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An Efficient Algorithm for Computing Zero-Order Hankel Transforms

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Abstract

Postnicov [22], in 2003, proposed an algorithm to compute the Hankel transforms of order zero and one by using Haar wavelets. But the proposed method faced problems in the evaluation of zero order Hankel transform. The purpose of this paper is to overcome this problem and obtain an efficient algorithm for evaluating Hankel transform of order zero by using rationalized Haar wavelets. Exact analytical representation of the Hankel transform, as series of the Bessel functions multiplied by the wavelet coefficients of the input function, is obtained. Numerical examples are given to illustrate the proposed algorithm.

Mathematics Subject Classification: 44A15; 65R10

Keywords: Finite Hankel transform; Bessel functions; Rationalized Haar Wavelets

1. Introduction

In many physical problems like propagation of light beams through systems with cylindrical symmetry, one needs to compute the nth-order Hankel transform and the inverse Hankel transform,

$$F_n(p) = \int_0^\infty f(r) r J_n(pr)\,dr\ . \tag{1}$$

$$f(r) = \int_0^\infty F_n(p)\, p J_n(pr)\,dp\ , \tag{2}$$

where J_n the nth-order Bessel function of the first kind, r is the radial coordinate and $p/2\pi$ is the spatial frequency. Analytical evaluations of (1) and (2) are rare and their numerical computations are difficult because of the oscillatory behaviour of the Bessel function and the infinite length of the interval. Since seminal work by Siegman [1] in 1977, a number of algorithms for the numerical evaluation of the Hankel transform have been published for both zero-order [2-11] and high-order [12-22] Hankel transform. Unfortunately, the efficiency of a method for computing Hankel transform is highly dependent on the function to be transformed, and thus it is difficult to choose the optimal algorithm for given function. In [9], the authors used Filon quadrature Philosophy to evaluate zero-order Hankel transform. They separated the integrand into the product of (assumed) slowly varying component and a rapidly oscillating one (in this case, former is $f(r)$ and the latter is $rJ_0(pr)$). This methods works quite well for computing $F_0(p)$, for $p \geq 1$, but the calculation of inverse Hankel transform is more difficult, as $F_0(p)$ is no longer a smooth function but a rapidly oscillating one.

Postnikov [22] in 2003, gave the algorithm for evaluating the Hankel transform of zero and first order by representing the transforms as series of Bessel and Struve functions multiplied by the wavelet coefficients of the input function, but due to involvement of Struve functions in the representation of $F_0(p)$, the author could not compute it numerically. Recently in 2008, we [23] have given an efficient algorithm to compute Hankel transform using linear Legendre multi-wavelets.

As the zero-order Hankel transform naturally arises in a variety of applications of technological interest, including optics [6,10], acoustics [2], electromagnetics [24] and image processing [25] and the method [22] faced difficulties in its evaluation, it motivated us for the present work

The aim of this paper is to represent $F_0(p)$, as series of Bessel functions (only) multiplied by the Rationalized Haar (RH) wavelet coefficients of the input function, thereby getting an efficient algorithm for numerical evaluation of Hankel transforms of order zero. Numerical examples are given to illustrate the efficiency of proposed algorithm.

2. Preliminaries

The orthogonal set of Haar wavelets is a group of square waves with magnitudes $\pm 2^{j/2}$ and 0, $j = 0,1,2...$ [26]. The use of Haar wavelets comes from rapid convergence feature of Haar series in the expansions of function compared with that of Walsh series. Lynch et al. [27] have rationalized the Haar functions by deleting the irrational numbers and introducing the integral powers of two. The modification results in what is called the RH functions. The RH functions preserves all the properties of the original Haar functions and can be efficiently implemented using digital pipeline architecture [28]. The corresponding functions are known as RH functions.

The orthogonal set of RH functions is a group of square waves with magnitude of ± 1 in some interval and zeroes elsewhere [29]. The first function is $h_0(r) = 1$. The second function $h_1(r)$ is the fundamental square wave, or mother wavelet which also spans the whole interval [0,1),

$$h_1(r) = \begin{cases} 1 & 0 \le r < 1/2 \\ -1 & 1/2 \le r < 1 \end{cases}, \tag{3}$$

And others are generated as follows for $n > 1$,

$$h_i(r) = h_1(2^j r - k) \text{ .where } \; i = 2^j + k\,,\; j > 0,\; 0 \le \mathrm{k} < 2^{\mathrm{j}}.\; i, j, k \text{ are integers.} \tag{4}$$

The first function $h_0(r)$ is also included to make this set complete. The orthogonality property is given by

$$\int_0^1 h_n(r)\, h_i(r)\, dr = \begin{cases} 2^{-j}, & n = i = 2^j + k \\ 0\;\;, & n \ne i \end{cases}. \tag{5}$$

