) = \sup\{\frac{\|u + \tau v\| + \|u - \tau v\|}{2} - 1 : \ \|u\| = \|v\| = 1\}; \ \tau > 0.$$

It is well known that $\rho_V(\tau) \leq \tau \ \forall \ \tau > 0$ (see e.g.[8]). If $\rho_V(\tau) > 0 \ \forall \ \tau > 0$, then V is said to be smooth. If there exist a constant $C > 0$ and a real number $1 < q < \infty$, such that $\rho_V(\tau) \leq C\tau^q$, then V is said to be q-uniformly smooth. The Banach space V is called uniformly smooth if $\lim_{\tau \to 0} \frac{\rho_V(\tau)}{\tau} = 0$.

Typical examples of such p-uniformly smooth spaces are the Lebesgue L_p, the sequence l_p and the Sobolev $W_p^{(s)}$ spaces for $1 < p \leq 2$ (see e.g.[7]). For a given gauge function $\Phi(t) = t^{p-1}, \ 1 < p < \infty$, this means for a mapping $\Phi : R^+ \to R^+$ which is continuous and strictly increasing with $\Phi(0) = 0$ and $\lim_{t \to +\infty} \Phi(t) = +\infty$, the duality mapping $J : V \to V^\star$ with respect to Φ is given by

$$\langle u, Ju \rangle = \|u\| \|Ju\|, \quad \|Ju\| = \|u\|^{p-1}$$

for all $u \in V$. Such a mapping J is said to be weakly sequentially continuous if J is sequentially continuous relative to the weak topologies on both V and $V^\star$. The spaces $l_p, \ \ 1 < p < \infty$, posses duality mappings which are weakly sequentially continuous (see e.g.[2]). In [6] Opial, showed that no spaces $L_p, \ p > 1, \ p \neq 2$, possesses a weakly sequentially continuous duality mapping. It is well known (see e.g.[8],[4]) that J is single-valued odd, and is uniformly continuous on bounded sets if $V^\star$ is uniformly convex.

Therefore, we always suppose that V is a q-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and its dual space $V^\star$ is uniformly convex.

We always use the symbols $\to$ and $\rightharpoonup$ to indicate strong and weak convergence, respectively.

Let M be a nonempty closed convex set in V and F be a convex (not necessarily differentiable) functional on V. Given $A : V \to V$ nonlinear operator, we consider the problem of finding $u \in M$ such that

$$\langle Au, J(\eta - u) \rangle + F(\eta) - F(u) \geq \langle f, J(\eta - u) \rangle \ \ \forall \ \eta \in M, \tag{1}$$

where $f \in V$ is a given element. The inequality of the type (1) is called the mixed variational inequality.

Throught the following, we assume that the variational inequality (1) has a solution $u \in M$. To this end, it suffices, to assume that A is J-coercive, i.e.,

$$\langle Au, Ju \rangle \geq \rho_1(\|u\|)\|u\|^{p-1}, \quad p > 1, \quad \lim_{\xi \to +\infty} \rho_1(\xi) = +\infty;$$

and J-pseudomonotone, i.e.,

$$if \ u_n \rightharpoonup u \in M \ and \ \lim_{n \to +\infty} \sup \langle Au_n, J(u_n - u) \rangle \leq 0$$

imply

$$\lim_{n\to+\infty}\inf\langle Au_n, J(u_n - v)\rangle \geq \langle Au, J(u - v)\rangle \quad forall\ v \in M,$$

and J-Lipschitz Continuous, i.e., there exists a constant $L > 0$ such that

$$\|Au - Av\|^{q-1} \leq L^{q-1}\|J(u - v)\|, \quad q > 1,$$

and that the functional F satisfies the following condition:

$$|F(u) - F(v)| \leq \gamma\|u - v\|^{q-1}, \quad \forall\ u, v \in V,\ q > 1,\ \gamma > 0. \tag{2}$$

We note that if $V = H$ ($J \equiv I$, the identity operator and $q = 2$), then the above definition reduces to the standard definition of coercivity, pseudomonotonicity and Lipschitz continuity of the operator A (see e.g. [4]).

