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On a Nonlinear System

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Abstract

In this work we will prove that exists only a weak solution the mixed problem associated to the nonlinear system

$$\left| \begin{array}{l} u'' + \Delta^2 u - M(\|u\|^2)\Delta u + |u|^\rho u + \theta = f \\ \theta' - \Delta\theta + u' = g, \end{array} \right.$$

where M is a real function, ρ is a positive real number, f and g are know real functions.

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1 Introduction

In this work we consider the mixed system

$$\left| \begin{array}{l} u'' + \Delta^2 u - M(\|u\|^2)\Delta u + |u|^\rho u + \theta = f \quad \text{in} \quad Q \\ \theta' - \Delta\theta + u' = g \quad \text{in} \quad Q \\ u = \dfrac{\partial u}{\partial \eta} = \theta = 0 \quad \text{on} \quad \Sigma \\ u(x,0) = u_0(x);\ \theta(x,0) = \theta_0(x) \ \text{ and } \ u'(x,0) = u_1(x) \ \text{ in } \ \Omega, \end{array} \right. \tag{1}$$

where Ω is a non empty open bounded set of $\mathbb{R}^n$, for $n \geq 1$, with boundary Γ smooth, Q is the cylinder $\Omega \times (0,T)$ of $\mathbb{R}^{n+1}$ for $T > 0$, $|\nabla u(x,t)|$ is the norm in $\mathbb{R}^n$ of the vector $\nabla(x,t)$ and $\Delta u(x,t)$ is the usual Laplace operator in

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$\mathbb{R}^n$ of the function $u(x,t)$. We denote by $\dfrac{\partial}{\partial t} =', \ \Sigma = \Gamma \times (0,T)$ is the lateral boundary.

Our goal in this article is to study the existence of global weak solutions of problem (1) with initial conditions $u_0 \in H_0^2(\Omega),\ u_1$ and $\theta_0 \in L^2(\Omega)$, and also, the uniqueness of the solutions.

The dynamical part of the above system when $\theta = 0$ is a nonlinear perturbation of the beam equation, that has been extensively studied by several authors in different physical-mathematical contexts. Among then, we cite the following related works: Ball [1][2], Biber [3], Brito [4], Pereira [9] and Medeiros [8]. More recently we can cite the works Limaco et al [5], presented in the 56$^{\underline{\text{O}}}$ and 57$^{\underline{\text{O}}}$ SBA respectively.

2 Notation and main result

For the functional spaces we shall use, throughout this paper, the standard notation of the functional spaces used, for instance, in the books of Lions [6] or Medeiros-Milla Miranda [7].

In this section we shall assume the following hypothesis:

$M(\lambda)$ is a C^0 real function satifying

$$M(\lambda) \geq -\beta, 0 < \beta < \lambda_1,$$

where λ_1 is the first auto-value of the Spectral Problem:

$$(\Delta u, \Delta v) = \lambda\big((u,v)\big) \quad \forall \ \ v \in H_0^2(\Omega).$$

$0 < \rho$ if $n = 1, 2$ and $0 < \rho \leq \dfrac{2}{n-2}$ if $n \geq 3$.

Definition 2.1 *We say that the pair of functions* $\{u(x,t), \theta(x,t)\}$ *is solution of the problem* (1) *if*

$$u \in L^\infty(0,T; H_0^2(\Omega));$$

$$u' \in L^\infty(0,T; L^2(\Omega));$$

$$u'' \in L^2(0,T;H^{-2}(\Omega));$$

$$\theta \in L^2(0,T;H_0^1(\Omega));$$

$$\theta' \in L^2(0,T;H^{-1}(\Omega));$$

$$\begin{aligned}\frac{d}{dt}(u'(t),w) + (\Delta^2 u(t),w) + M(\int_\Omega |\nabla u(t)|^2 dx)\big((u(t),w)\big) + \\ +(|u(t)|^\rho u(t),w) + (\theta(t),w) = (f(t),w) \\ \frac{d}{dt}(\theta(t),w) + \big((\theta(t),w)\big) + (u'(t),w) = (g(t),w)\end{aligned}$$

for all $w \in H_0^2(\Omega)$ *in the sense of* $D'(0,T)$.

$$u(0) = u_0,\ u'(0) = u_1,\ \theta(0) = \theta_0$$

3 Main Result

The main result of the present work is given in the following theorem.

