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The Results on Fixed Points in Dislocated and Dislocated Quasi-Metric Space

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Abstract

In this paper we have proved fixed point theorems for Kannan mappings and generalized contraction mappings in dislocated quasi-metric space. Also discussed a common fixed point theorem in complete dislocated metric space.

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1 Introduction

Banach[1922] proved a celebrated fixed point theorem for contractions mappings in complete metric space. It is well-known as a Banach fixed point theorem. It has many applications in various branches of mathematics such as differential equation, integral equation etc. But Kannan[1968] proved a fixed point theorem for new types of contraction mappings called Kannan mappings in a complete metric space. We have to note here that Kannan mapping may not be continuous. Lj. B. Ciric [1974] gave a generalization of Banach contraction principle in metric space space. In this paper we study the mapping refereed by Kannan and Lj. B. Ciric and obtained fixed point theorems in dislocated quasi-metric metric.

2 Preliminary Notes

Definition 2.1 *Let X be a nonempty set and let $d : X \times X \to [0, \infty)$ be a function satisfying following conditions:*
(i) $d(x, y) = d(y, x) = 0$, implies $x = y$,
(ii)$d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.
Then d is called a dislocated quasi-metric on X. If d satisfies $d(x, x) = 0$, then it is called a quasi-metric on X. If d satisfies $d(x, y) = d(y, x)$, then it is called dislocated metric.

Definition 2.2 *A sequence $\{x_n\}$ in dq-metric space (dislocated quasi-metric space) (X, d) is called Cauchy if for given $\epsilon > 0$, $\exists n_0 \in N$ such that $\forall m, n \geq n_0$, implies $d(x_m, x_n) < \epsilon$ or $d(x_n, x_m) < \epsilon$ i.e. $\min\{d(x_m, x_n), d(x_n, x_m)\} < \epsilon$.*

In above definition if we replace $d(x_m, x_n) < \epsilon$ or $d(x_n, x_m) < \epsilon$ by $\max\{d(x_m, x_n), d(x_n, x_m)\} < \epsilon$, the sequence $\{x_n\}$ is called 'bi'Cauchy. Note that every bi Cauchy sequence is Cauchy.

Definition 2.3 *A sequence $\{x_n\}$ dislocated quasi-converges to x if*

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0.$$

In this case x is called a dq-limit of $\{x_n\}$ and we write $x_n \to x$.

Proposition 2.4 *Every convergent sequence in a dq-metric space is 'bi 'Cauchy.*

Converse of proposition 2.4 may not be true. Proof of the following lemma is obvious.

Lemma 2.5 *Every subsequence of dq-convergent sequence to a point x_0 is dq-convergent to x_0*

Definition 2.6 *A dq-metric space space (X, d) is called complete if every Cauchy sequence in it is a dq-convergent.*

The notion of the dislocated topologies is useful in the context of logic programming. Recently, Zeyada *et al.*[2005] have established a fixed point theorem in a complete dislocated quasi-metric space, as stated in following lemma and theorem.

Lemma 2.7 *Let (X, d) be a dq-metric space. If $f : X \to X$ is a contraction function, then $\{(f^n(x_0))\}$ is a Cauchy sequence for each $x_0 \in X$.*

Theorem 2.8 *Let (X, d) be a complete dq-metric space and let $f : X \to X$ be a continuous contraction function. Then f has a unique fixed point.*

3 Main Results

The following definition is introduced by R. Kannan [1]

Definition 3.1 *A mapping $T : X \to X$ is called Kannan mapping if*

$$d(Tx, Ty) \leq \alpha\{d(x, Tx) + d(y, Ty)\}$$

for all $x, y \in X$ and $0 \leq \alpha < 1/2$.

The following definition is introduced by Lj. B. Ciric[2].

Definition 3.2 *A mapping $T : X \to X$ is said to generalized contraction iff for every $x, y \in X$ there exist non negative numbers $\alpha, \beta, \gamma,$ and δ, which may depends on both x and y, such that* $\sup\{\alpha + \beta + \gamma + 2\delta : x, y \in X\} < 1$ *and*

$$\begin{aligned} d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) \\ + \delta[d(x, Ty) + d(y, Tx)]. \end{aligned}$$

Theorem 3.3 *Let (X, d) be a complete dq-metric space. If $T : X \to X$ be a continuous mapping satisfying*

$$d(Tx, Ty) \leq \alpha\{d(x, Tx) + d(y, Ty)\} \tag{1}$$

for all $x, y \in X$ and $0 \leq \alpha < 1/2$. Then T has a unique fixed point.