Consequently, the problem (1) becomes

$$\begin{aligned} &find\ u \in M\ such\ that,\ forall\ \eta \in M \subset H \\ &\quad \langle Au, \eta - u\rangle + F(\eta) - F(u) \geq \langle f, \eta - u\rangle \end{aligned} \tag{3}$$

where $f \in H$ is a given element.

This problem occurs, in particular, in the descriptions of stabilized filteration and equilibrium problems for soft shells (see [1], [5]).

The basic scope of this paper is to introduce the finite dimensional approximation and analyze its convergence for the mixed variational inequality (1).

2 An Internal Approximation

Let V_h be a finite dimensional subspace of the space V. Suppose V_h is an internal approximation of V:

$\forall\ v \in V$, there exists $v_h \in V_h$ such that $v_h \to v$ when $h \to 0$.

For each h we consider a nonempty closed convex subset $M_h \subset V_h$ (observe that, in general, the set M_h is not a subset of M) approximating M:

$$\forall\ \eta \in M,\ there\ exists\ \eta_h \in M_h\ such\ that\ \eta_h \to \eta\ when\ h \to 0, \tag{4}$$

and

$$if\ \eta_h \in M_h,\ \eta_h \rightharpoonup \eta\ in\ V\ when\ h \to 0,\ then\ \eta \in M. \tag{5}$$

It is obvious that (5) is satisfied if $M_h \subset M$ for all h. Indeed, in this case $\{\eta_h\} \subset M$ and M is closed and convex (hence weakly closed), it follows that $\eta \in M$.

The approximate problem corresponding to problem (1) is the following problem:

$$\begin{aligned} &find\ u_h \in M_h\ such\ that,\ forall\ \eta_h \in M_h \\ &\langle Au_h, J(\eta_h - u_h)\rangle + F(\eta_h) - F(u_h) \geq \langle f, J(\eta_h - u_h)\rangle. \end{aligned} \tag{6}$$

Under the conditions imposed on M_h, A, and F, there is a solution $u_h \in M_h$ to (6).

We now establish the convergence of the finite dimensional approximation in the following sense:

Lemma 1 *Let $F : V \to R^1$ be a convex continuous functional and let the operator $A : V \to V$ be J-coercive. If u_h is the solution of problem (6), then*

$$\|u_h\|_V \leq \tilde{c}, \tag{7}$$

where the constant $\tilde{c}$ is independent of h.

Proof. Since F is convex and continuous, it is bounded below by a continuous affine function (see [3]) which can be written:

$$\langle f^\star, J\eta\rangle - C^\star, \quad where\ f^\star \in V,\ \eta \in V\ and\ C^\star \in R^1. \tag{8}$$

Suppose that the assertion of the lemma is not true. In this case for each $N > 0$ we find h_N such that $\|u_{h_N}\|_V \geq N$. Let $\alpha_N = \frac{1}{\|u_{h_N}\|_V}$. Clearly $\alpha_N \in (0, 1)$ for $N \geq 2$. We may assume without less of generality that $0 \in M_h$. Therefore $\alpha_N u_{h_N} \in M_h$. Putting $\eta_{h_N} = \alpha_N u_{h_N}$ in (6), using (8) and the fact that $J(\beta u) = \beta|\beta|^{p-2}J(u)\ \ \forall\ u \in V,\ p > 1,\ \beta \in R^1$, we get