3. Outlines of algorithm

In practical applications, usually the function $f(r)$ has compact support and in many cases, though, the support may not be compact, given any $\varepsilon > 0$, there exist a compact interval I_ε such that $|f(r)| < \varepsilon$ for $r \notin I_\varepsilon$. Hence it is more appropriate to consider the finite Hankel transform. Suppose $f(r)$ is supported on $[0, h]$, then (1) reduces to

$$\hat{F}_n(p) = \int_0^1 f(r) r J_n(pr) dr\,, \tag{6}$$

known as the finite Hankel transform of $f(r)$, where r is replaced by r/h. Writing $g(r) = rf(r)$ in equation (6), we get

$$\hat{F}_n(p) = \int_0^1 g(r) J_n(pr) r dr\;. \tag{7}$$

The inverse finite Hankel transform is represented as Fourier-Bessel series [30].
The RH series representation of any function $g(r)$, which is square integrable in $[0,1)$ may be expanded as

$$g(r) = \sum_{i=0}^{\infty} c_i\, h_i(r)\,,\ r \in [0,1) \tag{8}$$

where,

$$c_0 = \int_0^1 g(r)dr\,, \tag{9}$$

and

$$c_i = 2^j \int_0^1 g(r)\, h_i(r)\, dr \quad \text{for} \quad i = 2^j + k \ \geq 1 \tag{10}$$

The series in equation (8) contains an infinite number of terms. We truncate it at level $i = m-1$, where $m = 2^{\alpha}$ for positive integer α, then equation (8) is reduced to

$$g(r) = \sum_{i=0}^{m-1} c_i\, h_i(r)\,. \tag{11}$$

Substituting (11) in (7), we get

$$\hat{F}_0(p) = \sum_{i=0}^{m-1} c_i \int_0^1 h_i(r)\, J_0(pr)dr$$

$$= \frac{1}{p}\left[c_0 \int_0^p J_0(t)dt + \sum_{i=1}^{m-1} c_i \left(2 \int_0^{2^{-j}p(k+1/2)} J_0(t)dt - \int_0^{2^{-j}pk} J_0(t)dt - \int_0^{2^{-j}p(k+1)} J_0(t)dt \right)\right]. \tag{12}$$

(By change of variable and (4)).

Using the following integral $\int_0^a J_0(t)dt = 2\sum_{n=0}^{\infty} J_{2n+1}(a)$, [31] in (12), one obtains

$$\hat{F}_0(p) = \frac{2}{p}\sum_{n=0}^{N}\left[c_0 J_{2n+1}(p) + \sum_{i=1}^{m-1} c_i \left(2J_{2n+1}(2^{-j}p(k+1/2)) - J_{2n+1}(2^{-j}pk) - J_{2n+1}(2^{-j}p(k+1))\right)\right]$$

$$\text{as } N \to \infty\,. \tag{13}$$

The corresponding representation for the inverse finite Hankel transform of $\hat{F}_0(p)$ is similar to the equation (13) with obvious modifications. If the integral in (9) and (10) evaluating c_i have closed form solutions, equation (13) gives us the full analytical solution for the zero-order Hankel transform. Otherwise the numerical solution is sought and our method provides an efficient algorithm for that.

4. Numerical examples

As $\hat{F}_0(p)$ contains only combination of Bessel functions and one can use their properties, such as orthogonality, known locality of the zeroes and the extremums, to obtain numerical solutions for $F_0(p)$ locally, by approximating them with truncated RH series (13) for finite Hankel transform $\hat{F}_0(p)$. We observe that the approximation is quite accurate from the illustrative numerical example possessing exact solution.

Example1. Let $f(r) = e^{-a^2 r^2}$ then $F_0(p) = \dfrac{1}{2a^2} e^{-p^2/4a^2}$ [31]. (14)

Here $f(r)$ is a rapidly decreasing function .The RH series representation (13) gives the exact analytical solutions of (14). It is observed that the Hankel transform of order zero, $F_0(p)$ and approximate transform $\hat{F}_0(p)$ truncated at level $\alpha = 4$ and $N = 31$ coincide as shown in fig.1. Note that, here replacement r to r/h is used. Also $F_0(p)$ and $\hat{F}_0(p)$ are marked as f0(p) (solid line) and F0(p) (dotted line) respectively in the figure. The error E(p) =F0(p) – f0(p) is shown in fig.2.

