Applied Mathematical Sciences, Vol. 2, 2008, no. 60, 2955 - 2961

Generalized Laguerre Polynomials Collocation Method for Solving Lane-Emden Equation

K. Parand and A. Taghavi

Department of Computer Sciences,
Shahid Beheshti University, Tehran, Iran
k_parand@sbu.ac.ir
amirtaghavims2@yahoo.com

Abstract

In this paper we propose, a collocation method for solving nonlinear singular Lane-Emden equation which is a nonlinear ordinary differential equation on semi-infnite interval. This approach is based on a generalized Laguerre polynomial collocation method. This method reduces the solution of this problem to the solution of a system of algebraic equations.We also present the comparison of this work with some well-known results and show that the present solution is highly accurate.

Mathematics Subject Classification: 05E35, 37C10

Keywords: Lane-Emden equation, Generalized Laguerre polynomials, Collocation method, Nonlinear ODE

1 Introduction

In the study of stellar structure [5] an important mathematical model described by the second-order ordinary differential equation

$$xy'' + 2y' + xg(y) = 0, \qquad x > 0, \tag{1}$$

arises, where $g(y)$ is some given function of y. Among the most popular form of $g(y)$ is

$$g(y) = y^m. \tag{2}$$

which is subject to the conditions

$$y(0) = 1, \qquad y'(0) = 0. \tag{3}$$

This equation is standard Lane-Emden equation. It was first proposed by Lane [18] and studied in more detail by Emden [7].
The Lane-Emden equation describes a variety of phenomena in theoretical physics and astrophysics, including aspects of stellar structure, the thermal history of a spherical cloud of gas, isothermal gas spheres, and thermionic currents, [4]. Since then the equation has been a centre of attention for many researchers.
The equation also appears in other contexts, e.g., in case of radiatively cooling, self-gravitating gas clouds, in the mean-field treatment of a phase transition in critical absorption or in the modeling of clusters of galaxies.
The physically interesting range of m is $0 \leq m \leq 5$. Numerical and perturbation approaches to solve equation (1) with $g(y) = y^m$ have been considered by various authors. It has been claimed in the literature that only for $m = 0, 1$ and 5 the solutions of the Lane-Emden equation(also called the polytropic differential equations) could be given in closed form.
In fact, for $m = 5$, only a 1-parameter family of solutions is presented. The so called generalized Lane-Emden equation of the first kind have been looked at in Goenner and Havas [9] and Goenner [10].
Recently, many analytic methods have been used to solve Lane-Emden equations, the main difficulty arises in the singularity of the equation at $x = 0$. Currently, most techniques in use for handling the Lane-Emden-type problems are based on either series solutions or perturbation techniques.
Bender et al. [3] proposed a perturbative technique for solving nonlinear differential equation such as Lane-Emden. Shawagfeh [16] applied a nonperturbative approximate analytic solution for the Lane-Emden equation using the Adomian decomposition method. Wazwaz [19] employed the Adomian decomposition method with an alternate framework designed to overcome the difficulty of the singular point. Liao [14] provided an analytic algorithm for Lane-Emden type equations. This algorithm logically contains the well-known Adomian decomposition method. Parand and Razzaghi [15] presented a numerical technique to solve higher ordinary deferential equations such as Lane-Emden. Their approach was based on a rational Legendre tau method.Bataineh et al. [1] obtained analytic solutions of singular initial value problems (IVPs) of the Emden-Fowler type by the homotopy analysis method (HAM).
Spectral methods have been successfully applied in the approximation of differential boundary value problems defined in unbounded domains.For problems whose solutions are sufficiently smooth, they exhibit exponential rates of convergence/spectral accuracy. There are three most commonly used spectral versions, namely the Galerkin-type, tau and collocation method. Among these, an approach consists in using the collocation method or the pseudospectral method based on the nodes of Gauss formulas related to unbounded intervals [13]. Collocation method has become increasingly popular for solving differen-

tial equations also they are very useful in providing highly accurate solutions to differential equations. In this paper, we aim to employ the collocation method to a singular form of Lane-Emden type initial value problems directly.