$$\begin{aligned} F(\eta_{h_N}) - \langle f, J\eta_{h_N}\rangle &\geq \langle Au_{h_N}, J(u_{h_N} - \eta_{h_N})\rangle + F(u_{h_N}) + \langle f, J(\eta_{h_N} - u_{h_N}\rangle - \langle f, J\eta_{h_N}\rangle \\ &\geq \langle Au_{h_N}, J(u_{h_N} - \eta_{h_N})\rangle + F(u_{h_N}) - \langle f, Ju_{h_N}\rangle \\ &\geq (1-\alpha_N)|1-\alpha_N|^{p-2}\langle Au_{h_N}, Ju_{h_N}\rangle + \langle f^\star, Ju_{h_N}\rangle - C^\star - \|f\|_V\|Ju_{h_N}\|_{V^\star} \\ &\geq (1-\alpha_N)|1-\alpha_N|^{p-2}\langle Au_{h_N}, Ju_{h_N}\rangle - (\|f^\star\|_V + \|f\|_V)\|Ju_{h_N}\|_{V^\star} - C^\star \\ &= (1-\alpha_N)|1-\alpha_N|^{p-2}\langle Au_{h_N}, Ju_{h_N}\rangle - (\|f^\star\|_V + \|f\|_V)\|u_{h_N}\|_V^{P-1} - C^\star. \end{aligned}$$

Further,

$$\|\eta_{h_N}\|_V = \alpha_N\|u_{h_N}\|_V = 1,$$

consequently, by the continuity of F and the weak continuity of J, we have

$$F(\eta_{h_N}) - \langle f, J\eta_{h_N}\rangle \leq \tilde{C} < +\infty,$$

where the constant $\tilde{C}$ is independent of h.

On the other hand, by the J-coercivity of the operator A, for sufficiently large N we obtain

$$\begin{aligned}&(1-\alpha_N)|1-\alpha_N|^{p-2}\langle Au_{h_N}, Ju_{h_N}\rangle - (\|f^\star\|_V + \|f\|_V)\|u_{h_N}\|_V^{p-1} - C^\star \\ &\geq [\frac{1}{2^{p-1}}\rho_1(\|u_{h_N}\|_V) - (\|f^\star\|_V + \|f\|_V)]\|u_{h_N}\|_V^{p-1} - C^\star \to +\infty\end{aligned}$$

which is a contradiction and the assertion of the Lemma is proved.

Theorem 1 *If u_h and u denote the solutions of problems (6) and (1) respectively, there exists a subsequence $\{h_k\}_{k=1}^{\infty}, h_k \to 0$ as $k \to \infty$, such that $\{u_{h_k}\}_{k=1}^{\infty}$ converges weakly to u in V and any weak limit point $u_\star$ of $\{u_h\}$ is a solution of problem (1). Moreover, if $\{u_{h_k}\}_{k=1}^{\infty}$ converges weakly to $u_\star$ in V, then*

$$\lim_{k\to\infty}\langle Au_{h_k} - Au_\star, J(u_{h_k} - u_\star)\rangle = 0. \tag{9}$$

Proof. From Lemma 1, it follows that $\{u_h\}$ is bounded and consequently there exists a subsequence $\{h_k\}_{k=1}^{\infty},\ h_k \to 0$, as $k \to \infty$, such that $u_{h_k} \rightharpoonup u_\star \in V$ as $k \to \infty$.

Let us show that $u_\star$ is a solution of problem (1). By virtue of (5), we have $u_\star \in M$. let η be an arbitrary element of M and let $\eta_h \in M_h$ be constructed according to (4). In this case, the sequence $\{\eta_h\}$, converges strongly to η in V when $h \to 0$. Since u_{h_k} is a solution of problem (6), then

$$\langle Au_{h_k}, J(\eta_{h_k} - u_{h_k})\rangle + F(\eta_{h_k}) - F(u_{h_k}) \geq \langle f, J(\eta_{h_k} - u_{h_k}\rangle,$$