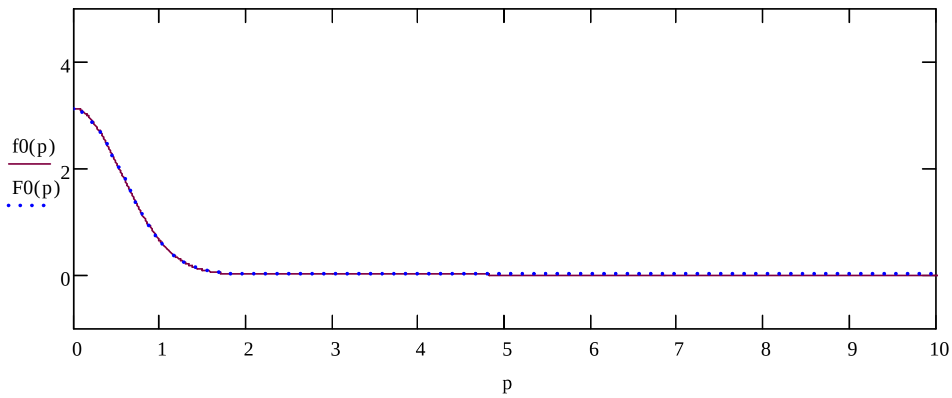

Fig.1.Exact transform, f0(p) (solid line), and the transform F0(p), (dotted line) truncated at level $\alpha = 4$ and $N = 31$.

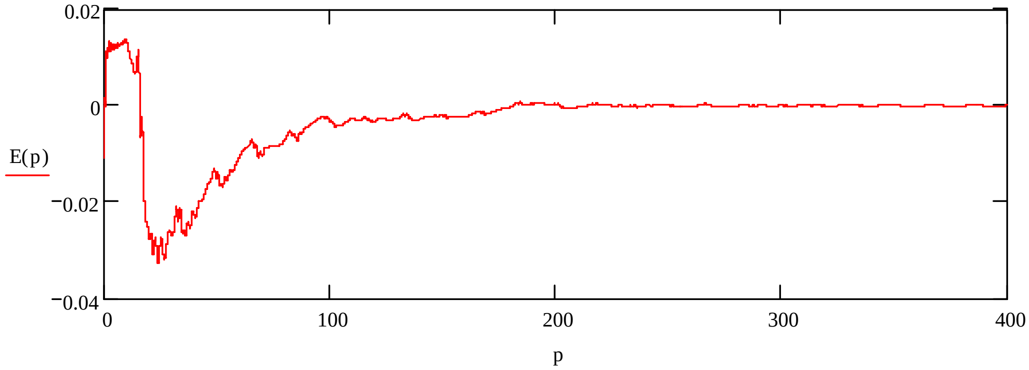

Fig.2. Error between approximated transform, F0 (p) and exact transform f0(p) for $\alpha = 4$ and $N = 31$.

Example 2: Let $f(r) = \frac{2}{\pi}\left[\arccos(r) - r(1-r^2)^{1/2}\right], \quad 0 \le r \le 1,$

then,

$$F_0(p) = 2\frac{J_1^2(p/2)}{p^2}, \qquad 0 \le p \le \infty . \text{ [9]} \tag{15}$$

Barakat et al. [9], evaluated $F_0(p)$ numerically using Filon quadrature philosophy but the associated error is appreciable for $p < 1$; whereas our methods give almost zero errors in that range. Note that $F_0(p)$ and $\hat{F}_0(p)$ are indicated by $f0(p)$ (solid line) and $F0(p)$ (dotted line) in the Figs.3 and 4 respectively.

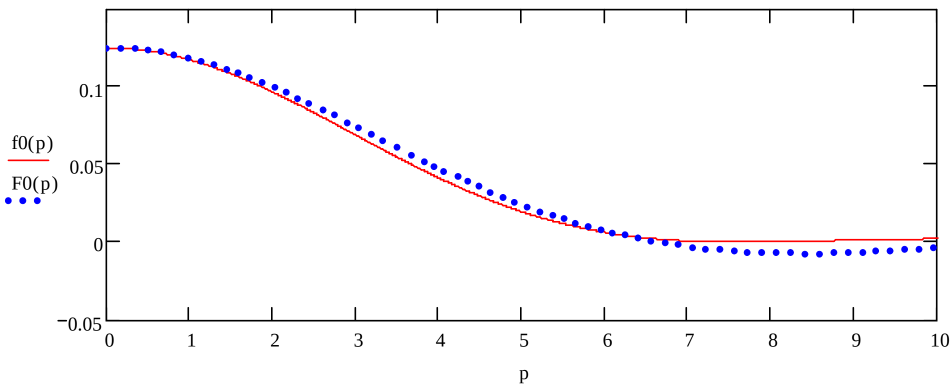

Fig.3. Error between approximated transform, F0(p) and exact transform f0(p) for $\alpha = 4$ and $N = 31$.