Proof: Let $\{x_n\}$ be a sequence in X, defined as follows.
Let $x_0 \in X, f(x_0) = x_1, f(x_1) = x_2, \cdots, f(x_n) = x_{n+1}, \cdots$.
Consider

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \alpha\{d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)\} \\ &= \alpha\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\} \end{aligned}$$

Therefore

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{\alpha}{1-\alpha} d(x_{n-1}, x_n) \\ &= \lambda d(x_{n-1}, x_n) \end{aligned}$$

where $\lambda = \dfrac{\alpha}{1-\alpha}$ *with* $0 \leq \lambda < 1$. Similarly we will show that

$$d(x_{n-1}, x_n) \leq \lambda d(x_{n-2}, x_{n-1}),$$

and

$$d(x_n, x_{n+1}) \leq \lambda^2 d(x_{n-2}, x_{n-1})$$

Thus

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_1, x_0).$$

Since $0 \leq \lambda < 1$, as $n \to \infty$, $\lambda^n \to 0$. Hence $\{x_n\}$ is a dq-cauchy sequence in X. Thus $\{x_n\}$ dislocated quasi-converges to some t_0. Since T is continuous, we have

$$T(t_0) = lim\ T(x_n) = lim\ x_{n+1} = t_0.$$

Thus $T(t_0) = t_0$. Thus T has a fixed point.
Uniqueness: Let x be a fixed point of T then by definition(3.1)
$d(x, x) = d(Tx, Tx) \leq \alpha\{d(x, x) + d(x, x)\}$
i.e. $d(x, x) \leq 2\alpha d(x, x)$ which gives $d(x, x) = 0$, since $0 \leq 2\alpha < 1$ and $d(x, x) \geq 0$. Thus $d(x, x) = 0$ if x is a fixed point of T.
Let $x, y \in X$ be fixed point of T. That is , $Tx = x$, $Ty = y$ Then by definition (3.1),

$$\begin{aligned} d(x, y) = d(Tx, Ty) &\leq \alpha\{d(x, x) + d(y, y)\} \\ &= 0, \quad \text{since} \quad d(x, x) = 0 = d(y, y), \end{aligned}$$

so $d(x, y) \leq 0$ and $d(x, y) \geq 0$ Hence $d(x, y) = 0$. Similarly $d(y, x) = 0$ and hence $x = y$.
Thus fixed point of T is unique.

Theorem 3.4 *Let (X, d) be a complete dq-metric space. Let $T : X \to X$ be a continuous generalized contraction. Then T has a unique fixed point.*

Proof: Let $\{x_n\}$ be a sequence in X, defined as follows.
Let $x_0 \in X, f(x_0) = x_1, f(x_1) = x_2, \cdots, f(x_n) = x_{n+1}, \cdots$.
Consider

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, Tx_{n-1}) + \gamma d(x_n, Tx_n) \\ &\quad + \delta[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})] \text{[see def 3.2]} \\ &= \alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, x_n) + \gamma d(x_n, x_{n+1}) \\ &\quad + \delta[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] \\ &\leq \alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, x_n) + \gamma d(x_n, x_{n+1}) \\ &\quad + \delta d(x_{n-1}, x_n) + \delta d(x_n, x_{n+1}) \end{aligned}$$

Therefore,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{\alpha+\beta+\delta}{1-(\gamma+\delta)} d(x_{n-1}, x_n) \\ &= \lambda d(x_{n-1}, x_n), \end{aligned}$$

where $\lambda = \dfrac{\alpha+\beta+\delta}{1-(\gamma+\delta)}$. Similarly, we have $d(x_{n-1}, x_n) \leq \lambda d(x_{n-2}, x_{n-1})$. In this way, we get

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1).$$

Since $0 \leq \lambda < 1$, so for $n \to \infty$, $\lambda^n \to 0$ we have $d(x_n, x_{n+1}) \to 0$. Similarly we show that $d(x_{n+1}, x_n) \to 0$. Hence $\{x_n\}$ is a Cauchy sequence in the complete dislocated quasi-metric space X, so there is a point $t_0 \in X$, such that $x_n \to t_0$. Since T is continuous, so

$$T(t_0) = lim\ T(x_n) = lim\ x_{n+1} = t_0.$$

Thus $T(t_0) = t_0$, so T has a fixed point.