consequently,

$$\begin{aligned}\langle Au_{h_k}, J(u_{h_k} - \eta)\rangle &\leq \langle Au_{h_k}, J(u_{h_k} - \eta_{h_k})\rangle + \langle Au_{h_k}, J(\eta_{h_k} - \eta)\rangle \\ &\leq \langle Au_{h_k}, J(\eta_{h_k} - \eta)\rangle + F(\eta_{h_k}) - F(u_{h_k}) - \langle f, J(\eta_{h_k} - u_{h_k})\rangle \\ &\leq \|Au_{h_k}\|_V\|\eta_{h_k} - \eta\|_V^{p-1} + F(\eta_{h_k}) - F(u_{h_k}) - \langle f, J(\eta_{h_k} - u_{h_k})\rangle \\ &\leq \tilde{C}_1\|\eta_{h_k} - \eta\|_V^{p-1} + F(\eta_{h_k}) - F(u_{h_k}) - \langle f, J(\eta_{h_k} - u_{h_k})\rangle.\end{aligned}$$

On passing to the lim sup in the above formula we conclude

$$\begin{aligned}\limsup_{k\to+\infty}\langle Au_{h_k}, J(u_{h_k}-\eta)\rangle &\le \tilde{C}_1 \limsup_{k\to+\infty}\|\eta_{h_k}-\eta\|_V^{p-1} + \limsup_{k\to+\infty} F(\eta_{h_k}) \\ &\quad - \liminf_{k\to+\infty} F(u_{h_k}) - \liminf_{k\to+\infty}\langle f, J(\eta_{h_k}-u_{h_k})\rangle \\ &\le F(\eta) - F(u_\star) - \langle f, J(\eta-u_\star)\rangle,\end{aligned}$$

(since $\eta_{h_k} \to \eta,\ F$ is continuous as well weakly lower semicontinuous and since J is weakly continuous).

Thus,

$$\limsup_{k\to+\infty}\langle Au_{h_k}, J(u_{h_k}-\eta)\rangle \le F(\eta) - F(u_\star) - \langle f, J(\eta-u_\star)\rangle \quad \forall\, \eta \in M. \tag{10}$$

Putting $\eta = u_\star$ in (10), we obtain

$$\limsup_{k\to+\infty}\langle Au_{h_k}, J(u_{h_k}-u_\star)\rangle \le 0. \tag{11}$$

From the J-pseudomonicity of the operator A, we have

$$\begin{aligned}\langle Au_\star, J(u_\star-\eta)\rangle &\le \liminf_{k\to+\infty}\langle Au_{h_k}, J(u_{h_k}-\eta)\rangle \\ &\le F(\eta) - F(u_\star) - \langle f, J(\eta-u_\star)\rangle \quad \forall\, \eta \in M,\end{aligned} \tag{12}$$

that is $u_\star$ is a solution of (1).

In a similar way, one can show that any weak limit point of the set $\{u_h\}$ is a solution of the problem (1).

Finally, since $u_{h_k} \rightharpoonup u_\star$ in V and J is weakly continuous, it follows that

$$\lim_{k\to+\infty}\langle Au_\star, J(u_{h_k}-u_\star)\rangle = 0,$$

therefore, from (11) and (12), we have

$$\begin{aligned} 0 \leq \liminf_{k\to+\infty}\langle Au_{h_k}, J(u_{h_k}-u_\star)\rangle &= \liminf_{k\to+\infty}\langle Au_{h_k}, J(u_{h_k}-u_\star)\rangle \\ &\quad + \lim_{k\to+\infty}\langle -Au_\star, J(u_{h_k}-u_\star)\rangle \\ &\leq \liminf_{k\to+\infty}\langle Au_{h_k}-Au_\star, J(u_{h_k}-u_\star)\rangle \\ &\leq \limsup_{k\to+\infty}\langle Au_{h_k}-Au_\star, J(u_{h_k}-u_\star)\rangle \\ &\leq \limsup_{k\to+\infty}\langle Au_{h_k}, J(u_{h_k}-u_\star)\rangle \\ &\quad + \limsup_{k\to+\infty}\langle -Au_\star, J(u_{h_k}-u_\star)\rangle \\ &= \limsup_{k\to+\infty}\langle Au_{h_k}, J(u_{h_k}-u_\star)\rangle \\ &\leq 0, \end{aligned}$$

that is, (9) holds and the theorem is proved.

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