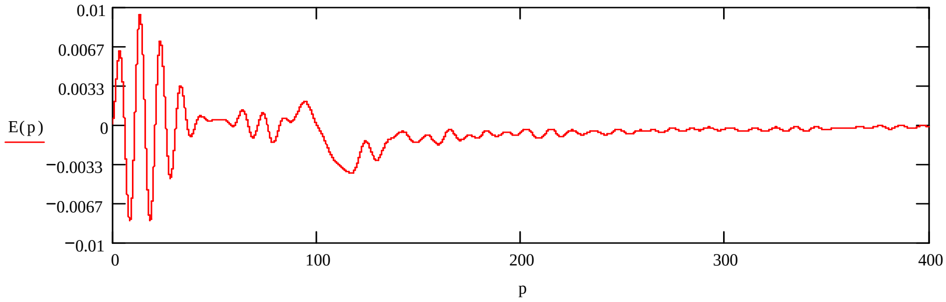

Fig.4. Error between approximated transform, F0(p) and exact transform f0(p) for $\alpha = 4$ and $N = 31$.

Example 3: Sombrero function

The following example was studied by [3,23,32], we apply our proposed method to solve it and it is found that the proposed method is better.

$$\mathrm{Circ}(r/a) = \begin{cases} 1, & r \le a, \\ 0, & r > a. \end{cases} \qquad (16)$$

The zero-order HT of $\mathrm{Circ}(r/a)$ is the Sombrero function, given by

$$S_0(p) = a^2 \frac{J_1(ap)}{ap}.$$

We use Eq. (13) to obtain approximation for the FHT $\widehat{F}_0(p)$ of the $\mathrm{Circ}(r/a)$. These approximations are compared with the exact HT $S_0(p)$ and are shown in Figs.5 and 6 respectively. Fig.6 represents the corresponding error $E(p) = \widehat{F}_0(p) - S_0(p)$. Note that $S_0(p)$ and $\widehat{F}_0(p)$ are indicated by $f0(p)$ (solid line) and $F0(p)$ (dotted line) in the Figs.5 and 6 respectively.

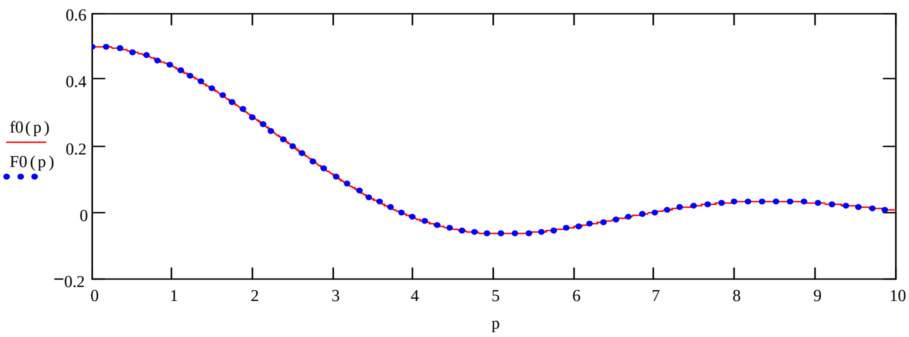

Fig.5.Exact transform, f0 (p) (solid line), and the transform F0 (p), (dotted line) truncated at level $\alpha = 4$ and $N = 31$.

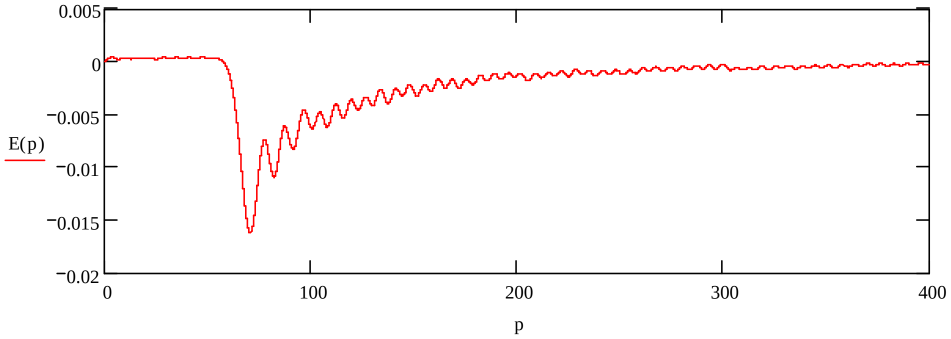

Fig.6. Error between approximated transform, F0 (p) and exact transform f0(p) for $\alpha = 4$ and $N = 31$.

Conclusion

The proposed algorithm is simple, efficient and is better than those given by [3,9,23,32].

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