Theorem 1 *Let be* $u_0 \in H_0^2(\Omega),\ u_1,\ \theta_0 \in L^2(\Omega)$ *and* $f,g \in L^2(0,T;L^2(\Omega))$. *Thus there exists only a pair of functions* $\{u,\theta\}$ *solutions of the problem* (1) *in the sense of the Definition* 2.1.

Proof of Theorem 1. Let $(w_m)_{m\in\mathbb{N}}$ be the eigenfunctions of the biharmonic operator on Ω and let V_m be the space generated by the first m eigenfunctions. Now let us consider the approximated system

$$(u_m''(t),w_k) + (\Delta^2 u_m(t),w_k) - M(\|u_m(t)\|^2)(\Delta u_m(t),w_k)$$
$$+(|u_m(t)|^\rho u_m(t),w_k) + (\theta_m(t),w_k) = (f(t),w_k) \tag{2}$$
$$(\theta_m'(t),w_k) - (\Delta\theta_m(t),w_k) + (u_m'(t),w_k) = (g(t),w_k) \tag{3}$$
$$u_m(0) = u_{0m} \longrightarrow u_0 \text{ strongly in } H_0^2(\Omega) \tag{4}$$
$$u_m'(0) = u_{1m} \longrightarrow u_1 \text{ strongly in } L^2(\Omega) \tag{5}$$
$$\theta_m(0) = \theta_{0m} \longrightarrow \theta_0 \text{ strongly in } L^2(\Omega) \tag{6}$$

where $1 \leq k \leq m$. Then there exist functions c_{km} and d_{km} such that

$$u_m(t) = \sum_{k=1}^{m} c_{km}(t) w_k \text{ and } \theta_m(t) = \sum_{k=1}^{m} d_{km}(t) w_k$$

are the unique local solutions of the above system on some interval $[0, t_m[$, where $t_m \in [0, T[$.
The estimates that we obtain below will allow us to extend the solutions $\{u_m, \theta_m\}$ to the interval $[0, T[$.

Estimate. Multiply (2) by $c'_{km}(t)$ and multiply (3) by $d_{km}(t)$ and sum over k we obtain:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|u'_m(t)|^2 + \frac{1}{2}\frac{d}{dt}|\Delta u_m(t)|^2 + \frac{1}{2}\frac{d}{dt}\hat{M}(\|u_m(t)\|^2) + \frac{1}{p}\frac{d}{dt}\|u_m(t)\|^p_{L^p(\Omega)} \\ = -(\theta_m(t), u'_m(t)) + (f(t), u'_m(t)) \end{aligned} \tag{7}$$

where $p = \rho + 2$ and $\hat{M}(\lambda) = \displaystyle\int_0^\lambda M(s)\, ds$.

$$\frac{1}{2}\frac{d}{dt}\{|\theta_m(t)|^2\} + \|\theta_m(t)\|^2 = -(u'_m(t), \theta_m(t)) + (g(t), \theta_m(t)) \tag{8}$$

Sum (7) and (8)and using the Poincaré's inequality we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\big\{|u'_m(t)|^2 + |\Delta u_m(t)|^2 + \hat{M}(\|u_m(t)\|^2) + \frac{2}{p}\|u_m(t)\|^p_{L^p(\Omega)} + |\theta_m(t)|^2\big\} + \|\theta_m(t)\|^2 \leq \\ \frac{3}{2}|\theta_m(t)|^2 + \frac{3}{2}|u'_m(t)|^2 + |f(t)|^2 + |g(t)|^2. \end{aligned}$$

Now integrating from 0 to $t \leq t_m$, we have

$$\begin{aligned} \frac{1}{2}\big\{|u'_m(t)|^2 + |\Delta u_m(t)|^2 + \hat{M}(\|u_m(t)\|^2) + \frac{2}{p}\|u_m(t)\|^p_{L^p(\Omega)} + |\theta_m(t)|^2\big\} + \\ + \int_0^t \|\theta_m(s)\|^2\, ds \leq \frac{1}{2}|u_{1m}|^2 + \frac{1}{p}\|u_{0m}\|^p + |\Delta u_{0m}|^2 + |\theta_{0m}|^2 + \hat{M}(\|u_{0m}\|^2) + \\ \frac{3}{2}\int_0^t \big\{|\theta_m(s)|^2 + |u'_m(s)|^2\big\}\, ds + \int_0^T \big\{|f(s)|^2 + |g(s)|^2\big\}\, ds. \end{aligned} \tag{9}$$