Uniqueness: If $x \in X$ is a fixed point of T then by given condition, we have

$$d(x, x) = d(Tx, Tx) \leq (\alpha+\beta+\gamma+2\delta) d(x, x)$$

which is true only if $d(x, x) = 0$, since $0 \leq \alpha+\beta+\gamma+2\delta < 1$ and $d(x, x) \geq 0$. Thus $d(x, x) = 0$ for a fixed point x of T.

Let x, y be fixed point of T. that is $x = Tx$ and $y = Ty$. Then by (3.1)

$$\begin{aligned} d(x, y) = d(Tx, Ty) &\leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) \\ &\quad + \delta[d(x, Ty) + d(y, Tx)] \\ &= \alpha d(x, y) + \beta d(x, x) + \gamma d(y, y) + \delta[d(x, y) + d(y, x)] \\ &= \alpha d(x, y) + \delta[d(x, y) + d(y, x)] \end{aligned} \tag{2}$$

Similarly we have

$$d(x, y) = \alpha d(y, x) + \delta[d(y, x) + d(x, y)].$$

Hence $|d(x, y) - d(y, x)| \leq \alpha |d(x, y) - d(y, x)|$, which implies $d(x, y) = d(y, x)$, since $0 \leq \alpha < 1$. Again from (3.2) $d(x, y) \leq (\alpha + 2\delta) d(x, y)$, which gives $d(x, y) = 0$, since $0 \leq \alpha + 2\delta < 1$. Further $d(x, y) = d(y, x) = 0$ gives $x = y$. Hence fixed point is unique. Hence the proof.

Remark: In theorem (3.4) if we put $\alpha = \delta = 0$ and $\beta = \gamma$ then we obtain theorem (3.3).

Theorem 3.5 *Let (X,d) be a complete dislocated metric space. Let $f : X \to X$ be continuous mapping satisfies;*

$$\begin{aligned} d(Tx,Ty) &\le \alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) \\ &\quad + \delta\frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \mu\frac{d(x,Ty)d(y,Tx)}{d(x,y)}, \end{aligned}$$

for all $x, y \in X$ and $\alpha, \beta, \gamma, \delta, \mu \ge 0$ with $\alpha + \beta + \gamma + \delta + 4\mu < 1$. Then T has a unique fixed point.

Proof: Let $\{x_n\}$ be a sequence in X, defined as follows.
Let $x_0 \in X, f(x_0) = x_1, f(x_1) = x_2, \cdots, f(x_n) = x_{n+1}, \cdots$.
Consider

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\le \alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, Tx_{n-1}) + \gamma d(x_n, Tx_n) \\ &\quad + \delta\frac{d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)}{d(x_{n-1}, x_n)} + \mu\frac{d(x_{n-1}, Tx_n)d(x_n, Tx_{n-1})}{d(x_{n-1}, x_n)} \\ &= \alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, x_n) + \gamma d(x_n, x_{n+1}) \\ &\quad + \delta\frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} + \mu\frac{d(x_{n-1}, x_{n+1})d(x_n, x_n)}{d(x_{n-1}, x_n)} \\ &\le (\alpha + \beta)d(x_{n-1}, x_n) + (\gamma + \delta)d(x_n, x_{n+1}) + 2\mu d(x_{n-1}, x_{n+1}) \\ &\le (\alpha + \beta + 2\mu)d(x_{n-1}, x_n) + (\gamma + \delta + 2\mu)d(x_n, x_{n+1}) \end{aligned}$$

Hence

$$\begin{aligned} d(x_n, x_{n+1}) &\le \frac{\alpha + \beta + 2\mu}{1 - (\gamma + \delta + 2\mu)} d(x_n, x_{n+1}) \\ &= \lambda d(x_{n-1}, x_n). \end{aligned}$$

where $\lambda = \dfrac{\alpha + \beta + 2\mu}{1 - (\gamma + \delta + 2\mu)}$, with $0 \le \lambda < 1$. Similarly we show that $d(x_{n-1}, x_n) \le \lambda d(x_{n-2}, x_{n-1})$. Continuing in this way we have $d(x_n, x_{n+1}) = \lambda^n d(x_0, x_1)$, since $0 \le \lambda < 1$, $\lambda^n \to 0 \quad n \to \infty$,. Hence $\{x_n\}$ is a Cauchy sequence in complete dislocated metric space X. So there ia point $u \in X$ such that $x_n \to u$. Since T is a continuous, so $T(u) = T(\lim x_n) = \lim T(x_n) = \lim x_{n+1} = u$. Thus u is a fixed point of T.
Uniqueness: Let u, v be fixed point of T. By condition we have