2 Properties of generalized Laguerre polynomials

This section is devoted to the introduction of the basic notions and working tools concerning orthogonal generalized Laguerre polynomials.
The Laguerre approximation has been widely used for numerical solutions of differential equations on semi-infinite intervals. Let $w(x)$ denotes a non-negative function over the interval $I = [0, \infty)$. We define

$$L^2_w(I) = \{v : I \to \mathbb{R} \mid v \; is \; measurable \; and \; \| v\|_w < \infty\}, \tag{4}$$

where

$$\| v\|_w = (\int_0^\infty \mid v(x) \mid^2 w(x)dx)^{\frac{1}{2}}, \tag{5}$$

is the norm induced by the scalar product

$$< u, v >_w = \int_0^{+\infty} u(x)v(x)w(x)dx. \tag{6}$$

Let

$$\Re_N = span\big\{1, x, \ldots, x^{2N-2}\big\}, \tag{7}$$

$L^\alpha_n(x)$ (generalized Laguerre polynomial) is the nth eigenfunction of the Sturm-Liouville problem [6],[8],[11]:

$$x\frac{d^2}{d^2x}L^\alpha_n(x) + (\alpha + 1 - x)\frac{d}{dx}L^\alpha_n(x) + nL^\alpha_n(x) = 0, \\ x \in I = [0, \infty), \qquad n = 0, 1, 2, ... \tag{8}$$

The generalized Laguerre polynomials are defined with the following recurrence formula:

$$L^\alpha_0(x) = 1, \quad L^\alpha_1(x) = 1 + \alpha - x, \\ nL^\alpha_n(x) = (2n - 1 + \alpha - x)L^\alpha_{n-1}(x) - (n + \alpha - 1)L^\alpha_{n-2}(x), n \geq 2, \alpha > -1, \tag{9}$$

with the normalizing condition:

$$L^\alpha_n(0) = \binom{n + \alpha}{n}. \tag{10}$$

These are orthogonal polynomials for the weight function $w_\alpha = x^\alpha e^{-x}$.

$$\int_0^{+\infty} L_n^\alpha(x) L_m^\alpha(x) w_\alpha(x) dx = \left(\frac{\Gamma(n+1+\alpha)}{n!} \right) \delta_{nm},$$

Let $N \geq 1$ be an integer and we define $x_{j,N}^\alpha$, $j = 0, \ldots, N-1$ to be zeroes of $\frac{d}{dx} L_N^\alpha$ and the point $x = 0$. It can be shown that $x_{j,N}^\alpha > 0$, $j = 0, \ldots, N-1$ and the corresponding weights are:

$$w_{0,N}^\alpha = \tfrac{(\alpha+1)\Gamma^2(\alpha+1)(N-1)!}{\Gamma(N+\alpha+1)}$$
$$w_{j,N}^\alpha = \tfrac{\Gamma(\alpha+N)}{N!} \left(L_N^\alpha(x_{j,N}^\alpha) \tfrac{d}{dx} L_{N-1}^\alpha(x_{j,N}^\alpha) \right)^{-1}, \quad j = 1, 2, ..., N-1.$$

The following quadrature formula is known:

$$\int_0^{+\infty} f(x) w_N(x) dx = \sum_{j=0}^{N} f(x_{j,N}^\alpha) w_{j,N}^\alpha + \left(\frac{\Gamma(N+\alpha+1)}{(N)!(2N)!} \right) f^{2N-1}(\xi), \quad 0 < \xi < \infty \tag{11}$$