From hypotheses we obtain

$$\hat{M}(\|u_m(t)\|^2) \geq -\frac{\beta}{\lambda_1}|\Delta u_m(t)|^2. \tag{10}$$

It follows from (9), (10) and from hypothesis

$$|u_m'(t)|^2 + \big(1 - \frac{\beta}{\lambda_1}\big)|\Delta u_m(t)|^2 + \frac{2}{p}\|u_m(t)\|^p_{L^p(\Omega)} + |\theta_m(t)|^2 + \\ + \int_0^t \|\theta_m(s)\|^2\, ds \leq C + 3\int_0^t \big\{|\theta_m(t)|^2 + |u_m'(t)|^2\big\}\, ds.$$

From Gronwall's inequality it follows that

$$|u_m'(t)|^2 + \big(1 - \frac{\beta}{\lambda_1}\big)|\Delta u_m(t)|^2 + \frac{2}{p}\|u_m(t)\|^p_{L^p(\Omega)} + |\theta_m(t)|^2 + \int_0^t \|\theta_m(s)\|^2\, ds \leq C. \tag{11}$$

where $C > 0$ constant independent of t and m.
Being $\|u_m(t)\| \leq C|\Delta u_m(t)|$, we obtain from (11)

$$(u_m)_m \text{ is bounded in } L^\infty(0,T; H_0^2(\Omega)); \tag{12}$$

$$(u_m')_m \text{ is bounded in } L^\infty(0,T; L^2(\Omega)); \tag{13}$$

$$(|u_m|^\rho u_m)_m \text{ is bounded in } L^{\frac{\rho+2}{\rho+1}}(Q); \tag{14}$$

$$(u_m)_m \text{ is bounded in } L^\infty(0,T; H_0^1(\Omega)); \tag{15}$$

$$(\theta_m)_m \text{ is bounded in } L^\infty(0,T; L^2(\Omega)); \tag{16}$$

$$(\theta_m)_m \text{ is bounded in } L^2(0,T; H_0^1(\Omega)); \tag{17}$$

Passage to Limit By Estimates above result that there exists subsequence, if necessary, denoted by (u_m) and (θ_m) and functions $u, \theta : Q \to \mathbb{R}$ such that:

$$\begin{array}{l} u_m \rightharpoonup u \text{ weak-star in } L^\infty(0,T; H_0^2(\Omega)) \\ u_m' \rightharpoonup u' \text{ weak-star in } L^\infty(0,T; L^2(\Omega)) \\ \theta_m \rightharpoonup \theta \text{ weak in } L^2(0,T; H_0^1(\Omega)). \end{array} \tag{18}$$

From $(18)_1$, $(18)_2$ and Aubin-Lions's Theorem, it followings that there exists a subsequence of (u_m) such that

$$u_m \to u \quad \text{strongly in} \quad L^2(0,T;H_0^1(\Omega)),$$

hence

$$\|u_m(t)\| \to \|u(t)\| \quad \text{a.e.} \quad in \;\; (0,T).$$

Consequently

$$M(\|u_m(t)\|^2) \to M(\|u(t)\|^2) \quad \text{a.e.} \quad in \;\; (0,T) \tag{19}$$

and

$$|M(\|u_m(t)\|^2)| < \infty, \quad \text{a.e.} \quad in \;\; (0,T). \tag{20}$$

From (19),(20) and Lebesgue's Theorem we obtain

$$M(\|u_m(t)\|^2) \to M(\|u(t)\|^2) \quad \text{in} \quad L^2(0,T). \tag{21}$$

From $(18)_1$ and (21) we it followings

$$M(\|u_m(t)\|^2)\Delta u_m \rightharpoonup M(\|u(t)\|^2)\Delta u \quad \text{in} \quad L^2(0,T;L^2(\Omega)). \tag{22}$$

Using Aubin-Lions's Theorem and Lions's Lemma we obtain

$$|u_m|^\rho u_m \rightharpoonup |u|^\rho u \quad \text{in} \quad L^{\frac{\rho+2}{\rho+1}}(Q). \tag{23}$$