$$\begin{aligned} d(u, v) &= d(Tu, Tv) \\ &\le \alpha d(u, v) + \beta d(u, u) + \gamma d(v, v) \\ &\quad + \delta\frac{d(u, u)d(v, v)}{d(u, v)} + \mu\frac{d(u, v)d(v, u)}{d(u, v)} \\ &= \alpha d(u, v) + \beta d(u, u) + \gamma d(v, v) + \delta\frac{d(u, u)d(v, v)}{d(u, v)} + \mu d(u, v) \end{aligned}$$

Now

$$\begin{aligned} d(u,u) &= d(Tu,Tu) \\ &\leq \alpha d(u,u) + \beta d(u,u) + \gamma d(u,u) + \delta d(u,u) + \mu d(u,u) \\ &= (\alpha+\beta+\gamma+\delta+\mu) d(u,u). \end{aligned}$$

Since $(0 \leq \alpha+\beta+\gamma+\delta+\mu) < 1$, we have $d(u,u) = 0$. Similarly $d(v,v) = 0$. Hence the fixed points u, v we get

$$d(u,v) \leq (\alpha+\mu) d(u,v)$$

Since $0 \leq \alpha+\mu < 1$. so we must have $d(u,v) = 0$. Similarly we have $d(v,u) = 0$. Hence $u = v$.

Theorem 3.6 *Let (X,d) be a complete dislocated metric space. Let $f, g : X \to X$ be continuous mappings satisfy;*

$$d(fx, gy) \leq h \max\{d(x,y), d(x,fx), d(y,gy)\},$$

for all $x, y \in X$ and $0 < h < 1$. Then f and g have common fixed point.

Proof: Let $x_0 \in X$. Define the sequence $\{x_n\}$ by $x_1 = f(x_0)$, $x_2 = g(x_1), \cdots, x_{2n} = g(x_{2n+1}), x_{2n+1} = f(x_{2n}), \cdots$.
Consider

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(fx_{2n}, gx_{2n+1}) \\ &\leq h \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, fx_{2n}), d(x_{2n+1}, gx_{2n+1})\} \\ &= h \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} \\ &= h d(x_{2n}, x_{2n+1}). \end{aligned}$$

Therefore

$$d(x_{2n+1}, x_{2n+2}) \leq h d(x_{2n}, x_{2n+1}).$$

Similarly

$$d(x_{2n}, x_{2n+1}) \leq h d(x_{2n-1}, x_{2n}).$$

and so

$$d(x_{2n+1}, x_{2n+2}) \leq h^2 d(x_{2n-1}, x_{2n}).$$

In this way we have

$$d(x_{2n+1}, x_{2n+2}) \leq h^{2n} d(x_0, x_1).$$

Since $h < 1$, as $h^{2n} \to 0 \ \ as \ \ n \to \infty$. Thus $\{x_n\}$ is a Cauchy sequence in a dislocated metric X. There exists a point $u \in X$ such that $x_n \to u$. Therefore the subsequences $\{fx_{2n}\} \to u$ and $\{gx_{2n+1}\} \to u$. Since f and g are continuous function, so we have $fu = u$ and $gu = u$.
Uniqueness of common fixed point: Let u, v be a common fixed point of f and g. Then

$$\begin{aligned} d(u,v) &= d(fu, gv) \\ &\leq \lim h \max\{u,v), d(u,fu), d(v,gv)\} \\ &= h \max\{d(u,v), d(u,u), d(v,v)\} \end{aligned}$$

Replacing v by u, we get $d(u,u) \leq hd(u,u)$, and hence $d(u,v) = 0$. Similarly $d(v,u) = 0$ and so $u = v$.

$$\begin{aligned} d(u,u) &= d(fu, \lim x_{2n+2}) \\ &= \lim d(fu, gx_{2n+1}) \\ &\leq \lim h \max\{d(fu, x_{2n+1}), d(u,fu), d(x_{2n+1}, x_{2n+2})\} \\ &= h \max\{d(u,u), d(u,u), d(u,u)\} \end{aligned}$$

Since $h < 1$, we have $d(u,u) = 0$. Similarly we have $d(v,v) = 0$. Therefore $d(u,v) < d(u,v)$, a contradiction. Hence the proof.

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