In particular, the second term on the right hand side vanishes when f is a polynomial of degree at most $2N-2$. For convenience, we shall set $x_{j,N}^\alpha = x_j$ and $w_{j,N}^\alpha = w_j$. We define

$$I_N u(x) = \sum_{j=0}^{N} a_j L_j^\alpha(x), \tag{12}$$

such that $I_N u(x_j) = u(x_j)$, $j = 0, \ldots, N$. $I_N u$ is the orthogonal projection of u upon $\mathfrak{R}_N$ with respect to the discrete inner product and discrete norm as:

$$< u, v >_{w_\alpha} = \sum_{j=0}^{N} u(x_j) v(x_j) w_\alpha, \tag{13}$$

$$\| u \|_{w_\alpha} = < u, u >_{w_\alpha}^{1/2}, \tag{14}$$

thus for the Gauss-Radau interpolation we have

$$< I_N u, v >_{w_\alpha} = < u, v >_{w_\alpha}, \qquad \forall u.v \in \mathfrak{R}_N. \tag{15}$$

In [17] Szegö approximated the zeroes of $L_n^\alpha(x)$ by a procedure.

3 Solution of Lane-Emden equation

To apply generalized Laguerre polynomials collocation method to the standard Lane-Emden Equation introduced in (1) and (2) with boundary conditions (3) at first by (12) we expand $y(x)$, as follows:

$$I_N y(x) = \sum_{j=0}^{N} a_j L_j^1(x/k). \quad k > 0. \tag{16}$$

Which $k > 0$ is a constant. To find the unknown coefficients a_j's, we substitute the truncated series into the (1) with $g(y)$ introduced in (2) and boundary conditions in (3) and applied (13)-(15). So we have

$$x\sum_{j=0}^{N} a_j L''^1_j(x/k) + 2\sum_{j=0}^{N} a_j L'^1_j(x/k) + x\left(\sum_{j=0}^{N} a_j {L_j}^1(x/k)\right)^m = 0, \tag{17}$$

$$\sum_{j=0}^{N} a_j {L_j}^1(0) = 1, \quad \sum_{j=0}^{N} a_j L'^1_j(0) = 0. \tag{18}$$

By replacing x in (17) with the $N-1$ collocation points which are roots of functions $\frac{d}{dx}L_N^1$, we have $N-1$ equations that generates a set of $N+1$ nonlinear equations with boundary equations in (3).

Table 1 shows the comparison of the first zero of y, between Padé approximation used by [3] and the present method for $m = 2, 3$.

Table 2 shows the approximations of $y(x)$ for standard Lane-Emden with $m = 3$ obtained by the method proposed in this paper for $N = 6$ and $k = 1.007$, and those obtained by Horedt [12].

Figure 1 shows the resulting graph of Lane-Emden for $N = 6$ and specific k's shown in table 1.

Table 1. Comparison of the first zero of y, between [3] and the present method for $m = 2, 3$.

m	N	k	Present method	Bender	Exact value
2	6	1.098	4.35280120	4.3603	4.35287460
3	6	1.007	6.89201052	7.0521	6.89684862

Table 2. Comparison of $y(x)$ for present method, solutions of Horedt [12] for $m = 3$

x	Present method	solutions of Horedt [12]
0.000	1.000000	1.000000
0.100	0.998313	0.998336
0.500	0.959811	0.959839
1.000	0.855084	0.855058
5.000	0.110820	0.110820
6.000	0.043708	0.043738
6.800	0.004155	0.004168
6.896	0.000026	0.000036

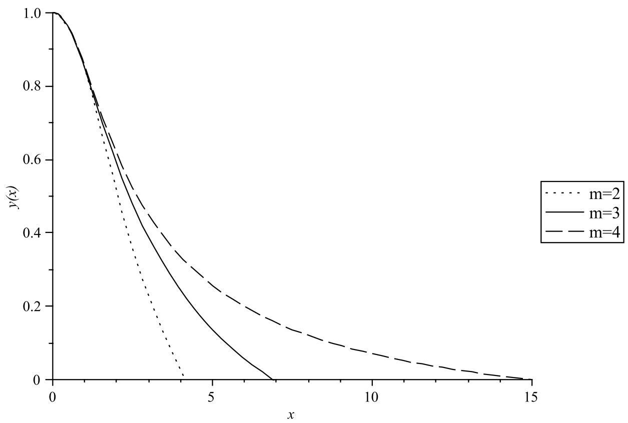

Figure 1: Lane-Emden equation graph obtained by present method .