From convergence (18), (22), (23) and observing that V_m is dense in $H_0^2(\Omega)$ we obtain

$$\begin{aligned} &\frac{d}{dt}(u'(t),w) + (\Delta^2 u(t),w) + M\Big(\int_\Omega |\nabla u(t)|^2 dx\Big)\big((u(t),w)\big) + \\ &\qquad\qquad + (|u(t)|^\rho u(t),w) + (\theta(t),w) = (f(t),w) \\ &\frac{d}{dt}(\theta(t),w) + \big((\theta(t),w)\big) + (u'(t),w) = (g(t),w) \end{aligned} \tag{24}$$

for all $w \in H_0^2(\Omega)$ in the sense of $D'(0,T)$.
From (24) and Teman's Theorem [10] we obtain

$$u'' \in L^2(0,T;H^{-2}(\Omega)) \quad \text{and} \quad \theta' \in L^2(0,T;H^{-1}(\Omega)).$$

4 Uniqueness

To Proof the uniqueness, we increase the hypotheses: M is C^1 real function and $M(\lambda) \geq 0, \quad \forall \ \lambda \in \mathbb{R}$.

Let $[u, \theta]$ and $[\hat{u}, \hat{\theta}]$ be solutions of (24) under the conditions of Theorem 1. Let $w = u - \hat{u}$ and $v = \theta - \hat{\theta}$. Then $[w, v]$ satisfies

$$\begin{aligned} &\frac{d}{dt}(w', z) + (\Delta w(t), \Delta z) + M(\int_\Omega |\nabla u|^2 dx)(\nabla w, \nabla z) + (|u|^\rho u - |\hat{u}|^\rho \hat{u}, z) + \\ &+ (v, z) = \quad M(\int_\Omega |\nabla \hat{u}|^2 dx)(\nabla \hat{u}, \nabla z) - M(\int_\Omega |\nabla u|^2 dx)(\nabla \hat{u}, \nabla z) \quad (25) \\ &\frac{d}{dt}(v, z) + (\nabla v, \nabla z) + (w', z) = 0 \quad (26) \\ &w(0) = 0, \quad w'(0) = 0 \text{ and } v(0) = 0 \quad (27) \end{aligned}$$

Taking $z = w'$ in (25) and $z = v$ in (26), we obtain

$$\begin{aligned} &\frac{d}{dt}\{|w'|^2 + \frac{1}{2}|\Delta w|^2\} + M(\int_\Omega |\nabla u|^2 dx)\frac{d}{dt}\|w\|^2 + \int_\Omega (|u|^\rho u - |\hat{u}|^\rho \hat{u})w' dx \\ &+ (v, w') = \quad M(\int_\Omega |\nabla \hat{u}|^2 dx)(\nabla \hat{u}, \nabla w') - M(\int_\Omega |\nabla u|^2 dx)(\nabla \hat{u}, \nabla w') \quad (28) \\ &\frac{d}{dt}|v|^2 + \|v\|^2 + (w', v) = 0 \quad (29) \end{aligned}$$

in the $D'(0, T)$ sense. Adding (28) to (29) we have

$$\begin{aligned} &\frac{d}{dt}\{|w'|^2 + |v|^2 + \frac{1}{2}|\Delta w|^2\} + M(\int_\Omega |\nabla u|^2 dx)\frac{d}{dt}\|w\|^2 + \|v\|^2 \\ &= \int_\Omega (|\hat{u}|^\rho \hat{u} - |u|^\rho u)w' dx - 2(v, w') + M(\int_\Omega |\nabla \hat{u}|^2 dx)(\nabla \hat{u}, \nabla w') \\ &\quad - M(\int_\Omega |\nabla u|^2 dx)(\nabla \hat{u}, \nabla w') \\ &\leq \left| \int_\Omega (|\hat{u}|^\rho \hat{u} - |u|^\rho u)w' dx \right| + 2|(v, w')| \\ &\quad + \left| M(\int_\Omega |\nabla \hat{u}|^2 dx) - M(\int_\Omega |\nabla u|^2 dx) \right| |(\nabla \hat{u}, \nabla w')| \end{aligned}$$

On the other hand, by Holder's inequality with $\frac{1}{q} + \frac{1}{n} + \frac{1}{2} = 1$, we have

$$\begin{aligned} \left| \int_\Omega (|\hat{u}|^\rho \hat{u} - |u|^\rho u)w' dx \right| &\leq (\rho + 1) \int_\Omega \sup(|u|^\rho, |\hat{u}|^\rho)|w|\,|w'| dx \\ &\leq C\left(\|\,|u|^\rho\|_{L^n(\Omega)} + \|\,|\hat{u}|^\rho\|_{L^n(\Omega)}\right) \|w\|_{L^q(\Omega)} |w'|_{L^2(\Omega)} \end{aligned}$$

By hypotheses, we have $\rho n \leq q$ and from the immersion $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ with $1/q = 1/2 - 1/n$, we have

$$\left| \int_\Omega (|\hat{u}|^\rho \hat{u} - |u|^\rho u) w' dx \right| \leq C(\|u\|^\rho + \|\hat{u}\|^\rho)\, \|w\|\, |w'|$$

and since $u,\, \hat{u} \in L^\infty(0,T; H_0^2(\Omega))$, we have

$$\begin{aligned} \left| \int_\Omega (|\hat{u}|^\rho \hat{u} - |u|^\rho u) w' dx \right| &\leq C\|w\|\, |w'| && (30)\\ 2|(v, w')| &\leq 2|v|\, |w'| && (31)\end{aligned}$$

Observe that

$$\begin{aligned} &\left| M(\int_\Omega |\nabla \hat{u}|^2 dx) - M(\int_\Omega |\nabla u|^2 dx) \right| |(\nabla \hat{u}, \nabla w')| \\ &\leq |M'(\xi)|\, \big||\nabla \hat{u}|^2 - |\nabla u|^2\big| |(-\Delta)\hat{u}|\, |w'| \end{aligned}$$

where ξ is between $|\nabla \hat{u}|^2$ and $|\nabla u|^2$. Then we have

$$\begin{aligned} &\left| M(\int_\Omega |\nabla \hat{u}|^2 dx) - M(\int_\Omega |\nabla u|^2 dx) \right| |(\nabla \hat{u}, \nabla w')| \\ &\leq C\big||\nabla \hat{u}| + |\nabla u|\big|\, \big||\nabla \hat{u}| - |\nabla u|\big| |(-\Delta)\hat{u}|\, |w'| \qquad (32)\\ &\leq C\|\hat{u} - u\|\, |(-\Delta)\hat{u}|\, |w'| \\ &\leq C\|w\|\, |w'| \end{aligned}$$

Substituting (30)–(32) in (28) and noting that

$$\begin{aligned} &M(\int_\Omega |\nabla u|^2 dx) \frac{d}{dt} |\nabla w|^2 \\ &= \frac{d}{dt}\left(M(\int_\Omega |\nabla u|^2 dx) |\nabla w|^2 \right) - \left[\frac{d}{dt} M(\int_\Omega |\nabla u|^2 dx) \right] |\nabla w|^2 \end{aligned}$$

we obtain:

$$\begin{aligned} &\frac{d}{dt}\left\{ |w'|^2 + |v|^2 + \frac{1}{2}|\Delta w|^2 + M(\int_\Omega |\nabla u|^2 dx)|\nabla w|^2 \right\} + \|v\|^2 \\ &\leq |v|^2 + C|w'|^2 + C\|w\|^2 + \left| \frac{d}{dt} M(\int_\Omega |\nabla u|^2 dx) \right| |\nabla w|^2 \qquad (33)\\ &\leq C\left\{ |v|^2 + |w'|^2 + \|w\|^2 \right\} \leq C\left\{ |v|^2 + |w'|^2 + |\Delta w|^2 \right\} \end{aligned}$$

Integrating (33) from 0 to $t \leq T$ and using the hypotheses on M we have

$$
\begin{aligned}
&|w'(t)|^2 + |v(t)|^2 + \frac{1}{2}|\Delta w(t)|^2 + \int_0^T \|v(s)\|^2 ds \\
\leq\; & C\int_0^t \left\{|v(s)|^2 + |w'(s)|^2 + |\Delta w(s)|^2\right\} ds
\end{aligned}
$$

By Gronwall's Lemma it follows that

$$|v(s)|^2 + |w'(s)|^2 + |\Delta w(s)|^2 \leq 0\,.$$

This implies that $v(t) = w(t) = 0 \ \forall t \in [0,T]$. Or $u(t) = \hat{u}(t)$ and $\theta(t) = \hat{\theta}(t) \ \forall t \in [0,T]$. This concludes the proof of uniqueness.

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