4 Conclusions

The fundamental goal of this paper has been to construct an approximation to the solution of nonlinear Lane-Emden equation in a semi-infinite interval which has singularity at $x = 0$. A set of orthoghonl polynomials are proposed to provide an effective but simple way to improve the convergence of the solution by collocation method. Through the comparisons among the solutions of Horedt and the approximate solutions of bender and the current work, it has been shown that the present work has provided more exact soloutions for Lane-Emden equations.

References

[1] A.S. Bataineh M.S.M. Noorani, I. Hashim, Homotopy analysis method for singular IVPs of Emden-Fowler type, *Communications in Nonlinear Science and Numerical Simulation,* (2008), doi:10.1016/j.cnsns.2008.02.004

[2] S. S. Bayin, *Mathemathical Methods in science And Engineering,* John Wiley & Sons, New York, 2006.

[3] C.M. Bender, K.A. Milton, S.S. Pinsky and Jr. L.M. Simmons, A new perturbative approach to nonlinear problems, *Journal of Mathematics and Physics* **30** (1989), 1447-1455.

[4] L. Bin and Y. Jiangong, Quasiperiodic solutions of Duffing's equations, *Nonlinear Analysis,* **33** (1998), 645-655.

[5] S. Chandrasekhar, *An introduction to the study of stellar structure,* Dover, New York 1957.

[6] O. Coulaud ,D. Funaro and O. Kavian, Computational aspects of pseudospectral Laguerre approximations, *Applied Numerical Mathematics,* **6** (1990), 451-458.

[7] R. Emden, *Gaskugeln,Teubner,* Leipzig and Berlin, 1907.

[8] D. Funaro, *Polynomial Approximation of Differential Equations,* Springer-Verlag, Berlin, 1992.

[9] H. Goenner and P. Havas, Exact solutions of the generalized Lane-Emden equation, *Journal of Mathematical Physics,* **41** (2000), 7029-7042.

[10] H. Goenner, Symmetry Transformations for the Generalized Lane-Emden Equation, *General Relativity and Gravitation,* **33** (2001), 833-841.

[11] B.Y. Guo ,J. Shen and C.L. Xu, Generalized Laguerre approximation and its applications to exterior problems, *Journal of computational Mathematics,* **23** (2005), 113-130.

[12] G. P. Horedt, *Polytropes Applications in Astrophysics and Related Fields,* Klawer Academic Publishers, 2004.

[13] V. Iranzo and A. Falques, Some spectral approximations for differential equations in unbounded domains, *Computational Method in Applied Mechanic,* **105** (1992), 105-126.

[14] S. Liao, A new analytic algorithm of Lane-Emden type equations, *Applied Mathematics and Computation,* 142 (2003), 1-16.

[15] K. Parand, M. Razzaghi, Rational Legendre approximation for solving some physical problems on semi-infinite intervals, *Physica Scripta* **69** (2004), 353-357

[16] N.T. Shawagfeh, Nonperturbative approximate solution for Lane-Emden equation, Journal of Mathematical Physics **34** (1993), 4364-4369.

[17] G. Szegö, *Orthogonal Polynomials,* AMS New York, 1939.

[18] W. Thomson, *Collected Papers,* Cambridge University Press, Cambridge, 1991.

[19] A. Wazwaz, A new algorithm for solving differential equations of Lane-Emden type, *Applied Mathematics and Computation,* **118** (2001), 287-310.

Received: April 1, 2008